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REMARKS ON PARTIAL b-METRIC SPACES AND FIXED POINT THEOREMS

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Abstract. In this paper, we prove some properties of a partial b-metric space in the sense of Shukla. As applications, we show that fixed point theorems on partial b-metric spaces can be implied from certain fixed point theorems on b-metric spaces. We also give examples to illustrate the results.

1. Introduction and preliminaries

In [4], Bakhtin introduced the notion of a b-metric space as a generalization of a metric space.

Definition 1.1. [4] Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a function satisfying:

1. $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. There exists $s \geq 1$ such that $d(x, y) \leq s\,[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a *b-metric* on X and (X, d) is called a *b-metric space* with a coefficient s.

This was previously studied in [6] for the case $s = 2$. A b-metric space is also called a *metric-type space* in the sense of [9, Definition 2.1]. b-metric spaces and fixed point theorems on b-metric spaces were investigated in many papers, see [8, 12–15] and some references therein.

In [11], Matthews introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. In that space, the usual metric was replaced by a partial metric with an interesting property that the self-distance of any point of space may not be zero.

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DEFINITION 1.2. [11] Let X be a non-empty set and $p : X \times X \to \mathbb{R}^+$ be a function satisfying:

1. $p(x, x) = p(x, y) = p(y, y)$ if and only if $x = y$ for all $x, y \in X$.
2. $p(x, x) \leq p(x, y)$ for all $x, y \in X$.
3. $p(x, y) = p(y, x)$ for all $x, y \in X$.
4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ for all $x, y, z \in X$.

Then p is called a *partial metric* on X and (X, p) is called a *partial metric space.*

Partial metric spaces and fixed point theorems on partial metric spaces were investigated by many authors, see [1, 3, 5] and some references therein.

Recently, Shukla introduced the notion of a partial b-metric space as a generalization of a partial metric and b-metric space in [17]. An analogue to Banach contraction principle, as well as a Kannan type fixed point theorem was proved in such space.

DEFINITION 1.3 ([17], Definition 3). Let X be a non-empty set and $b : X \times X \to \mathbb{R}^+$ be a function satisfying:

1. $b(x, x) = b(x, y) = b(y, y)$ if and only if $x = y$ for all $x, y \in X$.
2. $b(x, x) \leq b(x, y)$ for all $x, y \in X$.
3. $b(x, y) = b(y, x)$ for all $x, y \in X$.
4. There exists $s \geq 1$ such that $b(x, y) \leq s\,[b(x, z) + b(z, y)] - b(z, z)$ for all $x, y, z \in X$.

Then b is called a *partial b-metric* on X and (X, b) is called a *partial b-metric space* with coefficient s.

We see that the relation between a partial b-metric space and a b-metric space is alike the relation between a partial metric space and a metric space. As far as the relation between a partial metric space and a metric space is concerned, Samet et al. in [16] established some new fixed point theorems on metric spaces and analogous results on partial metric spaces were implied. Also, in [7], Haghi et al. showed that some fixed point generalizations to partial metric spaces can be obtained from the corresponding results in metric spaces.

In this paper, following the idea used in [7], we present a b-metric from a partial b-metric space and state some relationship between them. As applications, we show that some fixed point theorems on partial b-metric spaces can be implied from certain fixed point theorems on b-metric spaces. We also give examples to illustrate the results.

First we recall some notions and results which will be useful in what follows.

DEFINITION 1.4. [4] Let (X, b) be a b-metric space with coefficient s.

1. A sequence $\{x_n\}$ is called *convergent* to x in X, written as $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} b(x_n, x) = 0$.

2. A sequence $\{x_n\}$ is called a *Cauchy sequence* in X if $\lim_{n,m\to\infty} b(x_n, x_m) = 0$.

3. (X, b) is called *complete* if each Cauchy sequence in X is a convergent sequence.

DEFINITION 1.5. [10]

1. A point $w \in X$ is called a *point of coincidence* and a point $u \in X$ is called a *coincidence point* of two maps $T, g : X \to X$ if $Tu = gu = w$.

2. Two maps $T, g : X \to X$ are called *weakly compatible* if $Tgu = gTu$ for all their coincidence points u.

In [2], Aranđelović and Kečkić approached some fixed point theorems in symmetric spaces. The following Theorem 1.6 is a direct consequence of [2, Proposition 5] and [2, Theorem 3].

THEOREM 1.6. *Let (X, b) be a complete b-metric space with coefficient s and $T : X \to X$ be a map. If $b(Tx, Ty) \leq \lambda b(x, y)$ for all $x, y \in X$ and some $\lambda \in [0, 1)$, then T has a unique fixed point u.*

In [9], Jovanović et al. obtained several fixed point theorems on metric-type spaces, that is, on b-metric spaces. Some of the results are as follows.

THEOREM 1.7 ([9], Theorem 3.7). *Let (X, b) be a b-metric space with coefficient s and $T, g : X \to X$ be two maps such that $TX \subset gX$ and one of these subsets of X is complete. Suppose that there exist non-negative coefficients a_i, $i = 1, \ldots, 5$, such that*

$$2sa_1 + (s + 1)(a_2 + a_3) + (s^2 + s)(a_4 + a_5) < 2 \qquad (1)$$

and that for all $x, y \in X$,

$$b(Tx, Ty) \leq a_1 b(gx, gy) + a_2 b(gx, Tx) + a_3 b(gy, Ty) + a_4 b(gx, Ty) + a_5 . b(gy, Tx)$$

holds. Then T and g have a unique point of coincidence. If, moreover, the pair (T, g) is weakly compatible, then T and g have a unique common fixed point.

THEOREM 1.8 ([9], Theorem 3.11). *Let (X, b) be a b-metric space with coefficient s and $T, g : X \to X$ be two maps such that $TX \subset gX$ and one of these subsets of X is complete. Suppose that there exists $\lambda \in \left(0, \frac{1}{s}\right)$ such that for all $x, y \in X$,*

$$b(Tx, Ty) \leq \lambda \max\left\{ b(gx, gy), b(gx, Tx), b(gy, Ty), \frac{b(gx, Ty)}{2s}, \frac{b(gy, Tx)}{2s} \right\}.$$

Then T and g have a unique point of coincidence. If, moreover, the pair (T, g) is weakly compatible, then T and g have a unique common fixed point.

REMARK 1.9 ([17], Remarks 1 & 2).

1. In a partial b-metric space (X, b), if $b(x, y) = 0$, then $x = y$, but the converse may not be true.

2. Every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact need not hold.

EXAMPLE 1.10 ([17], Example 1). Let $X = \mathbb{R}^+$, $p > 1$ and $b : X \times X \to \mathbb{R}^+$ be defined by

$$b(x, y) = (\max\{x, y\})^p + |x - y|^p \text{ for all } x, y \in X.$$

Then (X, b) is a partial b-metric space with coefficient $s = 2^p > 1$, but it is neither a b-metric nor a partial metric space.

Some more examples of partial b-metrics can be constructed with the help of following propositions.

PROPOSITION 1.11 ([17], Proposition 1). *Let X be a non-empty set such that p is a partial metric and d is a b-metric with coefficient $s > 1$ on X. Then the function $b : X \times X \to \mathbb{R}^+$ defined by $b(x, y) = p(x, y) + d(x, y)$ for all $x, y \in X$ is a partial b-metric on X, that is, (X, b) is a partial b-metric space.*

PROPOSITION 1.12 ([17], Proposition 2). *Let (X, p) be a partial metric space, $q \geq 1$, then (X, b) is a partial b-metric space with coefficient $s = 2^{q-1}$, where b is defined by $b(x, y) = [p(x, y)]^q$ for all $x, y \in X$.*

DEFINITION 1.13 ([17], Definition 4). Let (X, b) be a partial b-metric space with coefficient s.

1. A sequence $\{x_n\}$ is called *convergent* to x in X, written $\lim_{n\to\infty} x_n = x$, if

$$\lim_{n\to\infty} b(x_n, x) = b(x, x).$$

2. A sequence $\{x_n\}$ is called a *Cauchy sequence* in X if $\lim_{n,m\to\infty} b(x_n, x_m)$ exists and is finite.

3. (X, b) is called *complete* if for each Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that

$$\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n\to\infty} b(x_n, x) = b(x, x).$$

Note that in a partial b-metric space, the limit of a convergent sequence may not be unique.

EXAMPLE 1.14 ([17], Example 2). Let $X = \mathbb{R}^+$, $a > 0$ be a constant and define $b : X \times X \to \mathbb{R}^+$ by $b(x, y) = \max\{x, y\} + a$ for all $x, y \in X$. Then (X, b) is a partial b-metric space with arbitrary coefficient $s \geq 1$. If $x_n = 1$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n = y$ for all $y \geq 1$.

2. Main results

First, we introduce the following notions on a partial b-metric space.

DEFINITION 2.1. Let (X, b) be a partial b-metric space.

1. A sequence $\{x_n\}$ is called a *0-Cauchy sequence* if $\lim_{n,m\to\infty} b(x_n, x_m) = 0$.
2. (X, b) is called *0-complete* if for each 0-Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that
$$\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n\to\infty} b(x_n, x) = b(x, x) = 0.$$

The relation between completeness and 0-completeness of a partial b-metric space is as follows.

LEMMA 2.2. *Let (X, b) be a partial b-metric space. If (X, b) is complete, then it is 0-complete.*

Proof. Let $\{x_n\}$ be a 0-Cauchy sequence in (X, b). Then $\lim_{n,m\to\infty} b(x_n, x_m) = 0$. This proves that $\{x_n\}$ is a Cauchy sequence in (X, b). Since (X, b) is complete, there exists $x \in X$ such that
$$\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n\to\infty} b(x_n, x) = b(x, x).$$
Since $\lim_{n,m\to\infty} b(x_n, x_m) = 0$, we have
$$\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n\to\infty} b(x_n, x) = b(x, x) = 0.$$
This proves that (X, b) is 0-complete. □

The converse of Lemma 2.2 does not hold as shown in the following example.

EXAMPLE 2.3. Let $X = (0, 1)$ and $b(x, y) = |x - y| + 1$ for all $x, y \in X$. Then (X, b) is a 0-complete, partial b-metric space with coefficient $s = 1$. Since
$$\lim_{n,m\to\infty} b\Big(\frac{1}{n}, \frac{1}{m}\Big) = \lim_{n,m\to\infty} \Big(\Big|\frac{1}{n} - \frac{1}{m}\Big| + 1\Big) = 1$$
we have $\{\frac{1}{n}\}$ is a Cauchy sequence in (X, b). Suppose on the contrary that $\lim_{n\to\infty} \frac{1}{n} = x$ in (X, b). Therefore,
$$\lim_{n\to\infty} b(x_n, x) = \lim_{n\to\infty} \Big(\Big|\frac{1}{n} - x\Big| + 1\Big) = b(x, x) = \big|x - x\big| + 1 = 1$$
which implies that $x = 0$. It is a contradiction since $0 \notin X$.

Now we state the relation between a partial b-metric b and certain b-metric on (X, b) as follows.

THEOREM 2.4. *Let (X, b) be a partial b-metric space with coefficient $s \geq 1$. For all $x, y \in X$, put*
$$d_b(x, y) = \begin{cases} 0 & \text{if } x = y \\ b(x, y) & \text{if } x \neq y. \end{cases}$$

Then we have

1. d_b *is a* b*-metric with coefficient* s *on* X.
2. *If* $\lim_{n\to\infty} x_n = x$ *in* (X, d_b), *then* $\lim_{n\to\infty} x_n = x$ *in* (X, b).
3. (X, b) *is 0-complete if and only if* (X, d_b) *is complete.*

Proof. 1. We have d_b is a function from $X \times X$ to $\mathbb{R}^+$. Moreover, $d_b(x, y) = 0$ if and only if $x = y$ and $d_b(x, y) = d_b(y, x)$ for all $x, y \in X$.

For all $x, y, z \in X$, if $x = y$ or $y = z$ or $z = x$, then $d_b(x, y) \leq d_b(x, z) + d_b(z, y)$. If $x \neq y \neq z$, then

$$\begin{aligned} d_b(x, y) = b(x, y) &\leq s\,[b(x, z) + b(z, y)] - b(z, z) \\ &\leq s\,[b(x, z) + b(z, y)] = s\,[d_b(x, z) + d_b(z, y)]\,. \end{aligned}$$

By the above, d_b is a b-metric with coefficient s on X.

2. If there exists n_0 such that $x_n = x$ for all $n \geq n_0$, then $\lim_{n\to\infty} b(x_n, x) = b(x, x)$. This proves that $\lim_{n\to\infty} x_n = x$ in (X, b). So we may assume that $x_n \neq x$ for all $n \in \mathbb{N}$. Then $d_b(x_n, x) = b(x_n, x)$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} x_n = x$ in (X, d_b), we have $\lim_{n\to\infty} d_b(x_n, x) = 0$. Therefore, $\lim_{n\to\infty} b(x_n, x) = \lim_{n\to\infty} d_b(x_n, x) = 0$. Note that $0 \leq b(x, x) \leq b(x_n, x)$ for all $n \in \mathbb{N}$, then $0 \leq b(x, x) \leq \lim_{n\to\infty} b(x_n, x) = 0$. This proves $\lim_{n\to\infty} b(x_n, x) = 0 = b(x, x)$, that is, $\lim_{n\to\infty} x_n = x$ in (X, b).

3. *Necessity.* Let $\{x_n\}$ be a Cauchy sequence in (X, d_b). Then $\lim_{n,m\to\infty} d_b(x_n, x_m) = 0$. If there exists n_0 such that $x_n = x$ for all $n \geq n_0$, then $\lim_{n\to\infty} x_n = x$ in (X, d_b). So, we may assume that $x_n \neq x_m$ for all $n \neq m$. It implies that

$$\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n,m\to\infty} d_b(x_n, x_m) = 0.$$

Then $\{x_n\}$ is a 0-Cauchy sequence in (X, b). Since (X, b) is 0-complete, there exists $x \in X$ such that

$$\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n\to\infty} b(x_n, x) = b(x, x) = 0.$$

Note that $0 \leq d_b(x_n, x) \leq b(x_n, x)$ for all $n \in \mathbb{N}$, then

$$0 \leq \lim_{n\to\infty} d_b(x_n, x) \leq \lim_{n\to\infty} b(x_n, x) = 0.$$

Then $\lim_{n\to\infty} d_b(x_n, x) = 0$. This proves that $\lim_{n\to\infty} x_n = x$ in (X, d_b). By the above, (X, d_b) is complete.

Sufficiency. Let $\{x_n\}$ be a 0-Cauchy sequence in (X, b). Then $\lim_{n,m\to\infty} b(x_n, x_m) = 0$. Since $0 \leq d_b(x_n, x_m) \leq b(x_n, x_m)$ for all $n, m \in \mathbb{N}$, we have $\lim_{n,m\to\infty} d_b(x_n, x_m) = 0$. This proves that $\{x_n\}$ is a Cauchy sequence in (X, d_b). Since (X, d_b) is complete, there exists $x \in X$ such that $\lim_{n\to\infty} d_b(x_n, x) = 0$. If there exists n_0 such that $x_n = x$ for all $n \geq n_0$, then $\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n\to\infty} b(x_n, x) = b(x, x)$. Since $\lim_{n,m\to\infty} b(x_n, x_m) = 0$, we get $\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n\to\infty} b(x_n, x) = b(x, x) = 0$. So, we may assume that

$x_n \neq x_m$ for all $n, m \in \mathbb{N}$. Then $\lim_{n\to\infty} b(x_n, x) = \lim_{n\to\infty} d_b(x_n, x) = 0$. Note that $0 \leq b(x,x) \leq b(x_n, x)$ for all $n \in \mathbb{N}$. Then $0 \leq b(x,x) \leq \lim_{n\to\infty} b(x_n, x) = 0$, that is, $b(x,x) = 0$. Therefore, we also have

$$\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n\to\infty} b(x_n, x) = b(x,x) = 0.$$

By the above, (X, b) is 0-complete. □

The following example shows that the converse of statement 2 from Theorem 2.4 does not hold.

EXAMPLE 2.5. Let $X = [0,1]$ and $b(x,y) = |x-y| + 1$ for all $x, y \in X$. Then (X, b) is a partial b-metric space with coefficient $s = 1$. We see that

$$\lim_{n\to\infty} b\Big(\frac{1}{n}, 0\Big) = \lim_{n\to\infty} \Big[\Big|\frac{1}{n} - 0\Big| + 1\Big] = 1 = b(0,0).$$

This proves that $\lim_{n\to\infty} \frac{1}{n} = 0$ in the partial b-metric space (X, b). On the other hand, we have

$$d_b(x,y) = \begin{cases} 0 & \text{if } x = y \\ |x-y| + 1 & \text{if } x \neq y. \end{cases}$$

Then

$$\lim_{n\to\infty} d_b\Big(\frac{1}{n}, 0\Big) = \lim_{n\to\infty} \Big[\Big|\frac{1}{n} - 0\Big| + 1\Big] = 1 \neq 0.$$

This proves that $\lim_{n\to\infty} \frac{1}{n} \neq 0$ in the b-metric space (X, d_b).

The relation between contraction conditions on partial b-metric spaces in [17] and certain contraction conditions on b-metric spaces is as follows.

THEOREM 2.6. *Let (X, b) be a partial b-metric space with coefficient s, d_b be defined as in Theorem 2.4 and $T : X \to X$ be a map. Then we have*

1. *If there exists $\lambda \in [0,1)$ such that $b(Tx, Ty) \leq \lambda b(x,y)$ for all $x, y \in X$, then $d_b(Tx, Ty) \leq \lambda d_b(x,y)$ for all $x, y \in X$.*
2. *If there exists $\lambda \in \big[0, \frac{1}{2}\big)$ such that $b(Tx, Ty) \leq \lambda\,[b(x, Tx) + b(y, Ty)]$ for all $x, y \in X$, then $d_b(Tx, Ty) \leq \lambda\,[d_b(x, Tx) + d_b(y, Ty)]$ for all $x, y \in X$.*
3. *If there exists λ such that $b(Tx, Ty) \leq \lambda \max\{b(x,y), b(x,Tx), b(y,Ty)\}$ for all $x \neq y \in X$, then $d_b(Tx, Ty) \leq \lambda \max\{d_b(x,y), d_b(x,Tx), d_b(y,Ty)\}$ for all $x, y \in X$.*

Proof. 1. If $x = y$, then $d_b(Tx, Ty) = 0 \leq \lambda d_b(x,y)$. If $x \neq y$, then $d_b(x,y) = b(x,y)$ and we have $d_b(Tx, Ty) \leq b(Tx, Ty) \leq \lambda b(x,y) = \lambda d_b(x,y)$. Therefore, $d_b(Tx, Ty) \leq \lambda d_b(x,y)$ for all $x, y \in X$.

2. If $x = Tx$, then $b(x, Tx) = b(Tx, Tx) \leq \lambda\big[b(x,Tx) + b(x,Tx)\big] = 2\lambda b(x, Tx)$. Since $2\lambda \in \big[0,1\big)$, we have $b(x, Tx) = 0 = d_b(x, Tx)$. It implies that $b(x, Tx) = d_b(x, Tx)$ for all $x \in X$. Therefore, for all $x, y \in X$,

$$d_b(Tx, Ty) \leq b(Tx, Ty) \leq \lambda\,[b(x,Tx) + b(y,Ty)] = \lambda\,[d_b(x,Tx) + d_b(y,Ty)]\,.$$

3. For all $x, y \in X$, we have

$$\max\{d_b(x,y), d_b(x,Tx), d_b(y,Ty)\} \leq \max\{b(x,y), b(x,Tx), b(y,Ty)\}. \tag{2}$$

In order to prove that

$$\max\{b(x,y), b(x,Tx), b(y,Ty)\} \leq \max\{d_b(x,y), d_b(x,Tx), d_b(y,Ty)\} \tag{3}$$

for all $x \neq y \in X$, we distinguish between two cases.

Case 1. *There exist* $x, y \in X$ *such that* $\max\{b(x,y), b(x,Tx), b(y,Ty)\} = b(x,y)$. Since $b(x,y) = d_b(x,y)$, we see that (3) holds.

Case 2. *There exist* $x, y \in X$ *such that* $\max\{b(x,y), b(x,Tx), b(y,Ty)\} = b(x,Tx)$. If $x = Tx$, then $b(x,Tx) = b(x,x) \leq b(x,y) = d_b(x,y)$. Therefore, (3) holds. If $x \neq Tx$, then $b(x,Tx) = d_b(x,Tx)$. It also implies that (3) holds.

By the above two cases, we see that (3) holds for all $x \neq y$. It follows from (2) and (3) that, for all $x \neq y$,

$$\max\{d_b(x,y), d_b(x,Tx), d_b(y,Ty)\} = \max\{b(x,y), b(x,Tx), b(y,Ty)\}.$$

Therefore,

$$\begin{aligned} d_b(Tx,Ty) \leq b(Tx,Ty) &\leq \lambda \max\{b(x,y), b(x,Tx), b(y,Ty)\} \\ &= \lambda \max\{d_b(x,y), d_b(x,Tx), d_b(y,Ty)\} \end{aligned}$$

for all $x \neq y$. If $x = y$, we have $d_b(Tx,Ty) = 0$. Then

$$d_b(Tx,Ty) \leq \lambda \max\{d_b(x,y), d_b(x,Tx), d_b(y,Ty)\}$$

for all $x, y \in X$. □

In what follows, by using Theorem 2.6, we show that fixed point theorems on partial b-metric spaces in [17] can be implied from certain fixed point theorems on b-metric spaces.

Corollary 2.7 ([17], Theorem 1). *Let (X,b) be a complete partial b-metric space with coefficient s and $T : X \to X$ be a map. If $b(Tx,Ty) \leq \lambda b(x,y)$ for all $x, y \in X$ and some $\lambda \in [0,1)$, then T has a unique fixed point u and $b(u,u) = 0$.*

Proof. From Lemma 2.2, since (X,b) is complete, (X,b) is 0-complete. Then (X,d_b) is complete by Theorem 2.4. From Theorem 2.6.(1), we have $d_b(Tx,Ty) \leq \lambda d_b(x,y)$ for all $x, y \in X$. It follows from Theorem 1.6 that T has a unique fixed point u. Since

$$b(u,u) = b(Tu,Tu) \leq \lambda b(u,u)$$

and $\lambda \in \big[0,1\big)$, we have $b(u,u) = 0$. □

In the proof of [17, Theorem 2], on page 6, at lines 19-20, we see that the inequality

$$b(u,Tu) \leq \frac{s}{1 - s\lambda} b(u, x_{n+1}) + \frac{s.\lambda}{1 - s.\lambda} b(x_n, x_{n+1})$$

only holds if $\lambda < \frac{1}{s}$. Therefore, the assumption $\lambda \neq \frac{1}{s}$ in [17, Theorem 2] may not be suitable. In what follows, we restate [17, Theorem 2], where the assumption $\lambda \neq \frac{1}{s}$ is replaced by $\lambda < \frac{1}{s}$.

Corollary 2.8. *Let (X,b) be a complete partial b-metric space with coefficient s and $T : X \to X$ be a map. If $b(Tx,Ty) \leq \lambda[b(x,Tx) + b(y,Ty)]$ for all $x, y \in X$ and some $\lambda \in \big[0, \frac{1}{2}\big)$ and $\lambda < \frac{1}{s}$, then T has a unique fixed point u and $b(u,u) = 0$.*

Proof. From Lemma 2.2, since (X, b) is complete, (X, b) is 0-complete. Then (X, d_b) is complete by statement 3 of Theorem 2.4. From statement 3 of Theorem 2.6, we have $d_b(Tx, Ty) \leq \lambda [d_b(x, Tx) + d_b(y, Ty)]$ for all $x, y \in X$.

Note that the condition (1) in Theorem 1.7 was used to prove the inequality (3.16) and the inequality

$$K(a_2 + a_3 + a_4 + a_5) < 2$$

at line -3, on page 7 in the proof of [9, Theorem 3.7], where K plays the role of s. These claims hold if $a_1 = 0$ and $a_2 + a_3 + s(a_4 + a_5) < \min\left\{1, \frac{2}{s}\right\}$. Therefore, using this modification of Theorem 1.7 with g being the identity and $a_2 = a_3 = \lambda$, we see that T has a unique fixed point u. Since

$$b(u, u) = b(Tu, Tu) \leq \lambda\big[b(u, Tu) + b(u, Tu)\big] = 2\lambda b(u, u)$$

and $2\lambda \in [0, 1)$, we have $b(u, u) = 0$. □

Corollary 2.9 ([17], Theorem 3). *Let (X, b) be a complete partial b-metric space with coefficient s and $T : X \to X$ be a map. If*

$$b(Tx, Ty) \leq \lambda \max\{b(x, y), b(x, Tx), b(y, Ty)\}$$

for all $x, y \in X$ and $\lambda \in \left[0, \frac{1}{s}\right)$, then T has a unique fixed point u and $b(u, u) = 0$.

Proof. From statement 3 of Theorem 2.6, we have

$$d_b(Tx, Ty) \leq \lambda \max\{d_b(x, y), d_b(x, Tx), d_b(y, Ty)\}$$

for all $x, y \in X$. By using Theorem 1.8 with g being the identity, we see that T has a unique fixed point u. Since

$$b(u, u) = b(Tu, Tu) \leq \lambda \max\{b(u, u), b(u, Tu), b(u, Tu)\} = \lambda b(u, u)$$

and $\lambda \in \left[0, \frac{1}{s}\right)$, we have $b(u, u) = 0$. □

The following example shows that for a partial b-metric space (X, b), the function d_b in Theorem 2.4 may not be a metric. Then the results of [7] may not be applicable to the above proofs.

Example 2.10. Let (X, b) be a partial b-metric space in Example 1.10 with $p = 2$. Then we have

$$d_b(x, y) = \begin{cases} 0 & \text{if } x = y \\ (\max\{x, y\})^2 + |x - y|^2 & \text{if } x \neq y. \end{cases}$$

We have $d_b(2, 0) = 2^2 + 2^2 = 8$, $d_b(2, 1) = 2^2 + 1^2 = 5$, $d_b(1, 0) = 1^2 + 1^2 = 2$. Then

$$d_b(2, 0) = 8 > 7 = d_b(2, 1) + d_b(1, 0).$$

This proves that d_b is not a metric on X.

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