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QUASI-REGULARITY OF HARMONIC MAPS BASED ON BLASCHKE PRODUCTS

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Abstract. The purpose of this paper is to find conditions which guarantee quasiregularity of a harmonic map of the unit disk $\mathbb{D}$ of the form $f(z) = \operatorname{Re} B_1(z) + i \operatorname{Im} B_2(z)$, where B_1, B_2 are automorphisms of $\mathbb{D}$.

1. Introduction

Quasiconformal harmonic maps are natural generalization of the notion of conformal map. A function w is called harmonic in a region D if it has the form $w = u+iv$ where u and v are real-valued harmonic functions in D. If D is simply-connected, then there are two analytic functions g and h defined on D such that w has the representation $w = g + \overline{h}$.

On the other hand, a quasiconformal map f is an orientation-preserving homeomorphism, which is at least partially differentiable almost everywhere on a domain D in $\mathbb{C}$, satisfying the Beltrami equation $f_{\overline{z}} = \mu f_z$.

More precisely, an analytic definition of quasiconformal map can be given in the following way. Let f be an orientation-preserving homeomorphism of a domain D into $\mathbb{C}$. Then f is quasiconformal if (see for instance [7]):

1. f is absolutely continuous on lines (ACL).

2. There exists a constant k, $0 \leq k < 1$, such that $|f_{\overline{z}}| \leq k\,|f_z|$ almost everywhere on D.

Setting $K = (1+k)/(1-k)$, we say that f is a K-quasiconformal mapping. We call the infimum of $K > 1$ such that f is K-qc, the maximal dilatation of f, and denote it by K_f.

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Quasiregular map is a map which satisfies the conditions 1. and 2. but which is not necessarily a homeomorphism. In this paper, we will consider quasiregular harmonic maps, i.e. we will not require map to be injective.

The first characterization of harmonic quasiconformal mappings with respect to the Euclidean metric for the unit disc was given by O. Martio, [4]. Thereafter this area has been studied by the participants of Belgrade Seminar for Analysis; for a partial review and further results see for example [1, 3, 5].

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk, and denote the circle of radius r and center z_0 by $C(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$. Let $B(z) = \lambda \prod_{j=1}^{n} \frac{a_j - z}{1 - \overline{a_j} z}$, $a_1, \dots, a_n \in \mathbb{D}, |\lambda| = 1$ be a finite Blaschke product, where $B : \mathbb{D} \to \mathbb{D}$, such that $|B(z)| < 1$ for $z \in \mathbb{D}$ and $|B(z)| = 1$ for $z \in \partial\mathbb{D}$.

In this paper, we consider the question, suggested by Matti Vuorienen in a personal communication, when a map of the form $f(z) = \operatorname{Re} B_1(z) + i \operatorname{Im} B_2(z)$ is harmonic quasi-regular, where B_1, B_2 are finite Blaschke products. We will give necessary and sufficient conditions for this (Theorem 2.1), when B_1 and B_2 are automorphisms of the unit disk, give a bound for quasi-regularity constant K (Theorem 2.2), as well as a simpler, sufficient condition (Corollary 2.3).

2. Main results

Our main result is the following theorem.

THEOREM 2.1. *Let $B_1, B_2 : \mathbb{D} \to \mathbb{D}$ be automorphisms of the unit disk given by $B_1(z) = \frac{a-z}{1-\overline{a}z}, B_2(z) = \frac{b-z}{1-\overline{b}z}$, $|a| < 1$, $|b| < 1$. Then the mapping $f : \mathbb{D} \to \mathbb{C}$, where $f(z) = \operatorname{Re} B_1(z) + i \operatorname{Im} B_2(z)$ is quasiregular if and only if*

$$\operatorname{Re}(1+i)(1-\overline{a}b) > \sqrt{2}|a-b| \quad \textit{and} \quad \operatorname{Im}(1+i)(1-\overline{a}b) > \sqrt{2}|a-b|, \tag{1}$$

and $f(z)$ is harmonic for all $a, b \in \mathbb{D}$.

Proof. We compute the real and the imaginary part of $f(z)$:

$$\operatorname{Re} \frac{a-z}{1-\overline{a}z} = \frac{1}{2}\Big(\frac{a-z}{1-\overline{a}z} + \frac{\overline{a}-\overline{z}}{1-a\overline{z}}\Big), \quad \operatorname{Im} \frac{b-z}{1-\overline{b}z} = \frac{1}{2i}\Big(\frac{(b-z)}{(1-\overline{b}z)} - \frac{\overline{b}-\overline{z}}{1-b\overline{z}}\Big).$$

Then $f(z)$ has the form:

$$f(z) = \frac{1}{2}\Big(\frac{a-z}{1-\overline{a}z} + \frac{\overline{a}-\overline{z}}{1-a\overline{z}}\Big) + \frac{1}{2}\Big(\frac{b-z}{1-\overline{b}z} - \frac{\overline{b}-\overline{z}}{1-b\overline{z}}\Big).$$

Now we find the partial derivatives of $f(z)$ with respect to z and $\overline{z}$ respectively:

$$f_z(z) = \frac{1}{2}\Big(\frac{|a|^2-1}{(1-\overline{a}z)^2} + \frac{|b|^2-1}{(1-\overline{b}z)^2}\Big), \quad f_{\overline{z}}(z) = \frac{1}{2}\Big(\frac{|a|^2-1}{(1-az)^2} + \frac{|b|^2-1}{(1-bz)^2}\Big).$$

Let us set

$$A(z) = \frac{|a|^2-1}{(1-\overline{a}z)^2}, \quad B(z) = \frac{|b|^2-1}{(1-\overline{b}z)^2},$$

Then we have:

$$f_z(z) = A(z) + B(z), \quad f_{\overline{z}}(z) = \overline{A(z)} - \overline{B(z)}.$$

Now we find the complex dilation μ of f and $k(z) = |\mu(z)|$:

$$\mu(z) = \frac{f_{\overline{z}}(z)}{f_z(z)} = \frac{\overline{A(z)} - \overline{B(z)}}{A(z) + B(z)},$$

$$k(z) = \Big|\frac{\overline{A(z)} - \overline{B(z)}}{A(z) + B(z)}\Big| = \frac{|\overline{A}(z) - \overline{B}(z)|}{|A(z) + B(z)|} = \frac{|A(z) - B(z)|}{|A(z) + B(z)|}.$$

Then $$k(z) = \frac{|1 - \frac{B(z)}{A(z)}|}{|1 + \frac{B(z)}{A(z)}|} = \frac{|1 - w|}{|1 + w|}, \text{ for } w = \frac{B(z)}{A(z)}.$$

To prove that $f(z)$ is harmonic quasiregular we have to show that for some $q < 1$:

$$\frac{|1 - w|}{|1 + w|} \leq q < 1. \tag{2}$$

The condition $\frac{|1-w|}{|1+w|} < 1$ is equivalent to $\operatorname{Re} w > 0$, where $w = \frac{B(z)}{A(z)} = \frac{|b|^2-1}{(1-\overline{b}z)^2} \frac{(1-\overline{a}z)^2}{|a|^2-1}$. Clearly we have $\operatorname{Re} w = \frac{|b|^2-1}{|a|^2-1} \operatorname{Re} \frac{(1-\overline{a}z)^2}{(1-\overline{b}z)^2}$, and $\frac{|b|^2-1}{|a|^2-1} > 0$, since $|a|, |b| < 1$, the condition $\operatorname{Re} w > 0$ is equivalent to $\operatorname{Re} \frac{(1-\overline{a}z)^2}{(1-\overline{b}z)^2} > 0$.

Let us introduce a map ϕ, where

$$\phi(z) = \frac{1 - \overline{a}z}{1 - \overline{b}z}. \tag{3}$$

Note that ϕ is a Moebius transformation which maps $C(0,1)$ to another circle C_f, the condition $\operatorname{Re} \frac{(1-\overline{a}z)^2}{(1-\overline{b}z)^2} > 0$ for $z \in \mathbb{D}$ is equivalent to C_f belonging to regions I or II, where the region I is $x > |y|$ and the region II is $x < -|y|$.

Let us find the center O_f and radius r_f of C_f:

$$\phi(z) = \frac{1 - \overline{a}z}{1 - \overline{b}z} = \frac{\overline{a}}{\overline{b}}\Big(\frac{z - \frac{1}{\overline{a}}}{z - \frac{1}{\overline{b}}}\Big) = \frac{\overline{a}}{\overline{b}}\Big(1 + \frac{\frac{1}{\overline{b}} - \frac{1}{\overline{a}}}{z - \frac{1}{\overline{b}}}\Big) = \frac{\overline{a}}{\overline{b}} + \frac{\frac{\overline{a}}{\overline{b}} - 1}{\overline{b}z - 1}.$$

We can write $\phi(z) = \phi_1 \circ \phi_2 \circ \phi_3 \circ \phi_4 \circ \phi_5(z)$ such that

$$\phi_5(z) = \overline{b}z,\ \phi_4(z) = z - 1,\ \phi_3(z) = \frac{1}{z},\ \phi_2(z) = (\frac{\overline{a}}{\overline{b}} - 1)z,\ \phi_1(z) = z + \frac{\overline{a}}{\overline{b}}.$$

Let us see what circle $C(0,1)$ is mapped to under ϕ, step by step:

1. map ϕ_5: the image of unit circle under this map is a circle $C(O_1, r_1)$ with center $O_1 = 0$ and radius $r_1 = |b|$, since the map is a homothety with cofficient $|b|$ composed with a rotation preserving the circle center 0.

2. map ϕ_4: this map is a translation by -1, and the image of our circle becomes $C(O_2, r_2)$ with center $O_2 = -1$ and radius $r_2 = |b|$.

3. map ϕ_3: application of the map $\phi_3(z) = \frac{1}{z}$ to circle from the previous step gives the circle $C(O_3, r_3)$ with center O_3 and radius r_3, where, using that the center

is on the real line,

$$O_3 = \frac{1}{2}\Big(\frac{-1}{1-|b|} - \frac{1}{1+|b|}\Big) = \frac{-1}{1-|b|^2},$$

and

$$r_3 = \frac{-1}{1+|b|} + \frac{1}{1-|b|^2} = \frac{|b|}{1-|b|^2}.$$

4. map ϕ_2: this map is multiplication by $(\alpha-1)$, where $\alpha = \frac{\overline{a}}{\overline{b}}$, and we get that image of $C(O_3, r_3)$ under this map is the circle $C(O_4, r_4)$ with center O_4 and radius r_4, where

$$O_4 = (\alpha-1)\frac{-1}{1-|b|^2} = \frac{1-\alpha}{1-|b|^2}, \quad \text{i.e. } O_4 = \frac{1-\overline{a}/\overline{b}}{1-|b|^2} = \frac{\overline{b}-\overline{a}}{\overline{b}(1-|b|^2)},$$

and $$r_4 = |\alpha-1|\frac{|b|}{1-|b|^2}, \quad \text{i.e. } r_4 = \frac{|a-b|}{|b|}\frac{|b|}{1-|b|^2} = \frac{|a-b|}{1-|b|^2}.$$

5. map ϕ_1: this map is a translation by α, where $\alpha = \frac{\overline{a}}{\overline{b}}$, and we get the final circle $C(O_f, r_f)$ with center O_f and radius r_f, where

$$O_f = \frac{\overline{b}-\overline{a}}{\overline{b}(1-|b|^2)} + \frac{\overline{a}}{\overline{b}} = \frac{\overline{b}-\overline{a}|b|^2}{\overline{b}(1-|b|^2)}, \tag{4}$$

and

$$r_f = \frac{|a-b|}{1-|b|^2}. \tag{5}$$

Now we got the final circle with center $O_f = \frac{\overline{b}-\overline{a}|b|^2}{\overline{b}(1-|b|^2)}$, and a radius $r_f = \frac{|a-b|}{1-|b|^2}$.

For convenience, we will apply homotethy with cofficient $1-|b|^2$, we get $O_h = 1-\overline{a}b$, $r_h = |a-b|$ (this homothety preserves regions I, II). Note that, since $|a\overline{b}| < 1$, O_h never lies in region II of Figure 1.

Next, we apply rotation R by $\pi/4$, $R(z) = e^{i\pi/4}z$, which takes the region I to the region $x > 0$, $y > 0$ (i.e. the first quadrant). Our circle will be mapped to circle with center $e^{i\pi/4}\,O_h$ and radius r_h . The circle will be in the first quadrant if and only if $\mathrm{Re}(e^{i\pi/4}\,O_h) > r_h, \mathrm{Im}(e^{i\pi/4}O_h) > r_h$ which is equivalent to:

$$\mathrm{Re}(1+i)(1-\overline{a}b) > \sqrt{2}|a-b|, \quad \mathrm{Im}(1+i)(1-\overline{a}b) > \sqrt{2}|a-b|. \qquad \square$$

THEOREM 2.2. *The quasi-regularity constant* K *for a map* $f(z) = \mathrm{Re}\, B_1(z) + i\,\mathrm{Im}\, B_2(z)$ *from Theorem 2.1 satisfies* $K < \frac{R^4+1}{2\epsilon^2}$, *where* $R = \frac{|1-\overline{a}b|+|a-b|}{\sqrt{(1-|a|^2)(1-|b|^2)}}$ *and* $\epsilon = \min\Big(\frac{(\mathrm{Re}(\frac{1+i}{\sqrt{2}})(1-\overline{a}b))-|a-b|}{\sqrt{(1-|a|^2)(1-|b|^2)}}, \frac{(\mathrm{Im}(\frac{1+i}{\sqrt{2}})(1-\overline{a}b))-|a-b|}{\sqrt{(1-|a|^2)(1-|b|^2)}}\Big)$.

Proof. We will use the notation from the proof of Theorem 2.1. Recall that $w(z) = \frac{1-|b|^2}{1-|a|^2}\Big(\frac{1-\overline{a}z}{1-\overline{b}z}\Big)^2$, and our map f is K-quasi-regular with $K = \frac{1+q}{1-q}$ if and only if (2) holds for all $z \in \mathbb{D}$.

If $M(w) = \frac{1-w}{1+w}$, then $C(0,q) \xrightarrow{M} C(O',r')$, where

$$O' = \frac{1}{2}\Big(\frac{1+q}{1-q} + \frac{1-q}{1+q}\Big) = \frac{1+q^2}{1-q^2}$$

and

$$r' = \Big(\frac{1+q}{1-q} - \frac{1-q}{1+q}\Big) = \frac{2q}{1-q^2}.$$

Since $M(M(w)) = w$, we have that the condition (2) is equivalent to the condition that $w(z)$ lies inside the circle $C(O',r')$.

Now $w(z) = \psi(z)^2$, where $\psi(z) = \sqrt{\frac{1-|b|^2}{1-|a|^2}\frac{1-\overline{a}z}{1-\overline{b}z}} = \sqrt{\frac{1-|b|^2}{1-|a|^2}}\phi(z)$, where the map $\phi(z)$ is given by formula (3) from Theorem 2.1. Since ϕ maps $C(0,1)$ to circle

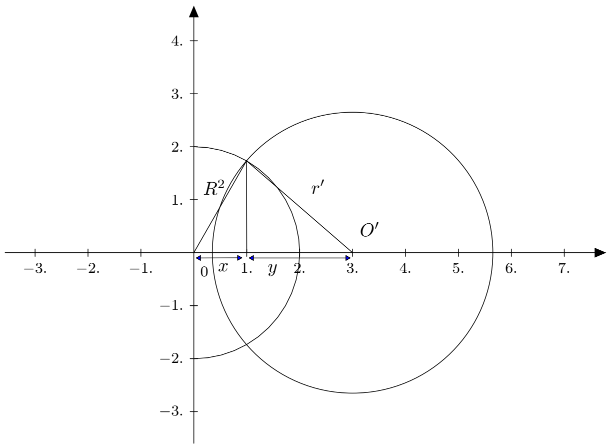

Figure 1: The intersections of circles $C(0,R^2)$ and $C(O',r')$.

$C(O_f, r_f)$, where O_f, r_f are given by (4) and (5), ψ maps $C(0,1)$ to circle $C(O,r)$, where

$$O = \sqrt{\frac{1-|b|^2}{1-|a|^2}}O_f = \frac{|1-\overline{a}b|}{\sqrt{(1-|a|^2)(1-|b|^2)}},$$

and

$$r = \sqrt{\frac{1-|b|^2}{1-|a|^2}}r_f = \frac{|a-b|}{\sqrt{(1-|a|^2)(1-|b|^2)}}.$$

So, we want to find a sufficient condition, so that $C(O,r)$ is mapped inside $C(O',r')$ under the square mapping.

Note that $C(O,r)$ lies inside $C(0,R)$, where $R = |O| + r = \frac{|1-\overline{a}b|+|a-b|}{\sqrt{(1-|a|^2)(1-|b|^2)}}$, and after rotation by the angle $\pi/4$ around 0, $C(O,r)$ is mapped to $C(\frac{1+i}{\sqrt{2}}O, r)$ which lies in the region $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > \epsilon \ \wedge \ \operatorname{Im} z > \epsilon\}$, where

$$\epsilon = \min\Big(\frac{(\operatorname{Re}(\frac{1+i}{\sqrt{2}})(1-\overline{a}b)) - |a-b|}{\sqrt{(1-|a|^2)(1-|b|^2)}}, \frac{(\operatorname{Im}(\frac{1+i}{\sqrt{2}})(1-\overline{a}b)) - |a-b|}{\sqrt{(1-|a|^2)(1-|b|^2)}}\Big).$$

Since $\operatorname{Im}(z^2) = 2\operatorname{Re}(z)\operatorname{Im}(z)$ and $(\frac{1+i}{\sqrt{2}})^2 = i$, the region Ω will be mapped inside the region $\{z \in \mathbb{C} : \operatorname{Re} z \geq 2\epsilon^2\}$ under the map $z \mapsto z^2/(\frac{1+i}{\sqrt{2}})^2$.

Thus, it is enough that region $\{z \in \mathbb{C} : \operatorname{Re} z \geq 2\epsilon^2 \wedge |z| < R^2\}$ is inside the circle $C(O', r')$, which is true if the intersections of circles $C(0, R^2)$ and $C(O', r')$ have real parts greater or equal to $2\epsilon^2$ (see Figure 1).

Now let us compute an upper bound for quasiregularity constant K.

From Figure 1, we have, by Pythagoras's theorem.

$$R^4 - x^2 = r'^2 - y^2 \tag{6}$$

and

$$x + y = O' \tag{7}$$

From (6) we have $R^4 - r'^2 = x^2 - y^2 = (x-y)(x+y)$, so $x - y = \frac{R^4 - r'^2}{x+y} = \frac{R^4 - r'^2}{O'}$, and therefore $x = \frac{R^4 - r'^2}{O'} + y = \frac{R^4 - r'^2}{O'} + O' - x$. Then $x = \frac{1}{2}\left[\frac{R^4 - r'^2 + O'^2}{O'}\right]$, so our condition is $\frac{1}{2}\left[\frac{R^4 - r'^2 + O'^2}{O'}\right] > 2\epsilon^2$, i.e. $R^4 - r'^2 + O'^2 > 4\epsilon^2 O'$. Substituting $O' = \frac{1+q^2}{1-q^2}$, $r' = \frac{2q}{1-q^2}$, we get:

$$R^4 + 1 > 4\epsilon^2 \frac{1+q^2}{1-q^2}.$$

or

$$\frac{1+q^2}{1-q^2} < \frac{R^4+1}{4\epsilon^2}.$$

Note that, for $|q| < 1$, $\frac{1+q}{1-q} = \frac{(1+q)^2}{1-q^2} = \frac{1+q^2}{1-q^2} + \frac{2q}{1-q^2} < 2\,\frac{1+q^2}{1-q^2}$. Since $\frac{1}{2}K = \frac{1}{2}\frac{1+q}{1-q} < \frac{1+q^2}{1-q^2}$, we get $K < \frac{R^4+1}{2\epsilon^2}$. $\square$

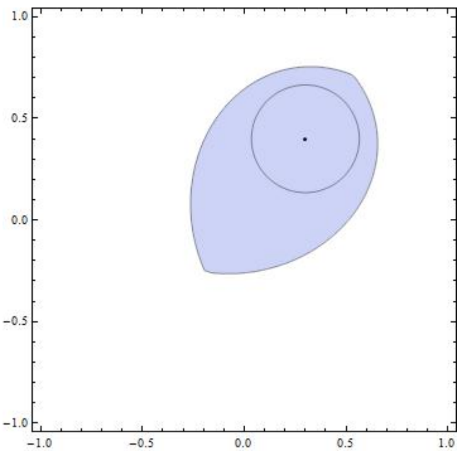

Figure 2: The region of points b around point a where $|a - b| < g(a, b)$ and a circle around $a = 0.3 + i\,0.4$ with radius $r = \frac{0.75}{2\sqrt{2}}$.

Now that we have estimated K, we give a simpler condition for quasiregularity of our map f. Our region, given by inequalities (1), is illustrated in Figure 2. Note that condition (1) is of the form $|a-b| < g(a,b)$, where $g(a,b)$ is the minimum of $\mathrm{Re}(1+i)(1-\overline{a}b)$ and $\mathrm{Im}\,(1+i)(1-\overline{a}b)$.

Corollary 2.3. *If* $|a-b| < \frac{1-|a|^2}{2\sqrt{2}}$ *then* $f(z) = \mathrm{Re}\, B_1(z) + i\,\mathrm{Im}\, B_2(z)$ *is quasiregular on* $\mathbb{D}$.

Proof. We need to prove that from $|a-b| < \frac{1-|a|^2}{2\sqrt{2}}$, it follows:

$$\mathrm{Re}(1+i)(1-\overline{a}b) > \sqrt{2}|a-b|, \quad \mathrm{Im}(1+i)(1-\overline{a}b) > \sqrt{2}|a-b|$$

It is sufficient to prove:

$$\mathrm{Re}\,\frac{(1+i)}{\sqrt{2}}(1-\overline{a}b) > \frac{1-|a|^2}{2\sqrt{2}} \quad \mathrm{Im}\,\frac{(1+i)}{\sqrt{2}}(1-\overline{a}b) > \frac{1-|a|^2}{2\sqrt{2}}.$$

Assume $b = a + (b-a) = a + \rho e^{i\theta}$, where $\rho < \frac{1-|a|^2}{2\sqrt{2}}$. Then the center

$$O = 1 - \overline{a}b = 1 - \overline{a}(a + \rho e^{i\theta})\overline{a}b = 1 - |a|^2 - \overline{a}\rho\, e^{i\theta}.$$

Now $\mathrm{Re}\,\frac{(1+i)}{\sqrt{2}}(1-\overline{a}b) = \frac{1-|a|^2}{\sqrt{2}} + \mathrm{Re}\,\frac{(1+i)}{\sqrt{2}}(-a\rho\ e^{i\theta})$. Let $e^{i\varphi} = \frac{(1+i)}{\sqrt{2}}e^{i\theta}$. Since $\rho < \frac{1-|a|^2}{2\sqrt{2}}, -\frac{1-|a|^2}{2\sqrt{2}} < \mathrm{Re}\,\rho e^{i\varphi} < \frac{1-|a|^2}{2\sqrt{2}}$, we get

$$\mathrm{Re}\,\frac{(1+i)}{\sqrt{2}}(1-\overline{a}b) > \frac{1-|a|^2}{\sqrt{2}} - \frac{1-|a|^2}{2\sqrt{2}} = (1-|a|^2)(\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}) = \frac{1-|a|^2}{2\sqrt{2}}. \qquad \square$$

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References

[1] D. Kalaj, M. Mateljević, *Quasiconformal harmonic mappings and generalizations*, J. Analysis, **18** (2010), 239–260.

[2] D. Kalaj, M. Mateljević, *(K, K')-quasiconformal harmonic mappings*, Potential Analysis, **36**(1), (2012), 117–135.

[3] V. Manojlović, *Harmonic quasiconformal mappings in domains in* $\mathbb{R}^n$, J. Analysis, **18** (2010), 297–316.

[4] O. Martio, *On harmonic quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I **425** (1968), 3–10.

[5] M. Mateljević, *Dirichlet's principle, uniqueness of harmonic maps and extremal qc mappings*, Zbornik radova 10(18), Two topics in mathematics, editor B. Stanković, 2004, 41–91; ISBN 86-80593-36-2.

[6] M. Mateljević, V. Božin, M. Knežević, *Quasiconformality of harmonic mappings between Jordan domains*, Filomat, **24**(3) (2010), 111–124.

[7] O. Lehto, K. Virtanen, *Quasiconformal Mappings in the Plane*, Translated from the German by K. W. Lucas, Second Edition, Springer-Verlag, 1973.

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