МATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 23.12.2016

research paper
оригинални научни рад

STRONG MIXED AND GENERALIZED FRACTIONAL CALCULUS FOR BANACH SPACE VALUED FUNCTIONS

George A. Anastassiou

Abstract. We present here a strong mixed fractional calculus theory for Banach space valued functions of generalized Canavati type. Then we establish several mixed fractional Bochner integral inequalities of various types.

1. Introduction

Here we use the Bochner integral for Banach space valued functions, which is a direct generalization of Lebesgue integral to this case. The reader may read about Bochner integral and its properties from [2, 6, 7, 9–12].

Using Bochner integral properties and the great article [12], we develop a right and left generalized Canavati type, [8], strong fractional theory for the first time in the literature, which is the direct analog of the real one, but now dealing with Banach space valued functions.

In the literature there are very few articles only about the left weak fractional theory of Banach space valued functions with one of the best [1].

However we found the left weak theory, using Pettis integral and functionals, complicated, less clear, dificult and unnecessary.

With this article and [4, 5] earlier, we try to simplify matters and put the related theory on its natural grounds and resemble the theory on real numbers.

2. Main results

Here $C([a,b],X)$ stands for the space of continuous functions from $[a,b]$ into X, where X is a Banach space.

2010 Mathematics Subject Classification: 26A33, 26D10, 26D15, 46B25
Keywords and phrases: Right and left fractional derivative; right and left fractional Taylor's formula; Banach space valued functions; integral inequalities; Bochner integral.

All integrals here are of Bochner type. By [4], we have that: if $f \in C([a,b],X)$, then $f \in L_\infty([a,b],X)$ and $f \in L_1([a,b],X)$. Derivatives for vector valued functions are defined according to [11, p. 83], similar to numerical ones.

We need the following reverse Taylor's formula:

THEOREM 2.1. *Let $n \in \mathbb{N}$ and $f \in C^n([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space. Then*

$$f(a) = \sum_{i=0}^{n-1} \frac{(a-b)^i}{i!} f^{(i)}(b) + \frac{1}{(n-1)!} \int_b^a (a-t)^{n-1} f^{(n)}(t)\, dt. \tag{1}$$

Proof. We consider

$$F(x) := \sum_{i=0}^{n-1} \frac{(a-x)^i}{i!} f^{(i)}(x), \quad x \in [a,b].$$

Clearly $F \in C([a,b],X)$ and $F(a) = f(a)$, with $F(b) = \sum_{i=0}^{n-1} \frac{(a-b)^i}{i!} f^{(i)}(b)$. Furthermore it holds

$$F'(x) = \frac{(a-x)^{n-1}}{(n-1)!} f^{(n)}(x), \quad \forall\, x \in [a,b],$$

and $F' \in C([a,b],X)$.

By the Fundamental Theorem of Calculus for Banach space valued functions and Bochner integral, see [12], we get $F(b) - F(a) = \int_a^b F'(t)\, dt$. That is we have

$$\begin{aligned} \sum_{i=0}^{n-1} \frac{(a-b)^i}{i!} f^{(i)}(b) - f(a) &= \int_a^b \frac{(a-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt \\ &= -\int_b^a \frac{(a-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt, \end{aligned} \tag{2}$$

proving (1). ☐

THEOREM 2.2. *Let $f \in C([a,b],X)$. Then the function*

$$F(t) = \int_b^t f(\tau)\, d\tau = -\int_t^b f(\tau)\, d\tau, \quad \forall\, t \in [a,b]$$

is continuous and $F'(t) = f(t)$, $\forall\, t \in [a,b]$, i.e. $F \in C^1([a,b],X)$.

Proof. Let $a \le t \le t_0 \le b$. We have that

$$\begin{aligned} \frac{F(t) - F(t_0)}{t - t_0} &\overset{(2)}{=} \frac{1}{t_0 - t} \int_t^{t_0} f(\tau)\, d\tau \text{ (see [2, p. 426, Theorem 11.43])} \\ &= \frac{1}{t_0 - t} \int_t^{t_0} f(t_0)\, dt + \frac{1}{t_0 - t} \int_t^{t_0} [f(\tau) - f(t_0)]\, d\tau \\ &= f(t_0) + \frac{1}{t_0 - t} \int_t^{t_0} [f(\tau) - f(t_0)]\, d\tau, \end{aligned}$$

where

$$\left\| \frac{1}{t_0 - t} \int_t^{t_0} [f(\tau) - f(t_0)]\, d\tau \right\| \leq \frac{1}{t_0 - t} \int_t^{t_0} \|f(\tau) - f(t_0)\|\, d\tau$$
$$\leq \max_{t \leq \tau \leq t_0} \|f(\tau) - f(t_0)\| \to 0,$$

as $t \to t_0$, since $f(t)$ is continuous at t_0. Thus there exists $F'(t_0) = f(t_0)$, for any $t_0 \in [a, b]$. By [11, p. 83], now F is continuous. □

THEOREM 2.3. *Let $g \in C([a, b], X)$, where X is a Banach space, $n \in \mathbb{N}$. Then*

$$g(t) = \frac{1}{(n-1)!} \left(\int_b^t (t - z)^{n-1} g(z)\, dz \right)^{(n)}$$
$$= \frac{(-1)^n}{(n-1)!} \left(\int_t^b (z - t)^{n-1} g(z)\, dz \right)^{(n)}, \quad \forall\, t \in [a, b].$$

Proof. We apply Theorem 2.2 repeatedly. We consider the function

$$f(t) = \int_b^t \left(\int_b^{t_1} \left(\int_b^{t_2} \left(\dots \left(\int_b^{t_{n-1}} g(t_n)\, dt_n \right) dt_{n-1} \right) \dots \right) dt_2 \right) dt_1$$
$$= (-1)^n \left(\int_t^b \left(\int_{t_1}^b \left(\int_{t_2}^b \left(\dots \left(\int_{t_{n-1}}^b g(t_n)\, dt_n \right) dt_{n-1} \right) \dots \right) dt_2 \right) dt_1 \right),$$

$\forall\, t \in [a, b]$. Hence it holds

$$f'(t) = \int_b^t \left(\int_b^{t_1} \left(\dots \left(\int_b^{t_{n-2}} g(t_{n-1})\, dt_{n-1} \right) dt_{n-2} \right) \dots \right) dt_1$$
$$= (-1)^{n-1} \left(\int_t^b \left(\int_{t_1}^b \left(\dots \left(\int_{t_{n-2}}^b g(t_{n-1})\, dt_{n-1} \right) dt_{n-2} \right) \dots \right) dt_1 \right),$$

$\forall\, t \in [a, b]$, etc. Continuing, similarly, we get

$$f^{(n-2)}(t) = \int_b^t \left(\int_b^{t_1} g(t_2)\, dt_2 \right) dt_1 = (-1)^2 \left(\int_t^b \left(\int_{t_1}^b g(t_2)\, dt_2 \right) dt_1 \right),$$

$\forall\, t \in [a, b]$, and

$$f^{(n-1)}(t) = \int_b^t g(t_1)\, dt_1 = -\int_t^b g(t_1)\, dt_1, \quad \forall\, t \in [a, b].$$

Finally, we have that $f^{(n)}(t) = g(t), \ \forall\, t \in [a, b]$. Clearly $f \in C^n([a, b], X)$ with $f^{(i)}(b) = 0$, for $i = 0, 1, \dots, n - 1$.

By Theorem 2.1 now we obtain

$$f(t) = \frac{1}{(n-1)!} \int_b^t (t - z)^{n-1} f^{(n)}(z)\, dz,$$

and finally

$$f^{(n)}(t) = \frac{1}{(n-1)!} \left(\int_b^t (t - z)^{n-1} f^{(n)}(z)\, dz \right)^{(n)}, \quad \forall\, t \in [a, b],$$

proving the claim. □

Definition 2.4. Let $f \in C([a,b], X)$, where X is a Banach space. Let $\nu > 0$, we define the right Riemann-Liouville fractional Bochner integral operator

$$\left(J_{b-}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} f(z)\, dz, \quad \forall\, x \in [a,b],$$

where Γ is the gamma function.

In [5], we have proved that $\left(J_{b-}^{\nu} f\right) \in C([a,b], X)$. Furthermore in [5], we have proved that

$$J_{b-}^{\nu} J_{b-}^{\mu} f = J_{b-}^{\nu+\mu} f = J_{b-}^{\mu} J_{b-}^{\nu} f,$$

for any $\mu, \nu > 0$; any $f \in C([a,b], X)$.

Lemma 2.5. *Let $f \in C([a,b], X)$, $\nu \geq 1$, $n = [\nu]$ ($[\cdot]$ integral part), $\alpha = \nu - n$. Then*

$$\left(\left(J_{b-}^{\nu} f\right)(x)\right)^{(k)} = (-1)^k J_{b-}^{\nu-k} f(x),$$

$k = 0, 1, \ldots, n-1$. *Also*

$$\left(\left(J_{b-}^{\nu} f\right)(x)\right)^{(n)} = (-1)^n J_{b-}^{\alpha} f(x), \quad \text{if } \alpha > 0,$$

and

$$\left(J_{b-}^{\nu} f\right)^{(n)} = (-1)^n f, \quad \text{if } \alpha = 0. \tag{3}$$

Proof. We notice that

$$\left(\left(J_{b-}^{\nu} f\right)(x)\right)^{(k)} = \left(D^k J_{b-}^{\nu} f\right)(x) = \left(D^k J_{b-}^{k} J_{b-}^{\nu-k} f\right)(x)$$
$$= (-1)^k \left(I J_{b-}^{\nu-k} f\right)(x) = (-1)^k \left(J_{b-}^{\nu-k} f\right)(x),$$

$k = 0, 1, \ldots, n-1$, where I is the identity operator. If $\alpha > 0$, we get

$$\left(\left(J_{b-}^{\nu} f\right)(x)\right)^{(n)} = \left(D^n J_{b-}^{\nu} f\right)(x) = \left(D^n J_{b-}^{n+\alpha} f\right)(x)$$
$$= \left(D^n J_{b-}^{n} J_{b-}^{\alpha} f\right)(x) = (-1)^n \left(I J_{b-}^{\alpha} f\right)(x) = (-1)^n \left(J_{b-}^{\alpha} f\right)(x).$$

Equality (3) is obvious by Theorem 2.3. □

Theorem 2.6. $J_{b-}^{\nu} : C([a,b], X) \to C([a,b], X)$, $\nu > 0$, *is 1–1.*

Proof. Let $f \in C([a,b], X)$ such that $J_{b-}^{\nu} f = 0$. If $0 < \nu < 1$, then $J_{b-}^{1} f = J_{b-}^{1-\nu} J_{b-}^{\nu} f = 0$, hence $J_{b-}^{1} f = 0$. That is by Theorem 2.3, $(-1) f = \left(J_{b-}^{1} f\right)' = 0$, and $f = 0$.

If now $\nu \geq 1$, then $\nu = n + \alpha$, (where $n = [\nu]$, $\alpha := \nu - n$, $n \geq 1$, and $0 \leq \alpha < 1$). If $\alpha = 0$, then $J_{b-}^{n} f = 0$, hence by Theorem 2.3, $(-1)^n f = \left(J_{b-}^{n} f\right)^{(n)} = 0$, so that $f = 0$.

If $\alpha > 0$, then

$$J_{b-}^{\alpha}\left(J_{b-}^{n} f\right) = J_{b-}^{n+\alpha} f = J_{b-}^{\nu} f = 0.$$

Hence by the first case of this proof we get $J_{b-}^{n} f = 0$. And as in the second case of this proof we get $f = 0$. The proof now is complete. □

Definition 2.7. Let $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $\nu \notin \mathbb{N}$. Define the subspace of functions

$$C_{b-}^{\nu}([a,b],X) := \left\{ f \in C^n([a,b],X) : J_{b-}^{1-\alpha} f^{(n)} \in C^1([a,b],X) \right\}.$$

Define the Banach space valued right generalized ν-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\nu} f := (-1)^{n-1} \left(J_{b-}^{1-\alpha} f^{(n)} \right)'.$$

Notice that

$$J_{b-}^{1-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^b (z-x)^{-\alpha} f^{(n)}(z)\, dz$$

exists for $f \in C_{b-}^{\nu}([a,b],X)$, and

$$\left(D_{b-}^{\nu} f \right)(x) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (z-x)^{-\alpha} f^{(n)}(z)\, dz,$$

i.e.

$$\left(D_{b-}^{\nu} f \right)(x) = \frac{(-1)^{n-1}}{\Gamma(n-\nu+1)} \frac{d}{dx} \int_x^b (z-x)^{n-\nu} f^{(n)}(z)\, dz.$$

If $\nu \in \mathbb{N}$, then $\alpha = 0$, $n = \nu$, and

$$\left(D_{b-}^{\nu} f \right)(x) = \left(D_{b-}^{n} f \right)(x) = (-1)^n f^{(n)}(x).$$

Notice that $D_{b-}^{\nu} f \in C([a,b],X)$.

We give the following right fractional Taylor's formula.

Theorem 2.8. *Let $f \in C_{b-}^{\nu}([a,b],X)$, $\nu > 0$, $n := [\nu]$. Then*

1) If $\nu \geq 1$, we get

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \left(J_{b-}^{\nu} D_{b-}^{\nu} f \right)(x), \quad \forall\, x \in [a,b]. \tag{4}$$

2) If $0 < \nu < 1$, we get

$$f(x) = J_{b-}^{\nu} D_{b-}^{\nu} f(x), \quad \forall\, x \in [a,b]. \tag{5}$$

We have that

$$J_{b-}^{\nu} D_{b-}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} \left(D_{b-}^{\nu} f \right)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{6}$$

Proof. Let $f \in C_{b-}^{\nu}([a,b],X)$. We see that

$$J_{b-}^{1}\left(D_{b-}^{\nu} f \right)(x) = \int_x^b \left(D_{b-}^{\nu} f \right)(z)\, dz = (-1)^{n-1} \int_x^b \frac{d}{dz} \left(J_{b-}^{1-\alpha} f^{(n)} \right)(z)\, dz$$

$$= (-1)^{n-1} \left[\left(J_{b-}^{1-\alpha} f^{(n)} \right)(b) - \left(J_{b-}^{(1-\alpha)} f^{(n)} \right)(x) \right] = (-1)^n J_{b-}^{1-\alpha} f^{(n)}(x).$$

That is

$$J_{b-}^{1-\alpha} f^{(n)}(x) = (-1)^n J_{b-}^{1}\left(D_{b-}^{\nu} f \right)(x) = (-1)^n J_{b-}^{1-\alpha}\left(J_{b-}^{\alpha}\left(D_{b-}^{\nu} f \right) \right)(x).$$

Hence since $J_{b-}^{1-\alpha}$ is 1–1 we get $f^{(n)}(x) = (-1)^n J_{b-}^{\alpha}\left(D_{b-}^{\nu} f\right)(x)$. Consequently

$$\begin{aligned} J_{b-}^{n} f^{(n)}(x) &= (-1)^n J_{b-}^{n} J_{b-}^{\alpha}\left(D_{b-}^{\nu} f\right)(x) \\ &= (-1)^n J_{b-}^{n+\alpha}\left(D_{b-}^{\nu} f\right)(x) = (-1)^n J_{b-}^{\nu}\left(D_{b-}^{\nu} f\right)(x). \end{aligned}$$

That is

$$J_{b-}^{n} f^{(n)}(x) = (-1)^n J_{b-}^{\nu}\left(D_{b-}^{\nu} f\right)(x). \tag{7}$$

Let now $\nu \geq 1$, hence $n \geq 1$, $n \in \mathbb{N}$. By Theorem 2.1 we have that

$$\begin{aligned} f(x) - \sum_{i=0}^{n-1} \frac{(x-b)^i}{i!} f^{(i)}(b) &= \frac{1}{(n-1)!} \int_b^x (x-t)^{n-1} f^{(n)}(t)\, dt \\ &= \frac{(-1)^n}{\Gamma(n)} \int_x^b (t-x)^{n-1} f^{(n)}(t)\, dt = (-1)^n J_{b-}^{n} f^{(n)}(x) \\ &\overset{(7)}{=} (-1)^{2n} J_{b-}^{\nu}\left(D_{b-}^{\nu} f\right)(x) = J_{b-}^{\nu}\left(D_{b-}^{\nu} f\right)(x). \end{aligned}$$

That is

$$f(x) - \sum_{i=0}^{n-1} \frac{(x-b)^i}{i!} f^{(i)}(b) = J_{b-}^{\nu}\left(D_{b-}^{\nu} f\right)(x), \quad \forall\, x \in [a,b],$$

proving (4).

If $0 < \nu < 1$, then $n = 0$. Then by (7) we get

$$f(x) = J_{b-}^{\nu}\left(D_{b-}^{\nu} f\right)(x), \quad \forall\, x \in [a,b],$$

proving (5). The theorem is proved. □

COROLLARY 2.9. *Let* $f \in C_{b-}^{\nu}([a,b], X)$, $\nu \geq 1$, *and* $f^{(i)}(b) = 0$, $i = 0, 1, \ldots, n-1$. *Then*

$$f(x) = \left(J_{b-}^{\nu} D_{b-}^{\nu} f\right)(x), \quad \forall\, x \in [a,b].$$

We give the following Taylor's formula:

THEOREM 2.10. *Let* $n \in \mathbb{N}$ *and* $f \in C^n([a,b], X)$. *Then*

$$f(b) = \sum_{i=0}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) + \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x)\, dx.$$

Proof. We consider

$$F(x) = \sum_{i=0}^{n-1} \frac{(b-x)^i}{i!} f^{(i)}(x), \quad x \in [a,b].$$

Clearly $F \in C([a,b], X)$ and $F(b) = f(b)$, and $F(a) = \sum_{i=0}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a)$. Furthermore it holds

$$F'(x) = \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x), \quad \forall\, x \in [a,b],$$

and $F' \in C([a,b], X)$.

By the Fundamental Theorem of Calculus for Banach space valued functions and Bochner integral, see [12], we get

$$F(b) - F(a) = \int_a^b F'(t)\, dt.$$

That is we have

$$f(b) - \left(\sum_{i=0}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) \right) = \int_a^b \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x)\, dx,$$

proving the claim. □

THEOREM 2.11. *Let $f \in C([a,b], X)$. Then the function*

$$F(t) = \int_a^t f(\tau)\, d\tau, \quad \forall\, t \in [a,b]$$

is continuous and $F'(t) = f(t)$, $\forall\, t \in [a,b]$, i.e. $F \in C^1([a,b], X)$.

Proof. Let $a \leq t_0 \leq t \leq b$. We have that

$$\begin{aligned} \frac{F(t) - F(t_0)}{t - t_0} &\overset{(2)}{=} \frac{1}{t-t_0} \int_{t_0}^t f(\tau)\, d\tau (\text{see [2, p 426, Theorem 11.43]}) \\ &= \frac{1}{t-t_0} \int_{t_0}^t f(t_0)\, d\tau + \frac{1}{t-t_0} \int_{t_0}^t [f(\tau) - f(t_0)]\, d\tau \\ &= f(t_0) + \frac{1}{t-t_0} \int_{t_0}^t [f(\tau) - f(t_0)]\, d\tau, \end{aligned}$$

where

$$\begin{aligned} \left\| \frac{1}{t-t_0} \int_{t_0}^t [f(\tau) - f(t_0)]\, d\tau \right\| &\leq \frac{1}{t-t_0} \int_{t_0}^t \| f(\tau) - f(t_0) \|\, d\tau \\ &\leq \max_{t_0 \leq \tau \leq t} \| f(\tau) - f(t_0) \| \to 0, \end{aligned}$$

as $t \to t_0$, since $f(t)$ is continuous at t_0.

Thus, there exists $F'(t_0) = f(t_0)$, for any $t_0 \in [a,b]$. By [11, p. 83], now F is continuous. □

THEOREM 2.12. *Let $g \in C([a,b], X)$, $n \in \mathbb{N}$. Then*

$$g(t) = \frac{1}{(n-1)!} \left(\int_a^t (t-z)^{n-1} g(z)\, dz \right)^{(n)}, \quad \forall\, t \in [a,b].$$

Proof. We apply Theorem 2.11 repeatedly. We consider the function

$$f(t) = \int_a^t \left(\int_a^{t_1} \left(\int_a^{t_2} \left(\dots \left(\int_a^{t_{n-1}} g(t_n)\, dt_n \right) dt_{n-1} \right) \dots \right) dt_2 \right) dt_1, \ \forall t \in [a,b].$$

Hence it holds

$$f'(t) = \int_a^t \left(\int_a^{t_1} \left(\dots \left(\int_a^{t_{n-2}} g(t_{n-1})\, dt_{n-1} \right) dt_{n-2} \right) \dots \right) dt_1, \ \forall\, t \in [a,b],$$

etc. Continuing, similarly, we get

$$f^{(n-2)}(t) = \int_a^t \left(\int_a^{t_1} g(t_2)\, dt_2 \right) dt_1, \quad \forall t \in [a,b],$$

and

$$f^{(n-1)}(t) = \int_a^t g(t_1)\, dt_1, \quad \forall\, t \in [a,b].$$

Finally, we have that $f^{(n)}(t) = g(t), \ \forall\, t \in [a,b]$ Clearly $f \in C^n([a,b], X)$ with $f^{(i)}(a) = 0$, for $i = 0, 1, \ldots, n-1$.

By Theorem 2.10 now we obtain

$$f(t) = \frac{1}{(n-1)!} \int_a^t (t-x)^{n-1} f^{(n)}(x)\, dx,$$

and finally

$$f^{(n)}(t) = \frac{1}{(n-1)!} \left(\int_a^t (t-x)^{n-1} f^{(n)}(x)\, dx \right)^{(n)}, \quad \forall\, t \in [a,b],$$

proving the claim. □

DEFINITION 2.13. Let $f \in C([a,b], X)$. Let $\nu > 0$, we define the left Riemann-Liouville fractional Bochner integral operator

$$(J_a^\nu f)(x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} f(z)\, dz, \quad \forall\, x \in [a,b].$$

In [4], we have proved that $(J_a^\nu f) \in C([a,b], X)$. Furthermore in [4], we have proved that

$$J_a^\nu J_a^\mu f = J_a^{\nu+\mu} f = J_a^\mu J_a^\nu f,$$

$\forall\, \mu, \nu > 0, \forall\, f \in C([a,b], X)$.

LEMMA 2.14. *Let $f \in C([a,b], X)$, $\nu \geq 1$, $n = [\nu]$, $\alpha = \nu - n$. Then*

$$((J_a^\nu f)(x))^{(k)} = J_a^{\nu-k} f(x),$$

$k = 0, 1, \ldots, n-1$. *Also*

$$((J_a^\nu f)(x))^{(n)} = J_a^\alpha f(x), \quad \text{if } \alpha > 0,$$

and

$$(J_a^\nu f)^{(n)} = f, \quad \text{if } \alpha = 0. \tag{8}$$

Proof. We notice that

$$\begin{aligned} ((J_a^\nu f)(x))^{(k)} &= \left(D^k J_a^\nu f\right)(x) = \left(D^k J_a^k J_a^{\nu-k} f\right)(x) \\ &= \left(I J_a^{\nu-k} f\right)(x) = \left(J_a^{\nu-k} f\right)(x), \end{aligned}$$

$k = 0, 1, \ldots, n-1$. If $\alpha > 0$, we get

$$\begin{aligned} ((J_a^\nu f)(x))^{(n)} &= (D^n J_a^\nu f)(x) = \left(D^n J_a^{n+\alpha} f\right)(x) \\ &= (D^n J_a^n J_a^\alpha f)(x) = (I J_a^\alpha f)(x) = (J_a^\alpha f)(x). \end{aligned}$$

Equality (8) is obvious by Theorem 2.12. □

THEOREM 2.15. $J_a^\nu : C([a,b],X) \to C([a,b],X)$, $\nu > 0$, *is 1–1.*

Proof. Let $f \in C([a,b],X)$ such that $J_a^\nu f = 0$. If $0 < \nu < 1$, then $J_a^1 f = J_a^{1-\nu} J_a^\nu f = 0$, hence $J_a^1 f = 0$. That is by Theorem 12, $f = (J_a^1 f)' = 0$, and $f = 0$.

If now $\nu \geq 1$, then $\nu = n + \alpha$, $0 \leq \alpha < 1$. If $\alpha = 0$, then $J_a^n f = 0$, hence by Theorem 12, $f = (J_a^n f)^{(n)} = 0$, so that $f = 0$.

If $\alpha > 0$, then

$$J_a^\alpha (J_a^n f) = J_a^{n+\alpha} f = J_a^\nu f = 0.$$

Hence by the first case of this proof we get $J_a^n f = 0$. And as in the second case of this proof we get $f = 0$. The theorem is proved. □

DEFINITION 2.16. Let $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $\nu \notin \mathbb{N}$. Define the subspace of functions

$$C_a^\nu([a,b],X) := \left\{ f \in C^n([a,b],X) : J_a^{1-\alpha} f^{(n)} \in C^1([a,b],X) \right\}.$$

Define the Banach space valued left generalized ν-fractional derivative of f over $[a,b]$ as

$$(D_a^\nu f) := \left(J_a^{1-\alpha} f^{(n)} \right)'.$$

Notice that

$$J_a^{1-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-z)^{-\alpha} f^{(n)}(z)\, dz$$

exists for $f \in C_a^\nu([a,b],X)$, and

$$(D_a^\nu f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-z)^{-\alpha} f^{(n)}(z)\, dz,$$

i.e.

$$(D_a^\nu f)(x) = \frac{1}{\Gamma(n-\nu+1)} \frac{d}{dx} \int_a^x (x-z)^{n-\nu} f^{(n)}(z)\, dz.$$

If $\nu \in \mathbb{N}$, then $\alpha = 0$, $n = \nu$, and

$$(D_a^\nu f)(x) = (D_a^n f)(x) = f^{(n)}(x).$$

Notice that $D_a^\nu f \in C([a,b],X)$.

We give the following left fractional Taylor's formula.

THEOREM 2.17. *Let $f \in C_a^\nu([a,b],X)$, $\nu > 0$, $n := [\nu]$. Then*

1) If $\nu \geq 1$, we get

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + (J_a^\nu D_a^\nu f)(x), \quad \forall\, x \in [a,b]. \tag{9}$$

2) If $0 < \nu < 1$, we get

$$f(x) = J_a^\nu D_a^\nu f(x), \quad \forall\, x \in [a,b]. \tag{10}$$

We have that

$$J_a^\nu D_a^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} (D_a^\nu f)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{11}$$

Proof. Let $f \in C_a^\nu([a,b],X)$. We see that

$$J_a^1(D_a^\nu f)(x) = \int_a^x (D_a^\nu f)(z)\, dz = \int_a^x \frac{d}{dz}\left(J_a^{1-\alpha} f^{(n)}\right)(z)\, dz$$
$$= \left[\left(J_a^{1-\alpha} f^{(n)}\right)(x) - \left(J_a^{1-\alpha} f^{(n)}\right)(a)\right] = \left(J_a^{1-\alpha} f^{(n)}\right)(x).$$

That is

$$\left(J_a^{1-\alpha} f^{(n)}\right)(x) = J_a^1(D_a^\nu f)(x) = J_a^{1-\alpha}(J_a^\alpha(D_a^\nu f))(x).$$

Since $J_a^{1-\alpha}$ is 1–1 we get

$$f^{(n)}(x) = (J_a^\alpha(D_a^\nu f))(x).$$

Consequently

$$J_a^n f^{(n)}(x) = (J_a^n J_a^\alpha(D_a^\nu f))(x) = \left(J_a^{n+\alpha}(D_a^\nu f)\right)(x) = (J_a^\nu(D_a^\nu f))(x).$$

That is

$$J_a^n f^{(n)}(x) = (J_a^\nu D_a^\nu f)(x), \quad \forall\, x \in [a,b]. \tag{12}$$

Let now $\nu \geq 1$, hence $n \geq 1$, $n \in \mathbb{N}$. By Theorem 2.10 we have that

$$f(x) - \sum_{i=0}^{n-1} \frac{(x-a)^i}{i!} f^{(i)}(a) = \frac{1}{(n-1)!}\int_a^x (x-z)^{n-1} f^{(n)}(z)\, dz$$
$$= \left(J_a^n f^{(n)}\right)(x) \overset{(12)}{=} (J_a^\nu D_a^\nu f)(x), \quad \forall\, x \in [a,b],$$

proving (9).

If $0 < \nu < 1$, then $n = 0$. Then by (12) we get

$$f(x) = (J_a^\nu D_a^\nu f)(x),$$

proving (10). □

Corollary 2.18. *Let $f \in C_a^\nu([a,b],X)$, $\nu > 0$, $n = [\nu]$, and $f^{(i)}(a) = 0$, $i = 0,1,\dots,n-1$. Then*

$$f(x) = (J_a^\nu D_a^\nu f)(x), \quad \forall\, x \in [a,b].$$

We give the following fractional Polya type integral inequality without any boundary conditions, see also [3, p. 4].

Theorem 2.19. *Let $0 < \nu < 1$, $f \in C([a,b],X)$. Assume that $f \in C_a^\nu\left(\left[a,\frac{a+b}{2}\right],X\right)$ and $f \in C_{b-}^\nu\left(\left[\frac{a+b}{2},b\right],X\right)$. Set*

$$M(f) = \max\left\{\left\|\|D_a^\nu f\|\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\|\|D_{b-}^\nu f\|\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right\}.$$

Then

$$\left\|\int_a^b f(x)\, dx\right\| \leq \int_a^b \|f(x)\|\, dx \leq M(f)\frac{(b-a)^{\nu+1}}{\Gamma(\nu+2)\, 2^\nu}. \tag{13}$$

Inequality (13) is sharp, namely it is attained by

$$f_*(x) = \begin{Bmatrix} (x-a)^\nu \overrightarrow{i}, & x \in \left[a,\frac{a+b}{2}\right], \\ (b-x)^\nu \overrightarrow{i}, & x \in \left[\frac{a+b}{2},b\right] \end{Bmatrix}, \quad 0 < \nu < 1,$$

$\overrightarrow{i} \in X : \|\overrightarrow{i}\| = 1$. *Clearly here non zero constant vector function f are excluded.*

Proof. By (5) and (6) we get that

$$\|f(x)\| = \frac{1}{\Gamma(\nu)} \left\| \int_x^b (z-x)^{\nu-1} \left(D_{b-}^{\nu} f\right)(z)\, dz \right\|$$
$$\leq \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} \left\| \left(D_{b-}^{\nu} f\right)(z) \right\| dz \leq \left\| \left\| D_{b-}^{\nu} f \right\| \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \frac{(b-x)^{\nu}}{\Gamma(\nu+1)}.$$

That is

$$\|f(x)\| \leq \left\| \left\| D_{b-}^{\nu} f \right\| \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \frac{(b-x)^{\nu}}{\Gamma(\nu+1)}, \ \ \forall\, x \in \left[\frac{a+b}{2}, b\right]. \tag{14}$$

Similarly by (10) and (11) we get:

$$\|f(x)\| \leq \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \left\| \left(D_a^{\nu} f\right)(z) \right\| dz \leq \frac{\| \| D_a^{\nu} f \| \|_{\infty, \left[a, \frac{a+b}{2}\right]}}{\Gamma(\nu+1)} (x-a)^{\nu}, \tag{15}$$

$\forall\, x \in \left[a, \frac{a+b}{2}\right]$. Hence we find

$$\int_a^b \|f(x)\|\, dx = \int_a^{\frac{a+b}{2}} \|f(x)\|\, dx + \int_{\frac{a+b}{2}}^b \|f(x)\|\, dx \text{ (by (14) and (15))}$$
$$\leq \frac{1}{\Gamma(\nu+1)} \left\{ \left(\int_a^{\frac{a+b}{2}} (x-a)^{\nu}\, dx \right) \| \| D_a^{\nu} f \| \|_{\infty, \left[a, \frac{a+b}{2}\right]} \right.$$
$$\left. + \left(\int_{\frac{a+b}{2}}^b (b-x)^{\nu}\, dx \right) \left\| \left\| D_{b-}^{\nu} f \right\| \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}$$
$$= \frac{1}{(\Gamma(\nu+1))(\nu+1)} \left\{ \left(\frac{b-a}{2} \right)^{\nu+1} \| \| D_a^{\nu} f \| \|_{\infty, \left[a, \frac{a+b}{2}\right]} \right.$$
$$\left. + \left(\frac{b-a}{2} \right)^{\nu+1} \left\| \left\| D_{b-}^{\nu} f \right\| \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}$$
$$= \frac{1}{\Gamma(\nu+2)} \left(\frac{b-a}{2} \right)^{\nu+1} \left\{ \| \| D_a^{\nu} f \| \|_{\infty, \left[a, \frac{a+b}{2}\right]} + \left\| \left\| D_{b-}^{\nu} f \right\| \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}.$$

So we have proved that

$$\int_a^b \|f(x)\|\, dx \leq \max \left\{ \| \| D_a^{\nu} f \| \|_{\infty, \left[a, \frac{a+b}{2}\right]}, \left\| \left\| D_{b-}^{\nu} f \right\| \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\} \frac{(b-a)^{\nu+1}}{\Gamma(\nu+2)\, 2^{\nu}},$$

proving (13).

Notice that

$$f_* \left(\left(\frac{a+b}{2} \right)_- \right) = f_* \left(\left(\frac{a+b}{2} \right)_+ \right) = \left(\frac{b-a}{2} \right)^{\nu} \overrightarrow{i},$$

so that $f_* \in C([a,b], X)$.

Here, very similarly, as in [3, p. 5–6], we get that

$$D_a^{\nu} \left((x-a)^{\nu} \overrightarrow{i} \right) = \Gamma(\nu+1) \overrightarrow{i}, \text{ for all } x \in \left[a, \frac{a+b}{2}\right].$$

Therefore it holds

$$\left\| \left\| D_a^{\nu} \left((\cdot - a)^{\nu} \overrightarrow{i} \right) \right\| \right\|_{\infty, \left[a, \frac{a+b}{2}\right]} = \Gamma(\nu + 1).$$

Similarly, it holds

$$\left\| \left\| D_{b-}^{\nu} \left((b - \cdot)^{\nu} \overrightarrow{i} \right) \right\| \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} = \Gamma(\nu + 1).$$

Consequently we find that $M(f_*) = \Gamma(\nu + 1)$. Applying f_* into (13) we obtain:

$$R.H.S.\ (13) = \frac{(b-a)^{\nu+1}}{(\nu+1)\, 2^{\nu}},$$

and

$$\begin{aligned} L.H.S.\ (13) &= \left\| \int_a^b f_*(x)\, dx \right\| = \int_a^b \|f_*(x)\|\, dx \\ &= \int_a^{\frac{a+b}{2}} (x-a)^{\nu}\, dx + \int_{\frac{a+b}{2}}^b (b-x)^{\nu}\, dx = \frac{(b-a)^{\nu+1}}{(\nu+1)\, 2^{\nu}}, \end{aligned}$$

proving optimality of (13). □

We present the following fractional Ostrowski type inequality, see also [3, p. 379–381].

THEOREM 2.20. *Let $\nu \geq 1$, $n = [\nu]$, $f \in C([a,b], X)$, $x_0 \in [a,b]$. Assume that $f|_{[a,x_0]} \in C_{x_0-}^{\nu}([a,x_0], X)$, $f|_{[x_0,b]} \in C_{x_0}^{\nu}([x_0,b], X)$, and $f^{(i)}(x_0) = 0$, for $i = 1, \ldots, n-1$, which is void when $1 \leq \nu < 2$. Then*

$$\begin{aligned} \left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(x_0) \right\| &\leq \frac{1}{(b-a)\,\Gamma(\nu+2)} \\ &\times \left\{ \left\| \left\| D_{x_0-}^{\nu} f \right\| \right\|_{\infty,[a,x_0]} (x_0 - a)^{\nu+1} + \left\| \left\| D_{x_0}^{\nu} f \right\| \right\|_{\infty,[x_0,b]} (b - x_0)^{\nu+1} \right\} \\ &\leq \frac{1}{(b-a)\,\Gamma(\nu+2)} \max\left(\left\| \left\| D_{x_0-}^{\nu} f \right\| \right\|_{\infty,[a,x_0]}, \left\| \left\| D_{x_0}^{\nu} f \right\| \right\|_{\infty,[x_0,b]} \right) \\ &\times \left[(b-x_0)^{\nu+1} + (x_0-a)^{\nu+1} \right] \\ &\leq \max\left(\left\| \left\| D_{x_0-}^{\nu} f \right\| \right\|_{\infty,[a,x_0]}, \left\| \left\| D_{x_0}^{\nu} f \right\| \right\|_{\infty,[x_0,b]} \right) \frac{(b-a)^{\nu}}{\Gamma(\nu+2)}. \end{aligned} \tag{16}$$

Proof. By 4 we get that

$$f(x) - f(x_0) = \frac{1}{\Gamma(\nu)} \int_x^{x_0} (z-x)^{\nu-1} \left(D_{x_0-}^{\nu} f \right)(z)\, dz, \quad \forall\, x \in [a, x_0]. \tag{17}$$

And from 9 we get that

$$f(x) - f(x_0) = \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-z)^{\nu-1} \left(D_{x_0}^{\nu} f \right)(z)\, dz, \quad \forall\, x \in [x_0, b]. \tag{18}$$

Hence (by 17)

$$\|f(x)-f(x_0)\| \le \frac{1}{\Gamma(\nu)}\int_x^{x_0}(z-x)^{\nu-1}\left\|\left(D_{x_0-}^{\nu}f\right)(z)\right\|\,dz$$
$$\le \left\|\left\|D_{x_0-}^{\nu}f\right\|\right\|_{\infty,[a,x_0]}\frac{(x_0-x)^{\nu}}{\Gamma(\nu+1)},\quad \forall\, x\in[a,x_0].$$

Furthermore it holds

$$\|f(x)-f(x_0)\| \overset{(18)}{\le} \frac{1}{\Gamma(\nu)}\int_{x_0}^{x}(x-z)^{\nu-1}\left\|\left(D_{x_0}^{\nu}f\right)(z)\right\|\,dz$$
$$\le \left\|\left\|D_{x_0}^{\nu}f\right\|\right\|_{\infty,[x_0,b]}\frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)},\quad \forall\, x\in[x_0,b].$$

Next we observe that

$$\left\|\frac{1}{b-a}\int_a^b f(x)\,dx-f(x_0)\right\| = \frac{1}{b-a}\left\|\int_a^b (f(x)-f(x_0))\,dx\right\|$$
$$\le \frac{1}{b-a}\int_a^b \|f(x)-f(x_0)\|\,dx$$
$$= \frac{1}{b-a}\left\{\int_a^{x_0}\|f(x)-f(x_0)\|\,dx+\int_{x_0}^b\|f(x)-f(x_0)\|\,dx\right\}$$
$$\le \frac{1}{(b-a)\Gamma(\nu+1)}\left\{\left\|\left\|D_{x_0-}^{\nu}f\right\|\right\|_{\infty,[a,x_0]}\int_a^{x_0}(x_0-x)^{\nu}\,dx\right.$$
$$\left.+\left\|\left\|D_{x_0}^{\nu}f\right\|\right\|_{\infty,[x_0,b]}\int_{x_0}^{b}(x-x_0)^{\nu}\,dx\right\}$$
$$= \frac{1}{(b-a)\Gamma(\nu+2)}\left\{\left\|\left\|D_{x_0-}^{\nu}f\right\|\right\|_{\infty,[a,x_0]}(x_0-a)^{\nu+1}\right.$$
$$\left.+\left\|\left\|D_{x_0}^{\nu}f\right\|\right\|_{\infty,[x_0,b]}(b-x_0)^{\nu+1}\right\}$$
$$\le \frac{1}{(b-a)\Gamma(\nu+2)}\max\left(\left\|\left\|D_{x_0-}^{\nu}f\right\|\right\|_{\infty,[a,x_0]},\left\|\left\|D_{x_0}^{\nu}f\right\|\right\|_{\infty,[x_0,b]}\right)$$
$$\times\left[(x_0-a)^{\nu+1}+(b-x_0)^{\nu+1}\right]$$
$$\le \frac{1}{\Gamma(\nu+2)}\max\left(\left\|\left\|D_{x_0-}^{\nu}f\right\|\right\|_{\infty,[a,x_0]},\left\|\left\|D_{x_0}^{\nu}f\right\|\right\|_{\infty,[x_0,b]}\right)(b-a)^{\nu}.$$

Notice here that $\left\|\left\|D_{x_0-}^{\nu}f\right\|\right\|_{\infty,[a,x_0]},\left\|\left\|D_{x_0}^{\nu}f\right\|\right\|_{\infty,[x_0,b]}<\infty$. The theorem is proved. □

Inequalities (16) are optimal.

Theorem 2.21. *All as in Theorem 2.20. Inequalities* (16) *are sharp, namely are* *attained by*

$$\overline{f}(z)=\left\{\begin{array}{c}(x_0-z)^{\nu}\overrightarrow{i},\quad z\in[a,x_0],\\(z-x_0)^{\nu}\overrightarrow{i},\ z\in[x_0,b]\end{array}\right\},$$

where $\overrightarrow{i}\in X:\left\|\overrightarrow{i}\right\|=1$, $\nu\geq 1$, $x_0\in[a,b]$ *is fixed.*

Proof. See that

$$\overline{f}^{(k)}(x_{0-})=\overline{f}^{(k)}(x_{0+})=0,\ \ k=0,1,\ldots,n-1.$$

We have that

$$\left\|\left\|D_{x_0-}^{\nu}\overline{f}\right\|\right\|_{\infty,[a,x_0]}=\left\|\left\|D_{x_0}^{\nu}\overline{f}\right\|\right\|_{\infty,[x_0,b]}=\Gamma(\nu+1).$$

The

$$\begin{aligned}\text{R.H.S. of (16)}&=\frac{1}{(b-a)\Gamma(\nu+2)}\max\left(\left\|\left\|D_{x_0-}^{\nu}\overline{f}\right\|\right\|_{\infty,[a,x_0]},\left\|\left\|D_{x_0}^{\nu}\overline{f}\right\|\right\|_{\infty,[x_0,b]}\right)\\&\quad\times\left[(b-x_0)^{\nu+1}+(x_0-a)^{\nu+1}\right]\\&=\frac{1}{(b-a)\Gamma(\nu+2)}\Gamma(\nu+1)\left[(b-x_0)^{\nu+1}+(x_0-a)^{\nu+1}\right]\\&=\frac{\left[(b-x_0)^{\nu+1}+(x_0-a)^{\nu+1}\right]}{(b-a)(\nu+1)}.\end{aligned}\tag{19}$$

The

$$\begin{aligned}\text{L.H.S. of (16)}&=\left\|\frac{1}{b-a}\int_a^b\overline{f}(x)\,dx-\overline{f}(x_0)\right\|=\left\|\frac{1}{b-a}\int_a^b\overline{f}(x)\,dx\right\|\\&=\frac{1}{b-a}\left[\int_a^{x_0}(x_0-z)^{\nu}\,dz+\int_{x_0}^b(z-x_0)^{\nu}\,dz\right]\\&=\frac{\left((x_0-a)^{\nu+1}+(b-x_0)^{\nu+1}\right)}{(b-a)(\nu+1)}.\end{aligned}\tag{20}$$

By (19) and (20) we get optimality of (16). □

We continue with a right fractional Poincaré type inequality.

THEOREM 2.22. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$, $m=[\alpha]$. *Let* $f\in C_{b-}^{\alpha}([a,b],X)$. *Assume that* $f^{(k)}(b)=0$, $k=0,1,\ldots,m-1$, *when* $\alpha\geq 1$. *Then*

$$\|f\|_{L_q([a,b],X)}\leq\frac{(b-a)^{\alpha}\left\|D_{b-}^{\alpha}f\right\|_{L_q([a,b],X)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}(q\alpha)^{\frac{1}{q}}}.$$

Proof. We have that (by (4)–(6))

$$f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(z-x)^{\alpha-1}\left(D_{b-}^{\alpha}f\right)(z)\,dz,\quad\forall\,x\in[a,b].$$

Hence

$$\|f(x)\| = \frac{1}{\Gamma(\alpha)} \left\| \int_x^b (z-x)^{\alpha-1} \left(D_{b-}^{\alpha} f\right)(z)\, dz \right\|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} \left\| D_{b-}^{\alpha} f(z) \right\| dz$$

$$\leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (z-x)^{p(\alpha-1)}\, dz \right)^{\frac{1}{p}} \left(\int_x^b \left\| \left(D_{b-}^{\alpha} f\right)(z) \right\|^q dz \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b],X)}.$$

We have proved that

$$\|f(x)\| \leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b],X)}, \quad \forall\, x \in [a,b].$$

Then

$$\|f(x)\|^q \leq \frac{(b-x)^{(p(\alpha-1)+1)\frac{q}{p}}}{(\Gamma(\alpha))^q (p(\alpha-1)+1)^{\frac{q}{p}}} \left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b],X)}^q, \quad \forall\, x \in [a,b].$$

Hence it holds

$$\int_a^b \|f(x)\|^q\, dx \leq \frac{(b-a)^{q\alpha}}{(\Gamma(\alpha))^q (p(\alpha-1)+1)^{\frac{q}{p}} q\alpha} \left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b],X)}^q.$$

The last inequality implies

$$\left(\int_a^b \|f(x)\|^q\, dx \right)^{\frac{1}{q}} \leq \frac{(b-a)^{\alpha} \left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b],X)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} (q\alpha)^{\frac{1}{q}}},$$

proving the claim. □

We finish with a Poincaré like left fractional inequality:

THEOREM 2.23. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $\nu > \frac{1}{q}$, $n = [\nu]$. *Let* $f \in C_a^{\nu}([a,b], X)$. *Assume that* $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$, *if* $\nu \geq 1$. *Then*

$$\|f\|_{L_q([a,b],X)} \leq \frac{(b-a)^{\nu}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} (q\nu)^{\frac{1}{q}}} \|D_a^{\nu} f\|_{L_q([a,b],X)}.$$

Proof. We have that (by (9)–(11))

$$f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} (D_a^{\nu} f)(z)\, dz, \quad \forall\, x \in [a,b].$$

Thus

$$\begin{aligned}\|f(x)\| &= \frac{1}{\Gamma(\nu)}\left\|\int_a^x (x-z)^{\nu-1}\left(D_a^{\nu}f\right)(z)\,dz\right\| \\ &\le \frac{1}{\Gamma(\nu)}\int_a^x (x-z)^{\nu-1}\left\|D_a^{\nu}f(z)\right\|\,dz \\ &\le \left(\frac{1}{\Gamma(\nu)}\int_a^x (x-z)^{p(\nu-1)}\,dz\right)^{\frac{1}{p}}\left(\int_a^x \left\|D_a^{\nu}f(z)\right\|^q\,dz\right)^{\frac{1}{q}} \\ &\le \frac{1}{\Gamma(\nu)}\frac{(x-a)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}\left\|D_a^{\nu}f\right\|_{L_q([a,b],X)}.\end{aligned}$$

We have proved that

$$\|f(x)\| \le \frac{(x-a)^{\nu-\frac{1}{q}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}\left\|D_a^{\nu}f\right\|_{L_q([a,b],X)}, \quad \forall\, x\in[a,b].$$

Then

$$\|f(x)\|^q \le \frac{(x-a)^{q\nu-1}}{(\Gamma(\nu))^q(p(\nu-1)+1)^{\frac{q}{p}}}\left\|D_a^{\nu}f\right\|^q_{L_q([a,b],X)},$$

and

$$\int_a^b \|f(x)\|^q\,dx \le \frac{(b-a)^{q\nu}\left\|D_a^{\nu}f\right\|^q_{L_q([a,b],X)}}{(\Gamma(\nu))^q(p(\nu-1)+1)^{\frac{q}{p}}\,q\nu}.$$

This last results into

$$\left(\int_a^b \|f(x)\|^q\,dx\right)^{\frac{1}{q}} \le \frac{(b-a)^{\nu}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}(q\nu)^{\frac{1}{q}}}\left\|D_a^{\nu}f\right\|_{L_q([a,b],X)}.$$

proving the claim. □

One can prove, with the above built fractional machinery, all kinds of inequalities but we choose to stop here.

References

[1] R. P. Agarwal, V. Lupulescu, D. O'Regan, G. Rahman, *Multi-term fractional differential equations in a nonreflexive Banach space*, Adv. Difference Equ. **2013**: 302 (2013).

[2] C. D. Aliprantis, K. C. Border, *Infinite Dimensional Analysis*, Springer, New York, 2006.

[3] G. A. Anastassiou, *Intelligent Comparisons: Analytic Inequalities*, Springer, New York, Heidelberg, 2016.

[4] G. A. Anastassiou, *A strong Fractional Calculus Theory for Banach space valued functions*, submitted, 2016.

[5] G. A. Anastassiou, *Strong Right Fractional Calculus for Banach space valued functions*, submitted, 2016.

[6] Appendix F, *The Bochner integral and vector-valued L_p-spaces*, `https://isem.math.kit.edu/images/f/f7/AppendixF.pdf`.

[7] Bochner integral. *Encyclopedia of Mathematics.* URL: `http://www.encyclopediaofmath.org/index.php?title=Bochner_integral& oldid=38659`.

[8] J. A. Canavati, *The Riemann-Liouville integral*, Nieuw Archief Voor Wiskunde, **5** (1) (1997), 53–75.

[9] M. Kreuter, *Sobolev Spaces of Vector-valued functions*, Ulm Univ., Master Thesis in Math., Ulm, Germany, 2015.

[10] J. Mikusinski, *The Bochner Integral*, Academic Press, New York, 1978.

[11] G. E. Shilov, *Elementary Functional Analysis*, Dover Publications, Inc., New York, 1996.

[12] C. Volintiru, *A proof of the fundamental theorem of Calculus using Hausdorff measures*, Real Analysis Exchange, **26** (1), (2000/2001), 381–390.

(received 27.10.2016; available online 23.12.2016)

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A.
E-mail: ganastss@memphis.edu