MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 13.06.2017

research paper
оригинални научни рад

ARITHMETIC PROPERTIES OF 3-REGULAR BI-PARTITIONS WITH DESIGNATED SUMMANDS

M. S. Mahadeva Naika and S. Shivaprasada Nayaka

Abstract. Recently Andrews, Lewis and Lovejoy introduced the partition functions $PD(n)$ defined by the number of partitions of n with designated summands and they found several modulo 3 and 4. In this paper, we find several congruences modulo 3 and 4 for $PBD_3(n)$, which represent the number of 3-regular bi-partitions of n with designated summands. For example, for each $n \geq 1$ and $\alpha \geq 0$ $PBD_3(4 \cdot 3^{\alpha+2}n + 10 \cdot 3^{\alpha+1}) \equiv 0 \pmod 3$.

1. Introduction

In 2002 Andrews, Lewis and Lovejoy [1] introduced a new class of partitions, partitions with designated summands which are constructed by taking ordinary partitions and tagging exactly one part among parts with equal size. With a convention that $PD(n) = 0$, for example there are 15 partitions of 5 with designated summands:

$$5', \quad 4'+1', \quad 3'+2', \quad 3'+1'+1, \quad 3'+1+1', \quad 2'+2+1', \quad 2+2'+1',$$
$$2'+1'+1+1, \quad 2'+1+1'+1, \quad 2'+1+1+1', \quad 1'+1+1+1+1,$$
$$1+1'+1+1+1, \quad 1+1+1'+1+1, \quad 1+1+1+1'+1, \quad 1+1+1+1+1'.$$

The authors [1] derived the following generating function of $PD(n)$.

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{f_6}{f_1 f_2 f_3}.$$

Throughout the paper, we use the standard q-series notation, and f_k is defined as

$$f_k := (q^k; q^k)_\infty = \lim_{n\to\infty} \prod_{l=1}^{n} (1 - q^{lk}).$$

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{1}$$

2010 Mathematics Subject Classification: 05A17, 11P83
Keywords and phrases: Partitions; designated summands; congruences.

Using Jacobi's triple product identity [4, Entry 19, p. 35], (1) becomes

$$f(a,b) = (-a,ab)_\infty (-b,ab)_\infty (ab,ab)_\infty .$$

The most important special cases of $f(a,b)$ are

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1.$$

The concept of partitions with designated summands goes back to MacMahon [9]. He considered partitions with designated summands and with exactly ℓ different sizes (see also Andrews and Rose [2]).

Andrews et al. [1] and N. D. Baruah and K. K. Ojah [3] have also studied $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd and the generating function is given by

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}.$$

Thus $PDO(5) = 8$ are

$$5', \quad 3'+1'+1, \quad 3'+1+1', \quad 1'+1+1+1+1, \quad 1+1'+1+1+1,$$
$$1+1+1'+1+1, \quad 1+1+1+1'+1, \quad 1+1+1+1+1'.$$

Chen, Ji, Jin and Shen [5] have established Ramanujan type identity for the partition function $PD(3n+2)$ which implies the congruence of Andrews et al. [1] and they also gave a combinatorial interpretation of the congruence for $PD(3n+2)$ by introducing a rank for partitions with designated summands. Recently Xia [14] extended the work of deriving congruence properties of $PD(n)$ by employing the generating functions of $PD(3n)$ and $PD(3n+2)$ due to Chen et al. [5].

Mahadeva Naika et al. [10] have studied $PD_3(n)$, the number of partitions of n with designated summands whose parts are not divisible by 3 and the generating function is given by

$$\sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_6^2 f_9}{f_1 f_2 f_{18}}.$$

In [11] Mahadeva Naika et al. have established many congruences for $PD_2(n)$, the number of bipartitions of n with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} PD_2(n)q^n = \frac{f_6^2}{f_1^2 f_2^2 f_3^2}.$$

Mahadeva Naika et al. [12] have derived $PD_{2,3}(n)$, the number of partitions of n with designated summands in which parts are not multiples of 2 or 3 and generating function is given by

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_4 f_6^2 f_9 f_{36}}{f_1 f_{12}^2 f_{18}^2}.$$

Motivated by the above work, in this paper, we study $PBD_3(n)$, the number of 3-regular bi-partitions of n with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} PBD_3(n)q^n = \frac{f_6^4 f_9^2}{f_1^2 f_2^2 f_{18}^2}. \tag{2}$$

To be precise by a bipartition with designated summands we mean a pair of partitions (μ, κ) where in partitions μ and κ are partitions with designated summands. Thus $PBD_3(4) = 35$ are

$(4', \emptyset)$, $(2'+2, \emptyset)$, $(2+2', \emptyset)$, $(2'+1'+1, \emptyset)$, $(2'+1+1', \emptyset)$, $(1'+1+1+1, \emptyset)$, $(1+1'+1+1, \emptyset)$, $(1+1+1'+1, \emptyset)$, $(1+1+1+1', \emptyset)$, $(2', 2')$, $(2', 1'+1)$, $(2', 1+1')$, $(1', 1'+1+1)$, $(1', 1+1'+1)$, $(1', 1+1+1')$, $(1'+1, 1'+1)$, $(1'+1, 1+1')$, $(1+1', 1'+1)$, $(1+1', 1+1')$, $(2'+1', 1')$, $(1', 2'+1')$, $(1'+1, 2')$, $(1+1', 2')$, $(1'+1+1, 1')$, $(1+1'+1, 1')$, $(1+1+1', 1')$, $(\emptyset, 4')$, $(\emptyset, 2'+2)$, $(\emptyset, 2+2')$, $(\emptyset, 2'+1'+1)$, $(\emptyset, 2'+1+1')$, $(\emptyset, 1'+1+1+1)$, $(\emptyset, 1+1'+1+1)$, $(\emptyset, 1+1+1'+1)$, $(\emptyset, 1+1+1+1')$.

In Section 3, we prove the following theorems.

Theorem 1.1. *For $n \geq 0$ we have*

$$\sum_{n=0}^{\infty} PBD_3(2n)q^n = \frac{f_3^2 f_6^6}{f_1^6 f_{18}^2} + q\frac{f_2^4 f_3^6 f_{18}^2}{f_1^8 f_6^2 f_9^2}, \tag{3}$$

$$\sum_{n=0}^{\infty} PBD_3(2n+1)q^n = 2\frac{f_2^2 f_3^4 f_6^2}{f_1^7 f_9}. \tag{4}$$

Theorem 1.2. *For each nonnegative integer n and $\alpha \geq 0$, we have*

$$PBD_3\left(4 \times 3^{\alpha+2} n + 10 \times 3^{\alpha+1}\right) \equiv 0 \pmod 3, \tag{5}$$

$$PBD_3\left(8 \times 3^{\alpha+2} n + 8 \times 3^{\alpha+2}\right) \equiv 0 \pmod 3, \tag{6}$$

$$PBD_3\left(2^{\alpha+3} n\right) \equiv 2^{\alpha} PBD_3(4n) \pmod 3, \tag{7}$$

$$\sum_{n=1}^{\infty} PBD_3(4n+2)q^n \equiv \psi(q)\psi(q^3) \pmod 3, \tag{8}$$

$$\sum_{n=1}^{\infty} PBD_3(8n+4)q^n \equiv 2\psi(q)\psi(q^3) \pmod 3. \tag{9}$$

Theorem 1.3. *Let p be a prime with $\left(\frac{-3}{p}\right) = -1$. Then for any nonnegative integer α,*

$$\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha} n + 2p^{2\alpha}\right) q^n \equiv \psi(q)\psi(q^3) \pmod 3, \tag{10}$$

and for $n \geq 0$, $1 \leq j \leq p-1$,

$$PBD_3\left(4p^{2\alpha+1}(pn+j)+2p^{2\alpha+2}\right) \equiv 0 \pmod{3}. \tag{11}$$

THEOREM 1.4. *Let* p *be a prime with* $\left(\frac{-3}{p}\right) = -1$. *Then for any nonnegative integer* α,

$$\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha}n+4p^{2\alpha}\right) q^n \equiv 2\psi(q)\psi(q^3) \pmod{3}, \tag{12}$$

and for $n \geq 0$, $1 \leq j \leq p-1$,

$$PBD_3\left(8p^{2\alpha+1}(pn+j)+4p^{2\alpha+2}\right) \equiv 0 \pmod{3}. \tag{13}$$

THEOREM 1.5. *For each* $n \geq 0$

$$PBD_3(12n+7) \equiv 0 \pmod{4}, \tag{14}$$

$$PBD_3(12n+11) \equiv 0 \pmod{4}, \tag{15}$$

$$PBD_3(24n+17) \equiv 0 \pmod{4}, \tag{16}$$

$$PBD_3(36n+27) \equiv 0 \pmod{4}, \tag{17}$$

$$PBD_3(72n+39) \equiv 0 \pmod{4}, \tag{18}$$

$$PBD_3(72n+57) \equiv 0 \pmod{4}, \tag{19}$$

$$PBD_3(216n+153) \equiv 0 \pmod{4}, \tag{20}$$

$$\sum\nolimits_{n=0}^{\infty} PBD_3(72n+3) \equiv 2f_1 \pmod{4}, \tag{21}$$

$$\sum\nolimits_{n=0}^{\infty} PBD_3(72n+15) \equiv 2f_1 f_4 \pmod{4}. \tag{22}$$

THEOREM 1.6. *For any prime* $p \geq 5$, $\alpha \geq 0$ *and* $n \geq 0$, *we have*

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha}n+3p^{3\alpha}\right) q^n \equiv 2f_1 \pmod{4}. \tag{23}$$

THEOREM 1.7. *For any prime* $p \geq 5$, $\alpha \geq 0$, $n \geq 0$ *and* $l = 1, 2, ...p-1$, *we have*

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha}(pn+l)+3p^{3\alpha}\right) \equiv 0 \pmod{4}. \tag{24}$$

THEOREM 1.8. *If* $p \geq 5$ *is a prime such that* $\left(\frac{-4}{p}\right) = -1$. *Then for all integers* $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha}n+15p^{2\alpha}\right) q^n \equiv 2f_1 f_4 \pmod{4}. \tag{25}$$

THEOREM 1.9. *Let* $p \geq 5$ *be prime and* $\left(\frac{-4}{p}\right) = -1$. *Then for all integers* $n \geq 0$ *and* $\alpha \geq 1$,

$$PBD_3\left(72p^{2\alpha}n+p^{2\alpha-1}(15p+72j)\right) \equiv 0 \pmod{4}, \tag{26}$$

where $j = 1, 2, \ldots, p-1$.

Theorem 1.10. *For each $n \geq 0$*

$$PBD_3(18n+15) \equiv 0 \pmod 6, \tag{27}$$

$$PBD_3(18n+3) \equiv 4f_1f_3 \pmod 6. \tag{28}$$

Theorem 1.11. *If $p \geq 5$ is a prime such that $\left(\frac{-3}{p}\right) = -1$. Then for all integers $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} PBD_3\left(18p^{2\alpha}n + 3p^{2\alpha}\right) q^n \equiv 4f_1f_3 \pmod 6. \tag{29}$$

Theorem 1.12. *Let $p \geq 5$ be prime and $\left(\frac{-3}{p}\right) = -1$. Then for all integers $n \geq 0$ and $\alpha \geq 1$,*

$$PBD_3\left(18p^{2\alpha}n + p^{2\alpha-1}(3p+18j)\right) \equiv 0 \pmod 6, \tag{30}$$

where $j = 1, 2, \ldots, p-1$.

2. Preliminaries

We list a few dissection formulas to prove our main results.

Lemma 2.1. *[4, Corollory, p. 49] We have*

$$\psi(q) = f(q^3, q^6) + q\psi(q^9) \tag{31}$$

Lemma 2.2. *The following 2-dissections hold:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3f_6^2}{f_2^2f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{32}$$

$$\frac{f_1}{f_3^3} = \frac{f_2f_4^2f_{12}^2}{f_6^7} - q\frac{f_2^3f_{12}^6}{f_4^2f_6^9}. \tag{33}$$

Hirschhorn, Garvan and Borwein [7] proved equation (32). Replacing q by $-q$ in (32), we obtain (33).

Lemma 2.3. *The following 2-dissections hold:*

$$\frac{1}{f_1f_3} = \frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2} + q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}, \tag{34}$$

$$f_1f_3 = \frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2} - q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}. \tag{35}$$

Equation (34) was proved by Baruah and Ojah [3]. Replacing q by $-q$ in (34) and using the fact that $(-q;-q)_\infty = \frac{f_2^3}{f_1f_4}$, we get (35).

Lemma 2.4. *The following* 3*-dissection holds:*

$$f_1f_2 = \frac{f_6f_9^4}{f_3f_{18}^2} - qf_9f_{18} - 2q^2\frac{f_3f_{18}^4}{f_6f_9^2}. \tag{36}$$

One can see this identity in [8].

Lemma 2.5. *The following 2-dissections hold:*

$$\frac{f_9}{f_1} = \frac{f_{12}^3f_{18}}{f_2^2f_6f_{36}} + q\frac{f_4^2f_6f_{36}}{f_2^3f_{12}} \tag{37}$$

and

$$\frac{f_1}{f_9} = \frac{f_2f_{12}^3}{f_4f_6f_{18}^2} - q\frac{f_4f_6f_{36}^2}{f_{12}f_{18}^3}. \tag{38}$$

Lemma 2.5 was proved by Xia and Yao [13]. Replacing q by $-q$ in (37) and using the relation $(-q;-q)_\infty = \dfrac{f_2^3}{f_1f_4}$, we obtain (38).

Lemma 2.6. *[6, Theorem 2.1] For any odd prime* p*,*

$$\psi(q) = \sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{39}$$

Furthermore, $\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod p$ *for* $0 \le m \le \frac{p-3}{2}$.

Lemma 2.7. *[6, Theorem 2.2] For any prime* $p \ge 5$*,*

$$f_1 = \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}.$$

Furthermore, for $-(p-1)/2 \le k \le (p-1)/2$ *and* $k \neq (\pm p-1)/6$, $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p$.

3. Proofs of main results

3.1 Proof of Theorems 1.1 and 1.2

Substituting (37) into (2), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} PBD_3(n)q^n &= \frac{f_6^4}{f_2^2f_{18}^2}\left(\frac{f_{12}^6f_{18}^2}{f_2^4f_6^2f_{36}^2} + 2q\frac{f_4^2f_{12}^2f_{18}}{f_2^5} + q^2\frac{f_4^4f_6^2f_{36}^2}{f_2^6f_{12}^2}\right)\\ &= \frac{f_6^2f_{12}^6}{f_2^6f_{36}^2} + 2q\frac{f_4^2f_6^4f_{12}^2}{f_2^7f_{18}} + q^2\frac{f_4^4f_6^6f_{36}^2}{f_2^8f_{12}^2f_{18}^2}.\end{aligned}$$

Extracting the terms involving q^{2n} and q^{2n+1} from the above equation, we obtain (3) and (4).

By the binomial theorem, it is easy to see that for positive integers k and m,

$$f_{2k}^{m} \equiv f_k^{2m} \pmod{2}, \tag{40}$$

$$f_{3k}^{m} \equiv f_k^{3m} \pmod{3} \tag{41}$$

and

$$f_{2k}^{2m} \equiv f_k^{4m} \pmod{4}. \tag{42}$$

Invoking (41) in (3), we find

$$\sum_{n=0}^{\infty} PBD_3(2n)q^n \equiv 1 + q\frac{f_1 f_6^6}{f_2^2 f_3^3} \pmod{3},$$

which implies that

$$\sum_{n=1}^{\infty} PBD_3(2n)q^n \equiv q\frac{f_1 f_6^6}{f_2^2 f_3^3} \pmod{3}. \tag{43}$$

Employing (33) into (43), we have

$$\sum_{n=1}^{\infty} PBD_3(2n)q^n \equiv q\frac{f_4^2 f_{12}^2}{f_2 f_6} - q^2\frac{f_2 f_{12}^6}{f_4^2 f_6^3} \pmod{3}. \tag{44}$$

Extracting the terms containing q^{2n+1}, dividing throughout by q and then replacing q^2 by q from (44) and using the fact that $\psi(q) = \dfrac{f_2^2}{f_1}$, we get (8).

Substituting (31) into (8), we obtain

$$\sum_{n=1}^{\infty} PBD_3(4n+2)q^n \equiv f(q^3, q^6)\psi(q^3) + q\psi(q^3)\psi(q^9) \pmod{3}, \tag{45}$$

implying

$$\sum_{n=1}^{\infty} PBD_3(12n+6)q^n \equiv \psi(q)\psi(q^3) \pmod{3}. \tag{46}$$

From equations (8) and (46), we get

$$PBD_3(12n+6) \equiv PBD_3(4n+2) \pmod{3}. \tag{47}$$

By using mathematical induction on α in (47), we have

$$PBD_3\left(4 \times 3^{\alpha+1}n + 2 \times 3^{\alpha+1}\right) \equiv PBD_3(4n+2) \pmod{3}. \tag{48}$$

Extracting the terms containing q^{3n+2} from (45) we obtain

$$PBD_3(12n+10) \equiv 0 \pmod{3}. \tag{49}$$

Using (49) in (48), we find (5).

Extracting the terms containing q^{2n} and replacing q^2 by q from (44), we get

$$\sum_{n=1}^{\infty} PBD_3(4n)q^n \equiv 2q\frac{f_1 f_6^6}{f_2^2 f_3^3} \pmod{3}. \tag{50}$$

Employing (33) into (50), we obtain

$$\sum_{n=1}^{\infty} PBD_3(4n)q^n \equiv 2q\frac{f_4^2 f_{12}^2}{f_2 f_6} - 2q^2\frac{f_2 f_{12}^6}{f_4^4 f_6^3} \pmod{3}. \tag{51}$$

Congruence (9) is obtained by extracting the terms containing q^{2n+1} from (51) and using the fact that $\psi(q) = \dfrac{f_2^2}{f_1}$.

Substituting (31) into (9), we have

$$\sum_{n=1}^{\infty} PBD_3(8n+4)q^n \equiv 2f(q^3,q^6)\psi(q^3) + 2q\psi(q^3)\psi(q^9) \pmod 3.$$

Extracting the terms containing q^{3n+1} and q^{3n+2} from the above equation, we obtain

$$\sum_{n=1}^{\infty} PBD_3(24n+12)q^n \equiv 2\psi(q)\psi(q^3) \pmod 3 \tag{52}$$

and

$$PBD_3(24n+20) \equiv 0 \pmod 3. \tag{53}$$

In view of the congruences (9) and (52), we get

$$PBD_3(24n+12) \equiv PBD_3(8n+4) \pmod 3. \tag{54}$$

Utilizing (54) and by mathematical induction on α, we arrive at

$$PBD_3\left(8\times 3^{\alpha+1}n + 8\times 3^{\alpha+1}\right) \equiv PBD_3(8n+4) \pmod 3. \tag{55}$$

Using (53) in (55), we obtain (6).

Extracting the terms containing q^{2n} and replacing q^2 by q from (51), we have

$$\sum_{n=1}^{\infty} PBD_3(8n)q^n \equiv q\frac{f_1 f_6^6}{f_2^4 f_3^3} \pmod 3. \tag{56}$$

In view of the congruences (56) and (50), we obtain

$$PBD_3(8n) \equiv 2\cdot PBD_3(4n) \pmod 3. \tag{57}$$

Utilizing (57) and by mathematical induction on α, we arrive at (7). □

3.2 Proof of Theorem 1.3

Equation (8) is the $\alpha = 0$ case of (10). If we assume that (10) holds for some $\alpha \geq 0$, then, substituting (39) in (10),

$$\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha}n + 2p^{2\alpha}\right)q^n$$
$$\equiv \left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right) \tag{58}$$
$$\times\left(\sum_{m=0}^{\frac{p-3}{2}} q^{3\frac{m^2+m}{2}} f\left(q^{3\frac{p^2+(2m+1)p}{2}}, q^{3\frac{p^2-(2m+1)p}{2}}\right) + q^{3\frac{p^2-1}{8}}\psi(q^{3p^2})\right) \pmod 3.$$

For any odd prime p, and $0 \leq m_1, m_2 \leq (p-3)/2$, consider the congruence

$$\frac{m_1^2+m_1}{2} + 3\frac{m_2^2+m_2}{2} \equiv \frac{4p^2-4}{8} \pmod p,$$

which implies that

$$(2m_1+1)^2 + 3(2m_2+1)^2 \equiv 0 \pmod p. \tag{59}$$

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the congruence (59) is $m_1 = m_2 = \dfrac{p-1}{2}$. Therefore, equating the coefficients of $q^{pn+\frac{4p^2-4}{8}}$ from both sides of (58), dividing throughout by $q^{\frac{4p^2-4}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha}\left(pn+\frac{4p^2-4}{8}\right)+2p^{2\alpha}\right)q^n \equiv \psi(q^p)\psi(q^{3p}) \pmod 3. \tag{60}$$

Equating the coefficients of q^{pn} on both sides of (60) and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha+2}n+2p^{2\alpha+2}\right)q^n \equiv \psi(q)\psi(q^3) \pmod 3,$$

which is the $\alpha+1$ case of (10). Extracting the terms involving q^{pn+j} for $1 \leq j \leq p-1$ in (60), we get (11). □

3.3 Proof of Theorem 1.4

Equation (9) is the $\alpha = 0$ case of (12). If we assume that (12) holds for some $\alpha \geq 0$, then, substituting (39) in (12),

$$\begin{aligned}
&\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha}n+4p^{2\alpha}\right)q^n \\
&\equiv 2\left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right) \\
&\times\left(\sum_{m=0}^{\frac{p-3}{2}} q^{3\frac{m^2+m}{2}} f\left(q^{3\frac{p^2+(2m+1)p}{2}}, q^{3\frac{p^2-(2m+1)p}{2}}\right)+q^{3\frac{p^2-1}{8}}\psi(q^{3p^2})\right) \pmod 3.
\end{aligned} \tag{61}$$

For any odd prime p, and $0 \leq m_1, m_2 \leq (p-3)/2$, consider the congruence

$$\frac{m_1^2+m_1}{2}+3\frac{m_2^2+m_2}{2} \equiv \frac{4p^2-4}{8} \pmod p,$$

which implies that

$$(2m_1+1)^2+3(2m_2+1)^2 \equiv 0 \pmod p. \tag{62}$$

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the congruence (62) is $m_1 = m_2 = \dfrac{p-1}{2}$. Therefore, equating the coefficients of $q^{pn+\frac{4p^2-4}{8}}$ from both sides of (61), dividing throughout by $q^{\frac{4p^2-4}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha}\left(pn+\frac{4p^2-4}{8}\right)+4p^{2\alpha}\right)q^n \equiv 2\psi(q^p)\psi(q^{3p}) \pmod 3. \tag{63}$$

Equating the coefficients of q^{pn} on both sides of (63) and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha+2}n+4p^{2\alpha+2}\right)q^n \equiv 2\psi(q)\psi(q^3) \pmod 3,$$

which is the $\alpha+1$ case of (12). Extracting the terms involving q^{pn+j} for $1 \leq j \leq p-1$ in (63), we arrive at (13). □

3.4 Proof of Theorem 1.5

Invoking (42) in (4), we find

$$\sum_{n=0}^{\infty} PBD_3(2n+1)q^n \equiv 2\frac{f_1 f_6^4}{f_2^2 f_9} \pmod 8. \tag{64}$$

Employing (38) into (64), we obtain

$$\sum_{n=0}^{\infty} PBD_3(2n+1)q^n \equiv 2\frac{f_6^3 f_{12}^3}{f_2 f_4 f_{18}^2} - 2q\frac{f_4 f_6^5 f_{36}^2}{f_2^2 f_{12} f_{18}^3} \pmod 8. \tag{65}$$

Extracting the terms containing q^{2n+1}, dividing throughout by q and then replacing q^2 by q from the above equation, we get

$$\sum_{n=0}^{\infty} PBD_3(4n+3)q^n \equiv 6\frac{f_2 f_3^5 f_{18}^2}{f_1^2 f_6 f_9^3} \pmod 8), \tag{66}$$

but

$$6\frac{f_2 f_3^5 f_{18}^2}{f_1^2 f_6 f_9^3} \equiv 6\frac{f_2 f_3^5 f_9}{f_1^2 f_6} \pmod 8). \tag{67}$$

Invoking (40) in (67), we get

$$\sum_{n=0}^{\infty} PBD_3(4n+3)q^n \equiv 2f_3 f_6 f_9 \pmod 4. \tag{68}$$

Congruences (14) and (15) follow by extracting the terms containing q^{3n+1} and q^{3n+2} from (68).

Extracting the terms containing q^{3n} and replacing q^3 by q from (68). we obtain

$$\sum_{n=0}^{\infty} PBD_3(12n+3)q^n \equiv 2f_1 f_2 f_3 \pmod 4. \tag{69}$$

Substituting (36) into (69), we find

$$\sum_{n=0}^{\infty} PBD_3(12n+3)q^n \equiv 2\frac{f_6 f_9^4}{f_{18}^2} - 2q f_3 f_9 f_{18} \pmod 4. \tag{70}$$

Congruence (17) is obtained by extracting the terms containing q^{3n+2} from (70).

Extracting the terms containing q^{3n} and replacing q^3 by q from the above equation we arrive at

$$\sum_{n=0}^{\infty} PBD_3(36n+3)q^n \equiv 2\frac{f_2 f_3^4}{f_6^2} \pmod 4. \tag{71}$$

Using (40) in (71), we obtain

$$\sum_{n=0}^{\infty} PBD_3(36n+3)q^n \equiv 2f_2 \pmod 4. \tag{72}$$

Congruences (18) and (21) follow by extracting the terms containing q^{2n} and q^{2n+1}

from (72).

Extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (70), we obtain

$$\sum_{n=0}^{\infty} PBD_3(36n+15)q^n \equiv 2f_1f_3f_6 \pmod{4}. \tag{73}$$

Employing (35) into (73), we find

$$\sum_{n=0}^{\infty} PBD_3(36n+15)q^n \equiv 2\frac{f_2f_8^2f_{12}^4}{f_4^2f_{24}^2} - 2q\frac{f_4^4f_6^2f_{24}^2}{f_2f_8^2f_{12}^2} \pmod{4}. \tag{74}$$

Extracting the terms containing q^{2n} and then replacing q^2 by q from (74), we obtain

$$\sum_{n=0}^{\infty} PBD_3(72n+15)q^n \equiv 2\frac{f_1f_4^2f_6^4}{f_2^2f_{12}^2} \pmod{4}. \tag{75}$$

Using (40) in (75) we arrive at (22).

Extracting the terms containing q^{2n} and replacing q^2 by q from (65), we get

$$\sum_{n=0}^{\infty} PBD_3(4n+1)q^n \equiv 2\frac{f_3^3f_6^3}{f_1f_2f_9^2} \pmod{8}. \tag{76}$$

Using (40) in (76), we have

$$\sum_{n=0}^{\infty} PBD_3(4n+1)q^n \equiv 2\frac{f_3^3f_6^3}{f_1f_2f_{18}} \pmod{4}. \tag{77}$$

Substituting (32) into (77), we arrive at

$$\sum_{n=0}^{\infty} PBD_3(4n+1)q^n \equiv 2\frac{f_4^3f_6^5}{f_2^3f_{12}f_{18}} + 2q\frac{f_6^3f_{12}^3}{f_2f_4f_{18}} \pmod{4}. \tag{78}$$

Extracting the terms containing q^{2n} and replacing q^2 by q from (78), we obtain

$$\sum_{n=0}^{\infty} PBD_3(8n+1)q^n \equiv 2\frac{f_2^3f_3^5}{f_1^3f_6f_9} \pmod{4},$$

but

$$\frac{f_2^3f_3^5}{f_1^3f_6f_9} \equiv \frac{f_2^2f_3f_6}{f_1f_9} \pmod{2}.$$

This yields

$$\sum_{n=0}^{\infty} PBD_3(8n+1)q^n \equiv 2\frac{f_2^2f_3f_6}{f_1f_9} \pmod{4}. \tag{79}$$

Using Jacobi's triple product identity and $\psi(q) = \frac{f_2^2}{f_1}$ in (31), we arrive at

$$\frac{f_2^2}{f_1} = \frac{f_6f_9^2}{f_3f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{80}$$

Employing (80) into (79), we get

$$\sum_{n=0}^{\infty} PBD_3(8n+1)q^n \equiv 2\frac{f_6^2f_9}{f_{18}} + 2q\frac{f_3f_6f_{18}^2}{f_9^2} \pmod{4}. \tag{81}$$

Congruence (16) is obtained by extracting the terms containing q^{3n+2} from the above equation.

Extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (81), we obtain

$$\sum_{n=0}^{\infty} PBD_3(24n+9)q^n \equiv 2\frac{f_1f_2f_6^2}{f_3^2} \pmod{4}. \tag{82}$$

Using (40) in (82), we have

$$\sum_{n=0}^{\infty} PBD_3(24n+9)q^n \equiv 2f_1f_2f_6 \pmod{4}. \tag{83}$$

Substituting (36) into (83), we obtain

$$\sum_{n=0}^{\infty} PBD_3(24n+9)q^n \equiv 2\frac{f_6^2f_9^4}{f_3f_{18}^2} - 2qf_6f_9f_{18} \pmod{4}. \tag{84}$$

Congruence (19) follows from (84) and extracting the terms containing q^{3n} and replacing q^3 by q from the above equation. we find

$$\sum_{n=0}^{\infty} PBD_3(72n+9)q^n \equiv 2\frac{f_2^2f_3^4}{f_1f_6^2} \pmod{4}. \tag{85}$$

Using (40) in (85), we get

$$\sum_{n=0}^{\infty} PBD_3(72n+9)q^n \equiv 2\frac{f_2^2}{f_1} \equiv 2\psi(q) \pmod{4}. \tag{86}$$

Substituting (31) into (86) and extracting the terms containing q^{3n+2}, we arrive at (20). □

3.5 Proof of Theorem 1.6

Employing Lemma (2.7) into (21), it can be see that

$$\sum_{n=0}^{\infty} PBD_3\left(72\left(pn + \frac{p^2-1}{24}\right) + 3\right) q^n \equiv 2f_p \pmod{4}, \tag{87}$$

which implies that

$$\sum_{n=0}^{\infty} PBD_3\left(72p^2n + 3p^3\right) q^n \equiv 2f_1 \pmod{4}.$$

Therefore, $PBD_3\left(72p^2n + 3p^3\right) \equiv PBD_3(72n+3) \pmod{4}$.

Using the above relation and by induction on α, we arrive at (23). □

3.6 Proof of Theorem 1.7

Combining (87) with Theorem (1.6), we derive that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha+1}n + 3p^{3\alpha}\right) \equiv 2f_p \pmod{4}.$$

Therefore, it follows that

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha+1}(pn+l)+3p^{3\alpha}\right) \equiv 0 \pmod{4}.$$

where $l = 1, 2, ..., p-1$, and we obtain (24). □

3.7 Proof of Theorem 1.8

For a prime $p \geq 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$\frac{3k^2+k}{2} + 4 \times \frac{3m^2+m}{2} \equiv \frac{5p^2-5}{24} \pmod{p}.$$

This is equivalent to $(6k+1)^2 + 4(6m+1)^2 \equiv 0 \pmod{p}$. Since $\left(\frac{-4}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p - 1)/6$. Therefore, from Lemma 2.7,

$$\sum_{n=0}^{\infty} PBD_3\left(72\left(p^2 n + 5 \times \frac{p^2-1}{24}\right) + 15\right) q^n \equiv 2f_1 f_4 \pmod{4}. \qquad (88)$$

Using (22), (88), and induction on α, we get (25). □

3.8 Proof of Theorem 1.9

From Lemma 2.7 and Theorem 1.8, for each $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3\left(72\left(p^2 n + 5 \times \frac{p^2-1}{24}\right) + 15\right) q^n \equiv 2f_1 f_4 \pmod{4}.$$

That is,

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha+1}n + 15p^{2\alpha+2}\right) q^n \equiv 2f_p f_{4p} \pmod{4}. \qquad (89)$$

Since there are no terms on the right of (89) where the powers of q are congruent to $1, 2, \ldots, p-1$ modulo p,

$$PBD_3\left(72p^{2\alpha+1}(pn+j) + 15p^{2\alpha+2}\right) \equiv 0 \pmod{4},$$

for $j = 1, 2, \ldots, p-1$. Therefore, for $j = 1, 2, \ldots, p-1$ and $\alpha \geq 1$, we arrive at (26). □

3.9 Proof of Theorem 1.10

By the binomial theorem, it is easy to see that for positive integers k and m,

$$f_{3k}^{3m} \equiv f_k^{9m} \pmod{9}, \qquad (90)$$

Invoking (90) in (4), we have

$$\sum_{n=0}^{\infty} PBD_3(2n+1)q^n \equiv 2\frac{f_1^2 f_2^2 f_3 f_6^2}{f_9} \pmod{18}. \qquad (91)$$

Employing (36) into (91) and extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (91), we obtain

$$\sum_{n=0}^{\infty} PBD_3(6n+3)q^n \equiv 14\frac{f_2^3 f_3^4}{f_6} + 8q\frac{f_1^3 f_6^8}{f_3^5} \pmod{18}. \tag{92}$$

Invoking (41) in (92), we see that

$$\sum_{n=0}^{\infty} PBD_3(6n+3)q^n \equiv 4f_3^4 + 4q\frac{f_6^8}{f_3^4} \pmod{6}. \tag{93}$$

Congruence (27) follows by extracting the terms containing q^{3n+2} from the above equation.

Extracting the terms containing q^{3n} and replacing q^3 by q from (93), we arrive at

$$\sum_{n=0}^{\infty} PBD_3(18n+3)q^n \equiv 4f_1^4 \pmod{6},$$

which implies

$$\sum_{n=0}^{\infty} PBD_3(18n+3)q^n \equiv 4f_1 f_1^3 \pmod{6}. \tag{94}$$

Invoking (41) in (94) we get (28). □

3.10 Proof of Theorem 1.11

For a prime $p \geq 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$\frac{3k^2+k}{2} + 3 \times \frac{3m^2+m}{2} \equiv \frac{4p^2-4}{24} \pmod{p}.$$

This is equivalent to $(6k+1)^2 + 3(6m+1)^2 \equiv 0 \pmod{p}$.

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p - 1)/6$. Therefore, from Lemma 2.7,

$$\sum_{n=0}^{\infty} PBD_3\left(18\left(p^2 n + 4 \times \frac{p^2-1}{24}\right) + 3\right) q^n \equiv 4f_1 f_3 \pmod{6}. \tag{95}$$

Using (28), (95), and induction on α, we arrive at (29). □

3.11 Proof of Theorem 1.12

From Lemma 2.7 and Theorem 1.11, for each $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3\left(18\left(p^2 n + 4 \times \frac{p^2-1}{24}\right) + 3\right) q^n \equiv 4f_1 f_3 \pmod{6}.$$

That is,

$$\sum_{n=0}^{\infty} PBD_3\left(18p^{2\alpha+1} n + 3p^{2\alpha+2}\right) q^n \equiv 4f_p f_{3p} \pmod{6}. \tag{96}$$

Since there are no terms on the right of (96) where the powers of q are congruent to $1, 2, \ldots, p-1$ modulo p,

$$PBD_3\left(18p^{2\alpha+1}(pn+j) + 3p^{2\alpha+2}\right) \equiv 0 \pmod{6},$$

for $j = 1, 2, \ldots, p-1$. Therefore, for $j = 1, 2, \ldots, p-1$ and $\alpha \geq 1$, we obtain (30). □

Acknowledgement. The authors would like to thank the anonymous referee for helpful comments and suggestions and the second author would like to thank for UGC for providing National fellowship for higher education (NFHE), ref. no.F1-17.1/2015-16/NFST-2015-17-ST-KAR-1376.

References

[1] G. E. Andrews, R. P. Lewis, J. Lovejoy, *Partitions with designated summands*, Acta Arith. **105** (2002), 51–66.

[2] G. E. Andrews, S. C. F. Rose, *MacMahon's sum-of-divisors functions, Chebyshev polynomials, and quasi-modular forms*, J. Reine Angew. Math. (2013), in press; arXiv:1010.5769

[3] N. D. Baruah, K. K. Ojah, *Partitions with designated summands in which all parts are odd*, INTEGERS **#A9 15**, (2015).

[4] B. C. Berndt, *Ramanujan's Notebooks*, Part III, Springer-Verlag, New York, 1991.

[5] W. Y. C. Chen, K. Q. Ji, H. T. Jin, E. Y. Y. Shen, *On the number of partitions with designated summands*, J. Number Theory, **133** (2013), 2929–2938.

[6] S. P. Cui, N. S. S. Gu, *Arithmetic properties of ℓ-regular partitions*, Adv. Appl. Math. **51** (2013), 507–523.

[7] M. D. Hirschhorn, F. Garvan, J. Borwein, *Cubic analogs of the Jacobian cubic theta function* $\theta(z, q)$, Canad. J. Math. **45** (1993), 673–694.

[8] M. D. Hirschhorn, J. A. Sellers, *A congruence modulo 3 for partitions into distinct non-multiples of four*, Journal of Integer Sequences, **17** (2014), Article 14.9.6

[9] P. A. MacMahon, *Divisors of numbers and their continuations in the theory of partitions*, Proc. Lond. Math. Soc. Ser. **2 19** (1919), 75–113.

[10] M. S. Mahadeva Naika, D. S. Gireesh, *Congruences for 3-regular partitions with designated summands*, INTEGERS **#A25 16**, (2016).

[11] M. S. Mahadeva Naika, C. Shivashankar, *Arithmetic properties of bipartitions with designated summands*, Boletín de la Sociedad Matemática Mexicana. DOI: 10.1007/s40590–016–0140–8

[12] M. S. Mahadeva Naika, S. Shivaprasada Nayaka, *Congruences for* $(2, 3)$*-regular partition with designated summands*, Note Mat. **36 (2)** (2016), 99–123.

[13] E. X. W. Xia, O. X. M. Yao, *Some modular relations for the Göllnitz-Gordon functions by an even-odd method*, J. Math. Anal. Appl. **387** (2012), 126–138.

[14] E. X. W. Xia, *Arithmetic properties of partitions with designated summands*, J. Number Theory **159** (2016), 160–175.

(received 21.10.2016; in revised form 05.05.2017; available online 13.06.2017)

Department of Mathematics, Bangalore University, Central College Campus, Bangalore-560 001, Karnataka, India

E-mail: msmnaika@rediffmail.com, shivprasadnayaks@gmail.com