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Corrected proof
Available online 05.04.2017

research paper
оригинални научни рад

LINEAR HAMILTONIAN SYSTEM IN SCALE OF HILBERT SPACES AND THE POINCARÉ RECURRENCE THEOREM

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Abstract. We consider an initial value problem for linear Hamiltonian system in the scale of Hilbert spaces and prove an existence and uniqueness theorem. We also prove a version of the Poincaré Recurrence Theorem.

1. Statement of the problem

Linear differential equations in Banach spaces and the semigroup theory have become a classical topic of PDE since the fifties of the last century.

One of the central results of this theory is the Hille–Yosida theorem [10], [5]. This theorem provides necessary and sufficient conditions that a densely defined operator $A : E \to E$ of a Banach space E generates a strongly continuous semirgoup $\{e^{tA}\}$ or, in other words, the initial value problem

$$\dot{x} = Ax, \quad x(0) = \hat{x}$$

has a good enough solution $x(t) = e^{tA}\hat{x}$.

This theorem is formulated in terms of spectrum of the operator A. The point is that the spectrum of an operator is not always simple to comprehend.

In this short paper we consider a Hamiltonian system in the scale of Hilbert spaces. Apparently there are no direct ways to solve such systems by means of the standard spectral methods.

To motivate the statement of our main problem we first simulate the whole construction by means of the easiest example. The matrix J (see below) in this example is very simple therefore the corresponding result follows from the Hille–Yosida theorem. But, in general case, our main theorem is not deduced from the Hille–Yosida theorem.

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth enough boundary $\partial\Omega$. It is well known that the Laplace operator Δ has the countable system of eigenfunctions

2010 Mathematics Subject Classification: 35A01, 37K05, 35A02
Keywords and phrases: Linear PDE; infinite dimensional Hamiltonian system.

$e_j(x) \in C^\infty(\overline{\Omega})$,

$$-\Delta e_j = \lambda_j^2 e_j, \quad e_j(\partial\Omega) = 0, \quad j \in \mathbb{N}.$$

These functions form orthogonal basis of $L^2(\Omega)$ and

$$0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lambda_j \to \infty$$

as $j \to \infty$. We also assume that $\|e_j\|_{L^2(\Omega)} = 1$.

Introduce a space

$$Y_s = \Big\{ u(x) = \sum_{k=1}^{\infty} u_k e_k \mid \|u\|_s^2 = \sum_{k=1}^{\infty} u_k^2 \lambda_k^{2s} < \infty \Big\}, \quad s \in \mathbb{R}.$$

One may treat elements of Y_s as distributions:

$$\phi = \sum_{k=1}^{\infty} \phi_k e_k \in \mathcal{D}(\Omega), \quad (u, \phi) = \sum_{k=1}^{\infty} u_k \phi_k.$$

It is not hard to show that the last sum is convergent.

The spaces Y_s are Hilbert spaces (all the Hilbert spaces we use are over the field $\mathbb{R}$) with the inner products

$$(u, v)_s = \sum_{k=1}^{\infty} \lambda_k^{2s} u_k v_k, \quad v(x) = \sum_{k=1}^{\infty} v_k e_k \in Y_s.$$

The spaces Y_s are referred to as fractional power spaces associated to the Laplace operator. Observe that $Y_0 = L^2(\Omega),\ Y_1 = H_0^1(\Omega)$.

The power of the Laplace operator is defined in the natural way

$$(-\Delta)^\mu u = \sum_{k=1}^{\infty} \lambda_k^{2\mu} u_k e_k,$$

and $(-\Delta)^\mu : Y_s \to Y_{s-2\mu},\ \mu \in \mathbb{R}$, is an isometric isomorphism.

Consider the wave equation

$$u_{tt} = \Delta u, \quad u(t,x) \mid_{x \in \partial\Omega} = 0, \quad u(0,x) = \hat{u}(x), \quad u_t(0,x) = \hat{v}(x). \tag{1}$$

Introduce a vector-valued function $f(t,x) = (u(t,x), w(t,x))^T$ and rewrite our problem as follows

$$f_t = JBf, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} (-\Delta)^{1/2} & 0 \\ 0 & (-\Delta)^{1/2} \end{pmatrix}.$$

The initial conditions take the form

$$w(0,x) = (-\Delta)^{-1/2} \hat{v}(x).$$

It is convenient to solve problem (1) in $Y_s,\ u(t,\cdot), w(t,\cdot) \in Y_s$ with

$$\hat{u} \in Y_s, \quad \hat{v} \in Y_{s-1}.$$

Observe that the Schrödinger equation $i\psi_t = \Delta\psi,\ \psi = \psi_1 + i\psi_2$ can also be written in the form $\xi_t = JB\xi,\ \xi = (\psi_1, \psi_2)^T$ where B is a positively defined self-adjoint operator and J is a skew-symmetric operator.

1.1 Abstract construction

In this section we describe the above situation axiomatically. All the inessential positive constants we denote by the letters $c, c_1, c_2, \dots$ or by c with another subscript.

Let

$$X_{s+\delta} \subset X_s, \quad \delta > 0, \quad s \in \mathbb{R} \tag{2}$$

be a scale of real Hilbert spaces with inner products $(\cdot,\cdot)_s$ and the corresponding norms $\|\cdot\|_s,\ \|\cdot\|_s \le \|\cdot\|_{s+\delta}$. The space $X_{s+\delta}$ is dense in X_s. In the above example $X_s = Y_s \times Y_s$.

Let

$$J : X_s \to X_s, \quad (Jx, y)_s = -(Jy, x)_s$$

stand for a bounded skew-symmetric operator. This means that the operator J is defined on $\bigcup_{s\in\mathbb{R}} X_s$ and $J(X_s) \subset X_s$. The same holds for all other operators, yet we do not stress on this.

If it is not specified anything else, the parameter t belongs to the interval $I_T = [0, T],\ T > 0$.

Introduce a bounded operator $B : X_s \to X_{s-1}$. This operator enjoys the following properties.

1. the operator B is self-adjoint and non-negative
$$(Bx, y)_s = (x, By)_s, \quad (Bx, x)_s \ge 0, \quad x, y \in X_{s+1};$$
2. all the powers $B^\sigma : X_{s+\sigma} \to X_s$ are bounded and coercive
$$(B^\sigma x, x)_s \ge c\|x\|^2_{s+\sigma/2}, \quad \sigma \ge 0, \quad x \in X_{s+\sigma};$$
3. the operator $-B^\sigma$ generates a C_0-semigroup [10]
$$e^{-tB^\sigma} : X_s \to X_s$$
such that for any $x \in X_{s+\sigma}$ we have
$$\Big\|\frac{1}{t}\Big(e^{-tB^\sigma} - I\Big)x + B^\sigma x\Big\|_s \to 0,$$
as $t \to 0+$.
Moreover, $e^{-tB^\sigma}(X_s) \subset X_{s+\delta},\ t \in (0, T]$ and for any $x \in X_s$ and for any $\delta \in (0, c_\delta)$ the estimate
$$\|e^{-tB^\sigma} x\|_{s+\delta} \le \frac{c_1}{t^{\delta/\sigma}}\|x\|_s,\ c_1 = c_1(s, \delta, \sigma) \tag{3}$$
is fulfilled.

The main object of our study is the following Hamiltonian initial value problem

$$\dot{x} = JB^{\hat{\sigma}}x, \quad x(0) = \hat{x} \in X_{\hat{s}} \tag{4}$$

with some fixed constants $\hat{\sigma} > 0,\ \hat{s} \in \mathbb{R}$.

1.2 Several remarks on accepted hypotheses

In the above example all the hypotheses can be checked by direct calculation.

Observe that hypotheses on the operator J can be considerably relaxed. Actually, one can put $J : X_s \to X_{s-\mu},\ \mu > 0$ to be a bounded and skew-symmetric operator:

$$(x, Jy)_s = -(Jx, y)_s, \quad x, y \in X_{s+\mu}.$$

This generalization does not bring any essential difficulties in the proofs of the theorems but makes the formulas more complicated than we have now.

Formula (3) is very standard in the parabolic semigroup theory [9], [1], [11].

In the sequel we do not use all the Hilbert spaces contained in the scale $\{X_s\}$. Actually we use only the spaces with indexes

$$\hat{s}, \quad \hat{s} \pm \gamma, \quad \hat{s} - \gamma - \hat{\sigma}, \quad \hat{s} - \gamma - 3\hat{\sigma}/2, \quad \hat{s} - \gamma - \hat{2}\sigma, \quad \hat{s} - \hat{\sigma}/2, \quad \hat{s} + \hat{\sigma}.$$

But the scales arise in the applications. Furthermore we believe that the scale $\{X_s\}$ is more acceptable object than several different spaces which look like a cumbersome and artificial construction.

2. Main theorems

Let $C_w(I_T, X_s)$ stand for the space $C(I_T, \tilde{X}_s)$ and $\tilde{X}_s$ is the space X_s endowed with the weak topology.

THEOREM 2.1. *1. Problem* (4) *has a weak solution*

$$x(t) \in L^\infty(I_T, X_{\hat{s}}) \cap C_w(I_T, X_{\hat{s}-\gamma})$$

with arbitrary $T > 0$ and arbitrary $\gamma > \hat{\sigma}$. That is

(a) for any $\psi \in X_{\hat{s}-\gamma}$ it follows that

$$\big(\psi, x(t)\big)_{\hat{s}-\gamma} \to \big(\psi, x(t_0)\big)_{\hat{s}-\gamma}$$

as $t \to t_0 \in I_T$;

(b) for any $u(t) \in C^1(I_T, X_{\hat{s}+\gamma}),\ u(T) = u(0) = 0$ it follows that

$$\int_0^T \big(\dot{u}(t), x(t)\big)_{\hat{s}}\, dt = \int_0^T \big(B^{\hat{\sigma}} J u(t), x(t)\big)_{\hat{s}}\, dt.$$

For almost all $t \in I$ this solution enjoys the inequality

$$\|x(t)\|_{\hat{s}} \le c\|\hat{x}\|_{\hat{s}}. \tag{5}$$

The constant c does not depend on T.

2. Assume in addition that embeddings (2) *are compact. Then the solution*

$$x(t) \in C(I_T, X_{\hat{s}-\gamma}) \cap C^1(I_T, X_{\hat{s}-\gamma-\hat{\sigma}}) \tag{6}$$

is unique. This is a classical solution in the space $X_{\hat{s}-\gamma}$.

Indeed, under the conditions of the second part of the theorem system (4) possesses the Hamiltonian

$$H(x) = \frac{1}{2}\big(B^{\hat{\sigma}} x, x\big)_{\hat{s}-\gamma-2\hat{\sigma}} = \frac{1}{2}\big(B^{\hat{\sigma}/2} x, B^{\hat{\sigma}/2} x\big)_{\hat{s}-\gamma-2\hat{\sigma}}, \quad x = x(t) \tag{7}$$

which is the first integral to this system:

$$\dot H=\big(B^{\hat\sigma}x,\dot x\big)_{\hat s-\gamma-2\hat\sigma}=\big(B^{\hat\sigma}x,JB^{\hat\sigma}x\big)_{\hat s-\gamma-2\hat\sigma}=0.$$

Actually system (4) possesses the continuum set of Hamiltonians:

$$H_s(x)=\frac{1}{2}\big(B^{\hat\sigma}x,x\big)_s,\quad s\le\hat s-\gamma-2\hat\sigma.$$

Using formula (7) we see that any solution of the kind (6) satisfies the estimate

$$c_{17}\|x(t)\|^2_{\hat s-\gamma-3\hat\sigma/2}\le H(x(t))=H(\hat x)\le c_{18}\|\hat x\|^2_{\hat s-\gamma-3\hat\sigma/2}. \tag{8}$$

This implies the uniqueness in Theorem 2.1.

Let us formulate a version of the Poincaré Recurrence Theorem.

THEOREM 2.2. *Assume that both parts of Theorem 2.1 are fulfilled. Then there is an increasing sequence $t_k\to\infty$, such that*

$$\|x(t_k)-\hat x\|_{\hat s-\gamma-3\hat\sigma/2}\to 0.$$

Unlike classical Poincaré Recurrence Theorem this assertion deals with a dynamical system on non compact infinite dimensional phase space. Another version of the recurrence theorem for an infinite dimensional system on non compact space has been proved for a hydrodynamic problem in [6].

3. Proofs of the theorems

3.1 Auxiliary lemmas

LEMMA 3.1. *Let μ stand for the standard Lebesgue measure in $\mathbb{R}_+=(0,\infty)$. Let $\tau\subset\mathbb{R}_+$ be a full measure set $\mu(\mathbb{R}_+\backslash\tau)=0$. Then there is a number $a>0$ such that $\{ak\}_{k\in\mathbb{N}}\subset\tau$.*

Proof. Assume the converse: for any $a>0$ there exists $k\in\mathbb{N}$ such that $ak\notin\tau$. This implies that

$$\mathbb{R}_+=\bigcup_{k\in\mathbb{N}}M_k,\quad M_k=\{a>0\mid ak\notin\tau\}=\mathbb{R}_+\backslash(\tau/k).$$

It follows that $\mu(M_k)=0$. This gives a contradiction. □

LEMMA 3.2 (the Banach-Steinhaus Theorem [3], [8]). *Let*

$$A_\nu:X\to Y,\quad \nu\in[\nu_1,\nu_2]$$

be a set of bounded operators of a Banach space X to the Banach space Y. Assume that for each $x\in X$ we have

$$\|A_\nu x-Ax\|_Y\to 0 \tag{9}$$

as $\nu\to\nu_0\in[\nu_1,\nu_2]$. Then the operator $A:X\to Y$ is also bounded and the convergence (9) is uniform on any compact set of variable x.

Moreover, for any continuous function $f:[t_1,t_2]\to X$ it follows that $\|A_\nu f(t)-Af(t_0)\|_Y\to 0$ as $\nu\to\nu_0$ and $t\to t_0\in[t_1,t_2]$.

Proof. Observe only that $F = \{f(t) \mid t \in [t_1, t_2]\}$ is a compact set in X as an image of the compact set $[t_1, t_2]$ under the continuous mapping f. So that by the previous propositions $A_\nu x$ converges to Ax uniformly in $x \in F$. □

LEMMA 3.3. *Let $x \in X_{s-s_0}$, $s_0 > 0$. Then the mapping*

$$t \mapsto e^{-tB^\gamma} x$$

is a continuous mapping of $(0, T]$ to X_s.

Proof. Let $t \to \tilde{t}-$, $\tilde{t} > 0$. We have

$$\big(e^{-tB^\gamma} - e^{-\tilde{t}B^\gamma}\big)x = e^{-tB^\gamma/2}\big(I - e^{-(\tilde{t}-t)B^\gamma}\big)e^{-tB^\gamma/2}x.$$

One yields the estimate

$$\Big\|e^{-tB^\gamma/2}\big(I - e^{-(\tilde{t}-t)B^\gamma}\big)e^{-tB^\gamma/2}x\Big\|_s \leq \frac{c}{t^{s_0/\gamma}}\Big\|\big(I - e^{-(\tilde{t}-t)B^\gamma}\big)e^{-tB^\gamma/2}x\Big\|_{s-s_0}.$$

The mapping

$$t \mapsto e^{-tB^\gamma/2}x$$

is a continuous mapping of the small neighbourhood of the point $\tilde{t}$ to X_{s-s_0}. From Lemma 3.2 it follows that

$$\Big\|\big(I - e^{-(\tilde{t}-t)B^\gamma}\big)e^{-tB^\gamma/2}x\Big\|_{s-s_0} \to 0.$$

If $t \to \tilde{t}+$ then the argument is trivial

$$\big(e^{-tB^\gamma} - e^{-\tilde{t}B^\gamma}\big)x = \big(e^{-(t-\tilde{t})B^\gamma} - I\big)e^{-\tilde{t}B^\gamma}x, \quad e^{-\tilde{t}B^\gamma}x \in X_s.$$

□

LEMMA 3.4. *Assume that a function $u \in C([0, t_*))$ satisfies the inequality*

$$0 \leq u(t) \leq A + B\int_0^t \frac{u(\xi)}{(t-\xi)^\alpha}\, d\xi, \quad t > 0 \tag{10}$$

here A, B, α are positive constants, $\alpha < 1$. Then $\sup_{t\in[0,t_*)} u(t) < \infty$.

Proof. Let us take numbers p, q such that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p < \frac{1}{\alpha}.$$

Then the right-hand side of (10) is not greater than

$$A + B\Big(\int_0^t \frac{d\xi}{(t-\xi)^{\alpha p}}\Big)^{\frac{1}{p}}\Big(\int_0^t u^q\, d\xi\Big)^{\frac{1}{q}}.$$

So that

$$u(t) \leq A + C\Big(\int_0^t u^q\, d\xi\Big)^{\frac{1}{q}} \tag{11}$$

with some positive constant C.

Define a function

$$w(t)=\begin{cases}(u(t)-A)^q & \text{if } u(t)\ge A,\\ 0 & \text{if } u(t)<A.\end{cases}$$

The function w is non-negative and continuous in $[0,t_*)$. Observe also that $u\le A+w^{1/q}$.

Inequality (11) implies

$$w(t)\le C^q\int_0^t \Phi\big(w(\xi)\big)\,d\xi,\quad \Phi(\eta)=\big(A+\eta^{1/q}\big)^q. \tag{12}$$

Recall the following fact.

Proposition 3.5 ([2]). *Let a non negative function $v\in C([0,t_1])$ be such that*

$$v(t)\le K\int_0^t \Phi\big(v(\xi)\big)\,d\xi,\quad K=\mathrm{const}>0.$$

Assume the function $\Phi\in C[0,\infty)$ to be positive and non decreasing.

Assume also that

$$\Psi(\eta)=\int_0^\eta \frac{d\xi}{\Phi(\xi)}\to\infty$$

as $\eta\to\infty$. Then

$$v(t)\le\Psi^{-1}\big(Kt\big),$$

here Ψ^{-1} is the inverse function.

By this Proposition and due to formula (12) the function w is bounded in $[0,t_*)$. Thus u is also bounded. □

3.2 Proof of the first part of Theorem 2.1

We use a version of the classical parabolic regularization method [7].

We will approximate problem (4) with the following sequence of "paraboli" problems

$$\dot x_n=JB^{\hat\sigma}x_n-a_nB^\gamma x_n,\quad x_n(0)=\hat x_n=e^{-a_nB}\hat x, \tag{13}$$

here $a_n=1/n,\quad n\in\mathbb{N}$. Note that $\hat x_n\in X_s,\quad s\in\mathbb{R}$.

Consider corresponding integral equation

$$x_n(t)=e^{-a_ntB^\gamma}\hat x_n+\int_0^t e^{-a_n(t-\xi)B^\gamma}JB^{\hat\sigma}x_n(\xi)\,d\xi \tag{14}$$

and the operator

$$\mathcal{F}_n[x(\cdot)]=e^{-a_ntB^\gamma}\hat x_n+\int_0^t e^{-a_n(t-\xi)B^\gamma}JB^{\hat\sigma}x(\xi)\,d\xi.$$

Lemma 3.6. *The operator $\mathcal{F}_n$ takes the space $C(I_\tau,X_s)$ to itself. Here $\tau>0,\ s\in\mathbb{R}$ are arbitrary constants.*

Proof. First note that if $t \to 0+$ then $e^{-a_n t B^\gamma}\hat{x} \to \hat{x}$ in X_s and

$$\|\mathcal{F}_n[x(\cdot)](t)\|_s \le \int_0^t \left\| e^{-a_n(t-\xi)B^\gamma} JB^{\hat{\sigma}} x(\xi) \right\|_s d\xi$$
$$\le \int_0^t \frac{c_1}{\big(a_n(t-\xi)\big)^{\hat{\sigma}/\gamma}} \|JB^{\hat{\sigma}} x(\xi)\|_{s-\hat{\sigma}}\, d\xi$$
$$\le \int_0^t \frac{c_{14}}{\big(a_n(t-\xi)\big)^{\hat{\sigma}/\gamma}} \|x(\xi)\|_s\, d\xi \to 0.$$

Thus $\mathcal{F}_n[x(\cdot)](t)$ is a continuous function in $t = 0$.

For $0 \le t'' \le t' \le \tau$ one has the identity

$$\mathcal{F}_n[x(\cdot)](t') - \mathcal{F}_n[x(\cdot)](t'') = e^{-a_n t'' B^\gamma}(e^{-a_n(t'-t'')B^\gamma} - I)\hat{x}_n$$
$$+ \int_{t''}^{t'} e^{-a_n(t'-\xi)B^\gamma} JB^{\hat{\sigma}} x(\xi)\, d\xi$$
$$+ \int_0^\tau \chi_{[0,t'']}(\xi)(e^{-a_n(t'-t'')B^\gamma} - I)e^{-a_n(t''-\xi)B^\gamma} JB^{\hat{\sigma}} x(\xi)\, d\xi, \tag{15}$$

here χ is the indicator function.

Assume that $t'' > 0$. The integral in the middle of the right-hand side of formula (15) is not greater than

$$\int_{t''}^{t'} \|e^{-a_n(t'-\xi)B^\gamma} JB^{\hat{\sigma}} x(\xi)\|_s\, d\xi$$
$$\le \int_{t''}^{t'} \frac{c_2}{\big(a_n(t'-\xi)\big)^{\hat{\sigma}/\gamma}} \|JB^{\hat{\sigma}} x(\xi)\|_{s-\hat{\sigma}}\, d\xi$$
$$\le c_3 \int_{t''}^{t'} \frac{1}{\big(a_n(t'-\xi)\big)^{\hat{\sigma}/\gamma}} \|x(\xi)\|_s\, d\xi$$
$$\le c_3 \max_{\xi \in I_\tau} \|x(\xi)\|_s \int_{t''}^{t'} \frac{1}{\big(a_n(t'-\xi)\big)^{\hat{\sigma}/\gamma}}\, d\xi \to 0 \tag{16}$$

as $t' \to t''$ or $t'' \to t'$.

The expression under the last integral in the formula (15) is majorated by L^1-summable function of $\xi \in I_\tau$:

$$\|(e^{-a_n(t'-t'')B^\gamma} - I)e^{-a_n(t''-\xi)B^\gamma} JB^{\hat{\sigma}} x(\xi)\|_s \le \frac{c_5}{\big(a_n(t''-\xi)\big)^{\hat{\sigma}/\gamma}} \|x(\xi)\|_s.$$

Fix t' and fix $\xi < t'$ and let $\xi < a < t'$. Let a constant $a \in (\xi, t')$. By Lemma 3.3 the mapping

$$t'' \mapsto e^{-a_n(t''-\xi)B^\gamma} JB^{\hat{\sigma}} x(\xi)$$

is a continuous mapping of $[a, t']$ to X_s. By Lemma 3.2

$$(A_\nu = e^{-a_n \nu B^\gamma} - I, \quad \nu = t' - t'', \quad X = Y = X_s)$$

it follows that

$$\|(e^{-a_n(t'-t'')B^\gamma}-I)e^{-a_n(t''-\xi)B^\gamma}JB^{\hat\sigma}x(\xi)\|_s\to 0$$

as $t''\to t'$.

When $t'\to t''$, the asymptotics

$$\|(e^{-a_n(t'-t'')B^\gamma}-I)e^{-a_n(t''-\xi)B^\gamma}JB^{\hat\sigma}x(\xi)\|_s\to 0,\quad \xi<t''$$

is obtained directly. Thus by the dominated convergence theorem we have

$$\Big\|\int_0^\tau\chi_{[0,t'']}(\xi)(e^{-a_n(t'-t'')B^\gamma}-I)e^{-a_n(t''-\xi)B^\gamma}JB^{\hat\sigma}x(\xi)\,d\xi\Big\|_s\to 0$$

as $t'\to t''$ or $t''\to t'$.

Analogously one has

$$\|(e^{-a_n(t'-t'')B^\gamma}-I)e^{-a_nt''B^\gamma}\hat x_n\|_s\to 0$$

as $t'\to t''$ or $t''\to t'$. □

Lemma 3.7. *For any $n\in\mathbb{N}$ and for any $s\in\mathbb{R}$ equation* (14) *has a unique solution*

$$x_n(t)\in C(I_T,X_s).$$

Proof. After already prepared estimates it is clear that for sufficiently small $\tau>0$ the mapping $\mathcal{F}_n$ is a contraction of $C(I_\tau,X_s)$ and thus there is a fixed point $\mathcal{F}_n[x_n]=x_n$. This is true for all $s\in\mathbb{R}$ but τ depends on s.

The solution x_n can actually be extended to the whole interval I_T. Assume the converse: there is a positive constant $t_*<T$ such that the solution $x_n(t)\in C([0,t_*),X_s)$ but the limit $\lim_{t\to t_*-}x_n(t)$ does not exist in X_s. If this limit would exist we could take $x(t_*)$ as a new initial condition and use again the contraction mapping principle to prolong the solution forward over t_*.

Observe that

$$\Big\|e^{-tB^\sigma}x\Big\|_s\le c_{s,\sigma}\|x\|_s,\quad t\ge 0,\quad t\in I_T$$

see for example [10]. Thus taking in estimate (16) $t'=t$ and $t''=0$ and replacing x with x_n, from equation (14) we get

$$\|x_n(t)\|_s\le c_4\|\hat x_n\|_s+c_3\int_0^t\frac{\|x_n(\xi)\|_s}{\big(a_n(t-\xi)\big)^{\hat\sigma/\gamma}}\,d\xi.$$

Consequently, by Lemma 3.4 the solution $x_n(t)$ is bounded on $[t,t_*)$.

To get the contradiction it is sufficient to observe that if $t',t''\to t_*-$ then

$$\|x_n(t')-x_n(t'')\|_s\to 0.$$

Since x_n is a fixed point of $\mathcal{F}_n$, the proof repeats the argument of Lemma 3.6. □

Lemma 3.8. *The function $x_n(t)\in C^1(I_T,X_s)$, $s\in\mathbb{R}$ solves initial value problem* (13).

Proof. In this lemma we already know that the function $x(\xi)$ belongs to $C(I_T,X_s)$

for all $s \in \mathbb{R}$. One must check that for $s \in \mathbb{R}$ it follows that

$$\lim_{h\to 0}\Big\|\frac{x_n(t+h)-x_n(t)}{h}+\big(-JB^{\hat\sigma}+a_nB^\gamma\big)\mathcal{F}_n[x_n](t)\Big\|_s=0,\ \text{for } t>0 \text{ and}$$

$$\lim_{h\to 0+}\Big\|\frac{x_n(h)-\hat x_n}{h}-JB^{\hat\sigma}\hat x_n+a_nB^\gamma\hat x_n\Big\|_s=0.$$

This follows from the same argument which is employed in Lemma 3.2. One must use formula (15) taking once $t'=t+h,\quad t''=t>0,\quad h>0$ and after this checking the case $t''=t-h,\quad t'=t>0$.

The operator

$$\frac{1}{h}\Big(e^{-a_nhB^\gamma}-I\Big)$$

appears instead of the operator $e^{-a_n(t'-t'')B^\gamma}-I$. To pass to the corresponding limits one must use again the dominated convergence theorem and Lemma 3.2.

For example, let us show that

$$A_h=\frac{1}{h}\Big\|\int_t^{t+h}e^{-a_n(t+h-\xi)B^\gamma}JB^{\hat\sigma}x(\xi)\,d\xi-JB^{\hat\sigma}x(t)\Big\|_s\to 0$$

as $h\to 0+$. Observe that since $x(t)\in C(I_T,X_{s+\hat\sigma})$ the function

$$\xi\to\Big\|e^{-a_n(t+h-\xi)B^\gamma}JB^{\hat\sigma}x(\xi)d\xi-JB^{\hat\sigma}x(t)\Big\|_s$$

is continuous and therefore by the Mean Value Theorem for integrals we have

$$\begin{aligned}A_h&\le\frac{1}{h}\int_t^{t+h}\Big\|e^{-a_n(t+h-\xi)B^\gamma}JB^{\hat\sigma}x(\xi)-JB^{\hat\sigma}x(t)\Big\|_s\,d\xi\\&=\Big\|e^{-a_n(t+h-\eta)B^\gamma}JB^{\hat\sigma}x(\eta)-JB^{\hat\sigma}x(t)\Big\|_s,\quad \eta\in[t,t+h].\end{aligned}$$

We obtain $x(\eta)\to x(t)$ in $X_{s+\hat\sigma}$ as $\eta\to t$; thus

$$JB^{\hat\sigma}x(\eta)\to JB^{\hat\sigma}x(t),\quad e^{-a_n(t+h-\eta)B^\gamma}JB^{\hat\sigma}x(\eta)\to JB^{\hat\sigma}x(t)$$

in X_s. The last asymptotic goes from Lemma 3.2. □

Lemma 3.9. *The following inequality holds* $\sup_{n\in\mathbb{N}}\max_{t\in I_T}\|x_n(t)\|_{\hat s}\le c_7\|\hat x\|_{\hat s}$.

Proof. Due to equation (13) and since J is a skew symmetric operator we obtain

$$\frac{d}{dt}\big(B^{\hat\sigma}x_n(t),x_n(t)\big)_{\hat s-\hat\sigma/2}=-2a_n\big(B^{\hat\sigma+\gamma}x_n(t),x_n(t)\big)_{\hat s-\hat\sigma/2}\le 0.$$

Hence,

$$\begin{aligned}c_8\|x_n(t)\|_{\hat s}&\le\big(B^{\hat\sigma}x_n(t),x_n(t)\big)_{\hat s-\hat\sigma/2}\le\big(B^{\hat\sigma}x_n(0),x_n(0)\big)_{\hat s-\hat\sigma/2}\\&=\big(B^{\hat\sigma}e^{-a_nB}\hat x,e^{-a_nB}\hat x\big)_{\hat s-\hat\sigma/2}=\big(B^{\hat\sigma/2}e^{-a_nB}\hat x,B^{\hat\sigma/2}e^{-a_nB}\hat x\big)_{\hat s-\hat\sigma/2}\\&\le c_9\|\hat x\|_{\hat s}.\end{aligned}$$

□

Corollary 3.10. *The following inequality holds*

$$\sup_{n\in\mathbb{N}}\max_{t\in I_T}\|\dot x_n(t)\|_{\hat s-\gamma}\le c_{10}\|\hat x\|_{\hat s}.$$

Proof. This directly follows from Lemma 3.9 and equation (13). □

From Lemma 3.9 it follows that the sequence $\{x_n(t)\}$ is *-weakly relatively compact in $L^\infty(I_T, X_{\hat{s}})$. Recall that for any normed space X, the strongly closed ball of X' is a $\sigma(X', X)$-compact set [10]. Note also that $L^\infty(I_T, X_{\hat{s}}) = \big(L^1(I_T, X_{\hat{s}})\big)'$ [4].

Lemma 3.9, its Corollary 3.10 and the third Ascoli theorem [8] imply that the sequence $\{x_n(t)\}$ is relatively compact in $C_w(I_T, X_{\hat{s}-\gamma})$. Note also that each bounded in $X_{\hat{s}-\gamma}$ set is weakly relatively compact.

Furthermore since the closed ball of $X_{\hat{s}-\gamma}$ is weakly compact, it is also weakly complete.

Hence the sequence $\{x_n(t)\}$ contains a subnet $\{x_{n_\alpha}\}$ with some directed set $\mathcal{A} \ni \alpha$ such that

$$\lim_{\mathcal{A}} x_{n_\alpha} = x$$

in $C_w(I_T, X_{\hat{s}-\gamma})$ and in $L^\infty(I_T, X_{\hat{s}})$ equipped with *-weak topology; and $\|x\|_{L^\infty(I_T, X_{\hat{s}})} \le c_7 \|\hat{x}\|_{\hat{s}}$.

By virtue of equation (13) it remains to pass to the limit in

$$-\int_0^T \big((\dot{u}(t), x_{n_\alpha}(t)\big)_{\hat{s}}\, dt = \int_0^T \big((-B^{\hat{\sigma}}J - a_{n_\alpha}B^\gamma)u(t), x_{n_\alpha}(t)\big)_{\hat{s}}\, dt.$$

3.3 Proof of the second part of Theorem 2.1

From Lemma 3.9 and Corollary 3.10 by the Third Ascoli theorem [8] it follows that the sequence $\{x_n(t)\}$ is relatively compact in $C(I_T, X_{\hat{s}-\gamma})$. Here we use the hypothesis on compactness of embedding $X_{\hat{s}} \subset X_{\hat{s}-\gamma}$.

Thus there is a subnet $\{x_{n_\alpha}\} \subset \{x_n\}$ such that in addition to enumerated above properties it is convergent to $x(t)$ in $C(I_T, X_{\hat{s}-\gamma})$ as $\alpha \in \mathcal{A}$.

In the space $C(I_T, X_{\hat{s}-\gamma-\hat{\sigma}})$ we pass to the limit in (14) and obtain that $x(t)$ is a solution to integral equation

$$x(t) = \hat{x} + \int_0^t JB^{\hat{\sigma}}x(\xi)\, d\xi. \qquad (17)$$

Differentiating equation (17) in t in the space $X_{\hat{s}-\gamma-\hat{\sigma}}$ we obtain

$$\dot{x} = JB^{\hat{\sigma}}x, \quad \dot{x} \in C(I_T, X_{\hat{s}-\gamma-\hat{\sigma}}).$$

Theorem 2.1 is proved.

3.4 Proof of Theorem 2.2

We employ Lemma 3.1. Let τ stand for the set of values $t > 0$ such that $x(t) \in X_{\hat{s}}$ and inequality (5) holds. Then put $t'_i = ai$. Observe that $t'_i - t'_j \in \tau$ provided $i > j$.

Let $e^{tJB^{\hat{\sigma}}}\hat{x}$ stand for the solution with initial condition $\hat{x} \in X_{\hat{s}}$. Since $e^{(t'_i - t'_j)JB^{\hat{\sigma}}}\hat{x} \in X_{\hat{s}}$ we can write

$$e^{t'_iJB^{\hat{\sigma}}}\hat{x} - e^{t'_jJB^{\hat{\sigma}}}\hat{x} = e^{t'_jJB^{\hat{\sigma}}}\big((e^{(t'_i - t'_j)JB^{\hat{\sigma}}} - I)\hat{x}\big).$$

The sequence $\{x(t_i')\}$ is bounded in $X_{\hat{s}}$. Thus it contains a subsequence $\{x(\tilde{t}_i)\}$, $\{\tilde{t}_i\} \subset \{t_i'\}$ that is convergent in $X_{\hat{s}-\gamma-3\hat{\sigma}/2}$,

$$\|x(\tilde{t}_i) - x(\tilde{t}_j)\|_{\hat{s}-\gamma-3\hat{\sigma}/2} \to 0$$

as $i, j \to \infty$.

By the same argument as in formula (8) we can write

$$\begin{aligned} c_{17}\big\|e^{(\tilde{t}_i-\tilde{t}_j)JB^{\hat{\sigma}}}\hat{x} - \hat{x}\big\|^2_{\hat{s}-\gamma-3\hat{\sigma}/2} &\le H\Big(\big(e^{(\tilde{t}_i-\tilde{t}_j)JB^{\hat{\sigma}}} - I\big)\hat{x}\Big) \\ = H\Big(e^{\tilde{t}_j JB^{\hat{\sigma}}}\big(e^{(\tilde{t}_i-\tilde{t}_j)JB^{\hat{\sigma}}} - I\big)\hat{x}\Big) &\le c_{18}\|x(\tilde{t}_i) - x(\tilde{t}_j)\|^2_{\hat{s}-\gamma-3\hat{\sigma}/2}. \end{aligned} \tag{18}$$

It remains to choose a sequence $j = j(i)$ such that $t_i = \tilde{t}_i - \tilde{t}_{j(i)} \to \infty$ and $j(i) \to \infty$ as $i \to \infty$. Indeed, from inequality (18) it follows that

$$\big\|e^{t_i JB^{\hat{\sigma}}}\hat{x} - \hat{x}\big\|_{\hat{s}-\gamma-3\hat{\sigma}/2} \to 0.$$

Theorem 2.2 is proved.

Acknowledgement. The author wishes to thank Professor D. V. Treschev for useful discussions.

Partially supported by grants RFBR 12-01-00441., Science Sch.-2964.2014.1

References

[1] J. M. Arrieta, A. N. Carvalho, *Abstract parabolic problems with critical nonlinearities and applications to Navier – Stokes and heat equations*, Trans. Amer. Math. Soc. **352** (1), 285–310.

[2] J. Bihari, *A generalization of lemma of Bellman and its application to uniqueness problems of differential equations*, Acta Math. Acad. Scient. Hung. **7** (1) (1956), 81–94.

[3] N. Bourbaki, *Espaces vectoriels topologiques*, Paris, Hermann, 1981.

[4] R. E. Edwards, *Functional Analysis*, New York, Holt, Rinehart and Winston, 1965.

[5] E. Hille, R. S. Phillips, *Functional Analysis and Semi-groups*, Am. Math. Soc., Providence, Rhode Island, 1957.

[6] V. V. Kozlov, *On the equations of the hydrodynamic type*, J. Appl. Math. Mech. **80** (3) (2016), 209–214.

[7] J. L. Lions, *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod et Gauthier-Villars, Paris, 1969.

[8] L. Schwartz, *Analyse mathématique*, **2**, Hermann, 1967.

[9] M. E. Taylor, *Partial Differential Equations* **2, 3**, Springer, New York, 1996.

[10] K. Yosida, *Functional Analysis*, New York, Springer-Verlag, 1980.

[11] O. Zubelevich, *Peano type theorem for abstract parabolic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire (2009), **26** (4), 1407–1421.

(received 05.12.2016; in revised form 07.01.2017; available online 05.04.2017)

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