

FINITE GROUPS WHOSE COMMUTING GRAPHS ARE INTEGRAL

Jutirekha Dutta and Rajat Kanti Nath

Abstract. A finite non-abelian group G is called commuting integral if the commuting graph of G is integral. In this paper, we show that a finite group is commuting integral if its central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m} , where p is any prime integer and D_{2m} is the dihedral group of order $2m$.

1. Introduction

Let G be a non-abelian group with center $Z(G)$. The commuting graph of G , denoted by Γ_G , is a simple undirected graph whose vertex set is $G \setminus Z(G)$, and two vertices x and y are adjacent if and only if $xy = yx$. In recent years, many mathematicians have considered commuting graphs of different finite groups and studied various graph theoretic aspects (see [4, 7, 11–14]). A finite non-abelian group G is called *commuting integral* if the commuting graph of G is integral. It is natural to ask which finite groups are commuting integral. In this paper, we compute the spectrum of the commuting graphs of finite groups whose central quotients are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, for any prime integer p , or D_{2m} , the dihedral group of order $2m$. Our computation reveals that those groups are commuting integral.

Recall that the spectrum of a graph \mathcal{G} , denoted by $\text{Spec}(\mathcal{G})$, is the multiset $\{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_n^{k_n}\}$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the adjacency matrix of \mathcal{G} with multiplicities k_1, k_2, \dots, k_n respectively. A graph \mathcal{G} is called integral if $\text{Spec}(\mathcal{G})$ contains only integers. It is well known that the complete graph K_n on n vertices is integral and $\text{Spec}(K_n) = \{(-1)^{n-1}, (n-1)^1\}$. Further, if $\mathcal{G} = K_{m_1} \sqcup K_{m_2} \sqcup \dots \sqcup K_{m_l}$, where K_{m_i} 's are complete graphs on m_i vertices for $1 \leq i \leq l$, then

$$\text{Spec}(\mathcal{G}) = \{(-1)^{\sum_{i=1}^l m_i - l}, (m_1 - 1)^1, (m_2 - 1)^1, \dots, (m_l - 1)^1\}.$$

The notion of integral graph was introduced by Harary and Schwenk [9] in the year 1974. Since then many mathematicians have considered integral graphs, see for example [2, 10, 15]. A very impressive survey on integral graphs can be found in [6].

2010 Mathematics Subject Classification: 05C25, 05C50, 20D60

Keywords and phrases: Integral graph; commuting graph; spectrum of a graph.

Ahmadi et al noted that integral graphs are of some interest for designing the network topology of perfect state transfer networks, see [3] and the references therein.

For any element x of a group G , the set $C_G(x) = \{y \in G : xy = yx\}$ is called the centralizer of x in G . Let $|\text{Cent}(G)| = |\{C_G(x) : x \in G\}|$, that is the number of distinct centralizers in G . A group G is called an n -centralizer group if $|\text{Cent}(G)| = n$. In [8], Belcastro and Sherman characterized finite n -centralizer groups for $n = 4, 5$. As a consequence of our results, we show that 4-, 5-centralizer finite groups are commuting integral. Further, we show that a finite $(p+2)$ -centralizer p -group is commuting integral for any prime p .

2. Main results and consequences

We begin this section with the following theorem.

THEOREM 2.1. *Let G be a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime integer. Then*

$$\text{Spec}(\Gamma_G) = \{(-1)^{(p^2-1)|Z(G)|-p-1}, ((p-1)|Z(G)|-1)^{p+1}\}.$$

Proof. Let $|Z(G)| = n$. Then since $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ we have $\frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^p, b^p, aba^{-1}b^{-1} \in Z(G) \rangle$, where $a, b \in G$ with $ab \neq ba$. Then for any $z \in Z(G)$, we have

$$C_G(a) = C_G(a^i z) = Z(G) \sqcup aZ(G) \sqcup \dots \sqcup a^{p-1}Z(G) \text{ for } 1 \leq i \leq p-1,$$

$$C_G(a^j b) = C_G(a^j b z) = Z(G) \sqcup a^j b Z(G) \sqcup \dots \sqcup a^{(p-1)j} b^{p-1} Z(G) \text{ for } 1 \leq j \leq p.$$

These are the only centralizers of non-central elements of G . Also note that these centralizers are abelian subgroups of G . Therefore

$$\Gamma_G = K_{|C_G(a) \setminus Z(G)|} \sqcup \left(\bigsqcup_{j=1}^p K_{|C_G(a^j b) \setminus Z(G)|} \right).$$

Thus $\Gamma_G = K_{(p-1)n} \sqcup (\bigsqcup_{j=1}^p K_{(p-1)n})$, since $|C_G(a)| = pn$ and $|C_G(a^j b)| = pn$ for $1 \leq j \leq p$ where as usual K_m denotes the complete graph with m vertices. That is, $\Gamma_G = \bigsqcup_{j=1}^{p+1} K_{(p-1)n}$. Hence the result follows. \square

The above theorem shows that G is commuting integral if the central quotient of G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for any prime integer p . Some consequences of Theorem 2.1 are given below.

COROLLARY 2.2. *Let G be a non-abelian group of order p^3 , for any prime p . Then*

$$\text{Spec}(\Gamma_G) = \{(-1)^{p^3-2p-1}, (p^2-p-1)^{p+1}\}.$$

Hence, G is commuting integral.

Proof. Note that $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 2.1. \square

COROLLARY 2.3. *If G is a finite 4-centralizer group then G is commuting integral.*

Proof. If G is a finite 4-centralizer group then by Theorem 2 of [8] we have $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, by Theorem 2.1,

$$\text{Spec}(\Gamma_G) = \{(-1)^{3(|Z(G)|-1)}, (|Z(G)| - 1)^3\}.$$

This shows that G is commuting integral. \square

Further, we have the following result.

COROLLARY 2.4. *If G is a finite $(p+2)$ -centralizer p -group, for any prime p , then*

$$\text{Spec}(\Gamma_G) = \{(-1)^{(p^2-1)|Z(G)|-p-1}, ((p-1)|Z(G)| - 1)^{p+1}\}.$$

Hence, G is commuting integral.

Proof. If G is a finite $(p+2)$ -centralizer p -group then by Lemma 2.7 of [5] we have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Now the result follows from Theorem 2.1. \square

The following theorem shows that G is commuting integral if the central quotient of G is isomorphic to the dihedral group $D_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1} \rangle$.

THEOREM 2.5. *Let G be a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, for $m \geq 2$. Then*

$$\text{Spec}(\Gamma_G) = \{(-1)^{(2m-1)|Z(G)|-m-1}, (|Z(G)| - 1)^m, ((m-1)|Z(G)| - 1)^1\}.$$

Proof. Since $\frac{G}{Z(G)} \cong D_{2m}$ we have $\frac{G}{Z(G)} = \langle xZ(G), yZ(G) : x^2, y^m, xyx^{-1}y \in Z(G) \rangle$, where $x, y \in G$ with $xy \neq yx$. It is not difficult to see that for any $z \in Z(G)$,

$$C_G(y) = C_G(y^i z) = Z(G) \sqcup yZ(G) \sqcup \cdots \sqcup y^{m-1}Z(G), 1 \leq i \leq m-1$$

and

$$C_G(xy^j) = C_G(xy^j z) = Z(G) \sqcup xy^j Z(G), 1 \leq j \leq m$$

are the only centralizers of non-central elements of G . Also note that these centralizers are abelian subgroups of G . Therefore

$$\Gamma_G = K_{|C_G(y) \setminus Z(G)|} \sqcup \left(\bigsqcup_{j=1}^m K_{|C_G(xy^j) \setminus Z(G)|} \right).$$

Thus $\Gamma_G = K_{(m-1)n} \sqcup (\bigsqcup_{j=1}^m K_n)$, since $|C_G(y)| = mn$ and $|C_G(xy^j)| = 2n$ for $1 \leq j \leq m$, where $|Z(G)| = n$. Hence the result follows. \square

COROLLARY 2.6. *If G is a finite 5-centralizer group then G is commuting integral.*

Proof. If G is a finite 5-centralizer group then by Theorem 4 of [8] we have $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or D_6 . Now, if $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ then by Theorem 2.1 we have

$$\text{Spec}(\Gamma_G) = \{(-1)^{8|Z(G)|-4}, (2|Z(G)| - 1)^4\}.$$

Again, if $\frac{G}{Z(G)} \cong D_6$ then by Theorem 2.5 we have

$$\text{Spec}(\Gamma_G) = \{(-1)^{5|Z(G)|-4}, (|Z(G)| - 1)^3, (2|Z(G)| - 1)^1\}.$$

In both cases Γ_G is integral. Hence G is commuting integral. \square

We also have the following result.

COROLLARY 2.7. *Let G be a finite non-abelian group and $\{x_1, x_2, \dots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then G is commuting integral if $r = 3, 4$.*

Proof. By Lemma 2.4 of [1], we have that G is a 4-centralizer or a 5-centralizer group according as $r = 3$ or 4. Hence the result follows from Corollaries 2.3 and 2.6. \square

We now compute the spectrum of the commuting graphs of some well-known groups, using Theorem 2.5.

PROPOSITION 2.8. *Let $M_{2mn} = \langle a, b : a^m = b^{2n} = 1, bab^{-1} = a^{-1} \rangle$ be a metacyclic group, where $m > 2$. Then*

$$\text{Spec}(\Gamma_{M_{2mn}}) = \begin{cases} \{(-1)^{2mn-m-n-1}, (n-1)^m, (mn-n-1)^1\}, & \text{if } m \text{ is odd} \\ \{(-1)^{2mn-2n-\frac{m}{2}-1}, (2n-1)^{\frac{m}{2}}, (mn-2n-1)^1\}, & \text{if } m \text{ is even.} \end{cases}$$

Proof. Observe that $Z(M_{2mn}) = \langle b^2 \rangle$ or $\langle b^2 \rangle \cup a^{\frac{m}{2}} \langle b^2 \rangle$ depending whether m is odd or even. Also, it is easy to see that $\frac{M_{2mn}}{Z(M_{2mn})} \cong D_{2m}$ or D_m depending whether m is odd or even. Hence, the result follows from Theorem 2.5. \square

The above Proposition 2.8 also gives the spectrum of the commuting graph of the dihedral group D_{2m} , where $m > 2$, as given below:

$$\text{Spec}(\Gamma_{D_{2m}}) = \begin{cases} \{(-1)^{m-2}, 0^m, (m-2)^1\}, & \text{if } m \text{ is odd} \\ \{(-1)^{\frac{3m}{2}-3}, 1^{\frac{m}{2}}, (m-3)^1\}, & \text{if } m \text{ is even.} \end{cases}$$

PROPOSITION 2.9. *The spectrum of the commuting graph of dicyclic group or the generalized quaternion group $Q_{4m} = \langle a, b : a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1} \rangle$, where $m \geq 2$, is given by*

$$\text{Spec}(\Gamma_{Q_{4m}}) = \{(-1)^{3m-3}, 1^m, (2m-3)^1\}.$$

Proof. The result follows from Theorem 2.5 noting that $Z(Q_{4m}) = \{1, a^m\}$ and $\frac{Q_{4m}}{Z(Q_{4m})} \cong D_{2m}$. \square

PROPOSITION 2.10. *Consider the group $U_{6n} = \langle a, b : a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$. Then $\text{Spec}(\Gamma_{U_{6n}}) = \{(-1)^{5n-4}, (n-1)^3, (2n-1)^1\}$.*

Proof. Note that $Z(U_{6n}) = \langle a^2 \rangle$ and $\frac{U_{6n}}{Z(U_{6n})} \cong D_6$. Hence the result follows from Theorem 2.5. \square

We conclude the paper by noting that the groups M_{2mn}, D_{2m}, Q_{4m} and U_{6n} are commuting integral.

ACKNOWLEDGEMENT. The authors would like to thank the referee for his/her valuable comments and suggestions.

REFERENCES

- [1] A. Abdollahi, S. M. Jafarain and A. M. Hassanabadi, *Groups with specific number of centralizers*, Houston J. Math. **33**, 1 (2007), 43–57.
- [2] A. Abdollahi and E. Vatandoost, *Which Cayley graphs are integral*, Electron. J. Combin. **16**, 1 (2009), no. 1, 17 pp.
- [3] O. Ahmadi, N. Alon, I. F. Blake, I. E. Shparlinski, *Graphs with integral spectrum*, Linear Algebra Appl. **430**, 1 (2009), 547–552.
- [4] S. Akbari, A. Mohammadian, H. Radjavi, P. Raja, *On the diameters of commuting graphs*, Linear Algebra Appl. **418** (2006), 161–176.
- [5] A. R. Ashrafi, *On finite groups with a given number of centralizers*, Algebra Colloq. **7**, 2 (2000), 139–146.
- [6] K. Balińska, D. Cvetković, Z. Radosavljević, S. Simić, D. Stevanović, *A survey on integral graphs*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **13** (2003), 42–65.
- [7] C. Bates, D. Bundy, S. Hart, P. Rowley, *A note on commuting graphs for symmetric groups*, Electronic J. Combin. **16** (2009), 1–13.
- [8] S. M. Belcastro, G. J. Sherman, *Counting centralizers in finite groups*, Math. Magazine, **67**, 5 (1994), 366–374.
- [9] F. Harary, A. J. Schwenk, *Which graphs have integral spectra?*, Graphs and Combinatorics, Lect. Notes Math., Vol 406, Springer-Verlag, Berlin, 1974, 45–51.
- [10] G. Indulal, A. Vijayakumar, *Some new integral graphs*, Appl. Anal. Discrete Math. **1** (2007), 420–426.
- [11] A. Iranmanesh, A. Jafarzadeh, *Characterization of finite groups by their commuting graph*, Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis **23**, 1 (2007), 7–13.
- [12] A. Iranmanesh, A. Jafarzadeh, *On the commuting graph associated with the symmetric and alternating groups*, J. Algebra Appl. **7**, 1 (2008), 129–146.
- [13] G. L. Morgan, C. W. Parker, *The diameter of the commuting graph of a finite group with trivial center*, J. Algebra **393**, 1 (2013), 41–59.
- [14] C. Parker, *The commuting graph of a soluble group*, Bull. London Math. Soc., **45**, 4 (2013), 839–848.
- [15] L. Wang, X. Li, C. Hoede, *Two classes of integral regular graphs*, Ars Combin. **76** (2005), 303–319.

(received 09.12.2016; in revised form 25.01.2017; available online 07.02.2017)

Department of Mathematical Sciences, Tezpur University, Napaam-784028, Sonitpur, Assam India

E-mail: jutirekhadutt@yahoo.com, rajatkantinathyahoo.com