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EXISTENCE OF ONE WEAK SOLUTION FOR $p(x)$-BIHARMONIC EQUATIONS INVOLVING A CONCAVE-CONVEX NONLINEARITY

Rabil Ayazoglu (Mashiyev), Gülizar Alisoy and Ismail Ekincioglu

Abstract. In the present paper, using variational approach and the theory of the variable exponent Lebesgue spaces, the existence of nontrivial weak solutions to a fourth order elliptic equation involving a $p(x)$-biharmonic operator and a concave-convex nonlinearity the Navier boundary conditions is obtained.

1. Introduction and preliminary results

In this paper, we are concerned with the existence of weak solutions for the following nonlinear elliptic Navier boundary value problem involving the $p(x)$-biharmonic operator

$$\begin{cases} \Delta^2_{p(x)} u + a(x)\,|u|^{p(x)-2}\,u = \lambda b(x)\,|u|^{\alpha(x)-2}\,u - \lambda c(x)\,|u|^{\beta(x)-2}\,u \ \text{ in } \Omega, \\ u = \Delta u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 1$, is a bounded domain with smooth boundary, $p \in C(\overline{\Omega})$ with $p(x) > 1$, $x \in \overline{\Omega}$, $a, b, c, \alpha, \beta \in C(\overline{\Omega})$ are nonnegative functions, λ is a positive parameter and $\Delta^2_{p(x)} u = \Delta\big(|\Delta u|^{p(x)-2}\,\Delta u\big)$ is the so-called $p(x)$-biharmonic operator.

The nonlinear differential equations and variational problems involving the $p(x)$-growth conditions appear in a variety of scientific research areas, such as modeling of dynamical phenomena which arise from the study of electrorheological fluids or elastic mechanics, thermorheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium. For the detailed application background see [4, 19, 22, 28–30], and for some recent works on this subject see [7, 9, 23–25, 27]. Moreover, we point out that elliptic equations involving the $p(x)$-biharmonic equations are not trivial generalizations of similar problems studied in the constant case since the $p(x)$-biharmonic

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operator is not homogeneous and, thus, some techniques which can be applied in the case of the $p(x)$-biharmonic operators fail in that new situation, such as the Lagrange Multiplier Theorem.

Recently, in [2], the authors studied the following problem

$$\begin{cases} \Delta^2_{p(x)} u = \lambda \left| u \right|^{\alpha(x)-2} u \ \text{ in } \Omega, \\ u = \Delta u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{2}$$

under the assumption $p(x) = \alpha(x)$. In particular, by the Ljusternik-Schnirelmann principle on C^1-manifolds, the authors proved, among other things, the existence of a sequence of eigenvalues and that $\sup \Lambda = +\infty$, where Λ is the set of all nonnegative eigenvalues. In [3], the authors studied the problem (2) when $p(x) \neq \alpha(x)$. Using the Mountain Pass Lemma and Ekeland's variational principle, the authors further established several existence criteria for eigenvalues. In [14], by applying variational arguments, the author studied the existence of at least one weak solution of the problem (1) in the case of $1 < \beta^- \leq \beta^+ < \alpha^- \leq \alpha^+ < p^-$, for $\lambda > 0$ large enough. In [15], the existence of at least one weak solution was obtained for the problem

$$\begin{cases} \Delta^2_{p(x)} u + a(x) \left| u \right|^{p(x)-2} u = \lambda \omega(x) f(u) \ \text{ in } \Omega, \\ u = \Delta u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

for $\lambda > 0$ sufficiently small, where $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is a bounded domain with smooth boundary, $p \in C(\overline{\Omega})$ with $p(x) > N$ on $\overline{\Omega}$, $a \in C(\overline{\Omega})$ is positive, $f \in C(\mathbb{R})$ satisfy certain conditions and $\omega \in L^{r(x)}(\Omega)$ for some $r \in C(\Omega)$. In recent years many authors have looked for multiple solutions of elliptic equations involving $p(x)$-biharmonic type operators (see, for instance, [1, 11, 12, 14, 15, 17, 18]).

Note that when $p(x) = p$ is a positive constant, several variations of problem (2) have also been investigated in the literature (see, e.g. [5, 10, 13]). Also, in [13], the authors studied the combined effect of concave and convex nonlinearities on the number of nontrivial solutions for the p-biharmonic equation of the form

$$\begin{cases} \Delta^2_{p} u = \lambda \left| u \right|^{q-2} u + \lambda f(x) \left| u \right|^{r-2} u \ \text{ in } \Omega, \\ u = \Delta u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$,

$$1 < r < p < q < p^* = \begin{cases} \frac{Np}{N-2p} & \text{if } p < \frac{N}{2} \\ \infty & \text{if } p \geq \frac{N}{2} \end{cases},$$

$\lambda > 0$ and $f : \overline{\Omega} \to \mathbb{R}$ is a continuous function which changes sign in $\overline{\Omega}$.

In the present paper, considering four different ordering cases of the functions α, β and p, which makes problem (1) involving a concave-convex nonlinearity, we obtain four results for problem (1). Since each case has specific challenges, we do not use a unique straightforward technique. In this context, the presentation of the current paper is unique. We believe that the present paper will make a contribution to the related literature because considering a number of different cases for the functions α, β and p is very important for the representation of the various physical situations described by the model equation (1). Motivated by the ideas introduced in [22–26], the goal of this article is to study the existence of weak solutions of the problem (1) involving a concave-convex nonlinearities.

Now, we proceed with some definitions and basic properties of variable spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. For further reading, we refer to the papers [8, 16, 20] and references therein.

Set $C_+\left(\overline{\Omega}\right) = \left\{h : h \in C\left(\overline{\Omega}\right),\ h(x) > 1,\ x \in \overline{\Omega}\right\}$, and define

$$h^- = \min_{x\in\overline{\Omega}} h(x) \quad \text{and} \quad h^+ = \max_{x\in\overline{\Omega}} h(x)\,,\ \forall h \in C_+\left(\overline{\Omega}\right).$$

For any $p \in C_+\left(\overline{\Omega}\right)$, we define the *variable exponent Lebesgue space* by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\},$$

under the norm

$$|u|_{p(x)} = \inf\left\{\eta > 0 : \int_\Omega \left|\frac{u(x)}{\eta}\right|^{p(x)} dx \le 1\right\},$$

which makes $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)}\right)$ a Banach space.

The *variable exponent Sobolev space* $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\gamma u \in L^{p(x)}(\Omega),\ |\gamma| \le k\},$$

where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_N)$ is a multi-index, $|\gamma| = \sum_{i=1}^N \gamma_i$, and $D^\gamma u = \frac{\partial^{|\gamma|} u}{\partial^{\gamma_1} x_1 \cdots \partial^{\gamma_N} x_N}$. Then, the space $\left(W^{k,p(x)}(\Omega), \|\cdot\|_{k,p(x)}\right)$, equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\gamma|\le k} |D^\gamma u|_{p(x)}\,,$$

is a separable and reflexive Banach space, provided $1 < p^- \le p^+ < \infty$. We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Throughout this paper, we let $X = W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$. Define a norm $\|.\|_X$ of X by

$$\|u\|_X := \|u\|_{1,p(x)} + \|u\|_{2,p(x)}\,.$$

Moreover, it is well known that if $1 < p^- \le p^+ < \infty$, the space $(X, \|\cdot\|_X)$ is a separable and reflexive Banach space, $\|u\|_X$ and $|\Delta u|_{p(x)}$ are two equivalent norms on X (see [8, 16]).

Let

$$\|u\|_a = \inf\left\{\eta > 0 : \int_\Omega \left(\left|\frac{\Delta u(x)}{\eta}\right|^{p(x)} + a(x)\left|\frac{u(x)}{\eta}\right|^{p(x)}\right) dx \le 1\right\}$$

for all $u \in X$. In view of $a^- \ge 0$, it is easy to see that $\|u\|_a$ is equivalent to the norms $\|u\|_X$ and $|\Delta u|_{p(x)}$ in X. In this paper, for the convenience, we will use the norm $\|\cdot\|_a$ on the space X.

For any $x \in \overline{\Omega}$, let

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } p(x) < \frac{N}{2}, \\ \infty & \text{if } p(x) \ge \frac{N}{2}. \end{cases}$$

PROPOSITION 1.1. *[1, 8, 16] Let* $\Lambda_{p(x),a}(u) = \int_\Omega \left(|\Delta u(x)|^{p(x)} + a(x) |u(x)|^{p(x)} \right) dx$ *for any* $u \in X$*. Then, we have*

i) $\|u\|_a \le 1 \Longrightarrow \|u\|_a^{p^+} \le \Lambda_{p(x),a}(u) \le \|u\|_a^{p^-}$*;*

ii) $\|u\|_a \ge 1 \Longrightarrow \|u\|_a^{p^-} \le \Lambda_{p(x),a}(u) \le \|u\|_a^{p^+}$*.*

PROPOSITION 1.2. *[2, 8, 16] Assume that* $q \in C^+(\Omega)$ *satisfy* $q(x) < p^*(x)$ *on* Ω*. Then, there exists a continuous and compact embedding* $X \hookrightarrow L^{q(x)}(\Omega)$*.*

Let us proceed with the settling of the problem (1) in the variational structure. A function $u \in X$ is said to be a weak solution of (1) if

$$\begin{aligned} &\int_\Omega \left(|\Delta u|^{p(x)-2} \Delta u \Delta v + a(x) |u|^{p(x)-2} uv \right) dx \\ &\quad - \lambda \int_\Omega \left(b(x) |u|^{\alpha(x)-2} uv - c(x) |u|^{\beta(x)-2} uv \right) dx = 0, \end{aligned}$$

for all $u \in X$.

The energy functional $I_\lambda : X \to \mathbb{R}$ corresponding to the problem (1) is defined as

$$I_\lambda(u) = \int_\Omega \frac{1}{p(x)} \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx - \lambda \int_\Omega \left(\frac{b(x)}{\alpha(x)} |u|^{\alpha(x)} - \frac{c(x)}{\beta(x)} |u|^{\beta(x)} \right) dx.$$

At this point, let us define the functionals $I_\lambda, \Phi : X \to \mathbb{R}$ by

$$\begin{aligned} \Phi(u) &= \int_\Omega \frac{1}{p(x)} \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx, \\ I_\lambda(u) &= \Phi(u) - \lambda \int_\Omega \left(\frac{b(x)}{\alpha(x)} |u|^{\alpha(x)} - \frac{c(x)}{\beta(x)} |u|^{\beta(x)} \right) dx. \end{aligned}$$

PROPOSITION 1.3. *[1]* Φ *is sequentially weakly lower semicontinuous,* $\Phi \in C^1(X, \mathbb{R})$*, and its Gâteaux derivative* $\Phi'(u)$ *at* $u \in X$ *is given by*

$$\langle \Phi'(u), v \rangle = \int_\Omega \left(|\Delta u|^{p(x)-2} \Delta u \Delta v + a(x) |u|^{p(x)-2} uv \right) dx, \quad \textit{for all } v \in X.$$

Using the previous proposition, the following result can be obtained easily.

PROPOSITION 1.4. *The functional* I_λ *is well-defined,* $I_\lambda \in C^1(X, \mathbb{R})$*, and its Gâteaux derivative* $I'_\lambda(u)$ *at* $u \in X$ *is given by*

$$\begin{aligned} \langle I'_\lambda(u), v \rangle &= \int_\Omega \left(|\Delta u|^{p(x)-2} \Delta u \Delta v + a(x) |u|^{p(x)-2} uv \right) dx \\ &\quad - \lambda \int_\Omega \left(b(x) |u|^{\alpha(x)-2} uv - c(x) |u|^{\beta(x)-2} uv \right) dx, \end{aligned}$$

for all $v \in X$*.*

2. Main results

In this paper, we obtain four different results for the problem (1). For each result, the functions $\alpha, \beta \in C_+\left(\overline{\Omega}\right)$ and $p \in C_+\left(\overline{\Omega}\right)$ have different ordering cases. Therefore, we split up the results of the present paper into the four natural parts. Moreover, in the rest of the paper, we always assume that $a^- \geq 0,\ b^-,\ c^- > 0$.

THEOREM 2.1. *Suppose that* $p(x) < \min\left\{\frac{N}{2}, \frac{Np(x)}{N-2p(x)}\right\}$, *and the following holds:*

$$1 < \alpha^- \leq \alpha^+ < \beta^- \leq \beta^+ < p^- \ \text{ on } \overline{\Omega}. \tag{3}$$

Then for all $\lambda \in (0, \infty)$, *problem* (1) *has at least one nontrivial weak solution.*

In order to prove Theorem 2.1 we first show that for any $a_1, a_2 > 0$ and $0 < k < m$ the following inequality holds:

$$a_1 t^k - a_2 t^m \leq a_1 \left(\frac{a_1}{a_2}\right)^{\frac{k}{m-k}}, \forall t \geq 0. \tag{4}$$

Indeed, since the function $[0, \infty) \ni t \mapsto t^\theta$ is increasing for any $\theta > 0$ it follows that

$$a_1 - a_2 t^{m-k} < 0,\ \forall t > \left(\frac{a_1}{a_2}\right)^{\frac{1}{m-k}},$$

and
$$t^k\left(a_1 - a_2 t^{m-k}\right) \leq a_1 t^k < a_1 \left(\frac{a_1}{a_2}\right)^{\frac{k}{m-k}},\ \forall t \in \left[0, \left(\frac{a_1}{a_2}\right)^{\frac{1}{m-k}}\right].$$

The above inequalities show that (4) holds true.

We now proceed with the following auxiliary results.

LEMMA 2.2. *For any* $\lambda \in (0, \infty)$, *we have*

i) I_λ *is bounded from below and coercive on* X.

ii) I_λ *is sequentially weakly lower semicontinuous on* X.

Proof. $i)$ For any $u \in X$ with $\|u\|_a > 1$,

$$I_\lambda(u) \geq \frac{1}{p^+}\int_\Omega \left(|\Delta u|^{p(x)} + a(x)\,|u|^{p(x)}\right) dx - \lambda \int_\Omega \left(\frac{b^+}{\alpha^-}\,|u|^{\alpha(x)} - \frac{c^-}{\beta^+}\,|u|^{\beta(x)}\right) dx.$$

Applying (4) to the second term of the above inequality, we get

$$\lambda\left(\frac{b^+}{\alpha^-}\,|u|^{\alpha(x)} - \frac{c^-}{\beta^+}\,|u|^{\beta(x)}\right) \leq \frac{\lambda b^+}{\alpha^-}\left(\frac{b^+\beta^+}{\alpha^- c^-}\right)^{\frac{\alpha(x)}{\beta(x)-\alpha(x)}}$$
$$\leq \frac{\lambda b^+}{\alpha^-}\max\left\{\left(\frac{b^+\beta^+}{\alpha^- c^-}\right)^{\frac{\alpha^-}{\beta^+-\alpha^-}}, \left(\frac{b^+\beta^+}{\alpha^- c^-}\right)^{\frac{\alpha^+}{\beta^- -\alpha^+}}\right\} := K,$$

where K is a positive constant independent of u and x. Now we obtain that

$$I_\lambda(u) \geq \frac{1}{p^+}\,\|u\|_a^{p^-} - |\Omega|\,K.$$

Hence, I_λ is bounded from below and coercive, that is, $i)$ is proved.

$ii)$ Let $\{u_n\} \subset X$ be a sequence such that $u_n \rightharpoonup u \in X$. By Proposition 1.3, Φ is sequentially weakly lower semicontinuous. Then,

$$\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n). \tag{5}$$

Moreover, by Proposition 1.2, X is compactly embedded to $L^{\alpha(x)}(\Omega)$ and $L^{\beta(x)}(\Omega)$:

$$u_n \to u \text{ in } L^{\alpha(x)}(\Omega) \text{ and } u_n \to u \text{ in } L^{\beta(x)}(\Omega). \tag{6}$$

Then, from (5) and(6) it reads

$$\begin{aligned} I_\lambda(u) &\le \liminf_{n\to\infty} \Phi(u_n) - \lambda \lim_{n\to\infty} \int_\Omega \left(\frac{b(x)}{\alpha(x)} |u_n|^{\alpha(x)} - \frac{c(x)}{\beta(x)} |u_n|^{\beta(x)} \right) dx \\ &\le \liminf_{n\to\infty} \left(\Phi(u_n) - \lambda \int_\Omega \left(\frac{b(x)}{\alpha(x)} |u_n|^{\alpha(x)} - \frac{c(x)}{\beta(x)} |u_n|^{\beta(x)} \right) dx \right), \end{aligned}$$

that is, $I_\lambda(u) \le \liminf_{n\to\infty} I_\lambda(u_n)$. Thus, I_λ is sequentially weakly lower semicontinuous. □

Lemma 2.3. *For any $\lambda \in (0,\infty)$ it holds $\inf_{u\in X} I_\lambda(u) < 0$.*

Proof. If we consider the condition (3), it reads

$$\liminf_{t\to 0} \frac{\frac{b^-}{\alpha^+} |t|^{\alpha(x)} - \frac{c^+}{\beta^-} |t|^{\beta(x)}}{|t|^{p^-}} = +\infty$$

uniformly in $x \in \Omega$. Then, for any $H > 0$ there exists $\delta > 0$ such that

$$\left| \inf_{x\in\Omega} \left(\frac{b^-}{\alpha^+} |t|^{\alpha(x)} - \frac{c^+}{\beta^-} |t|^{\beta(x)} \right) \right| > H |t|^{p^-} \text{ for every } 0 < |t| \le \delta.$$

Take a nonzero nonnegative function $\vartheta \in C_0^\infty(\Omega)$ with $\inf_{x\in\Omega} \vartheta(x) > 0$, $\lambda \in (0,\infty)$, and put

$$H > \frac{\|\vartheta\|_a^{p^-}}{\lambda \int_\Omega |\vartheta|^{p^-} dx}.$$

Moreover, choose $\varepsilon > 0$ such that $\varepsilon \sup_{x\in\Omega} \vartheta(x) < \delta$, and let $u_0 = \varepsilon\vartheta$. Then, for any $\lambda \in (0,\infty)$ we have

$$\begin{aligned} I_\lambda(\varepsilon\vartheta) \le{}& \frac{1}{p^-} \int_\Omega \left(|\Delta\varepsilon\vartheta|^{p(x)} + a(x) |\varepsilon\vartheta|^{p(x)} \right) dx \\ &- \lambda \left(\frac{b^-}{\alpha^+} \int_\Omega |\varepsilon\vartheta|^{\alpha(x)} dx - \frac{c^+}{\beta^-} \int_\Omega |\varepsilon\vartheta|^{\beta(x)} dx \right) \\ \le{}& \frac{\varepsilon^{p^-}}{p^-} \|\vartheta\|_a^{p^-} - H\varepsilon^{p^-} \int_\Omega |\vartheta|^{p^-} dx < \varepsilon^{p^-} \left(\frac{1}{p^-} - 1 \right) \|\vartheta\|_a^{p^-}. \end{aligned}$$

So, we get $\inf_{u\in X} I_\lambda(u) < 0$, which completes the proof. □

Proof (of Theorem 2.1). From Lemma 2.2, it follows that for any $\lambda \in (0,\infty)$, I_λ has a global minimizer $u \in X$ such that $I'_\lambda(u) = 0$ (see [21]). Then, u is a weak solution of the problem (1). Moreover, since $I_\lambda(0) = 0$ and $I_\lambda(u) < 0$ (Lemma 2.3), $u \ne 0$, i.e. u is a nontrivial solution. □

REMARK 2.4. Due to the results obtained above, we know that for any $\lambda \in (0, \infty)$ the problem (1) has at least one nontrivial solution. Therefore, it is straightforward to show that (1) has both positive and negative solutions. Indeed, set

$$\Psi_\lambda (x, t) := \lambda \left(b(x) |t|^{\alpha(x)-2} u - c(x) |t|^{\beta(x)-2} u \right),$$

and define $\Psi_\lambda^+ : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\Psi_\lambda^+ (x, t) = \begin{cases} \Psi_\lambda (x, t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Then, applying the similar arguments, it can be shown that the following problem

$$\begin{cases} \Delta^2_{p(x)} u + a(x) |u|^{p(x)-2} u = \Psi_\lambda^+ (x, t) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

has a nontrivial solution u, which is a critical point of the corresponding functional I_λ^+. Therefore, $\left\langle I_\lambda^+ (u), u \right\rangle = \int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx - \int_\Omega \Psi_\lambda^+ (x, u) u \, dx = 0$ holds, provided $u \geq 0$. This implies that u is a solution of (1) as well. Then, for any nonempty compact subset $\Omega_1 \subset \Omega$, there exists a positive constant c such that $u(x) \geq c > 0$, i.e. $x \in \Omega_1$ (the strong maximum principle), and hence u is a positive solution of (1). The existence of a negative solution of (1) can be obtained similarly.

THEOREM 2.5. *Suppose that* $\beta(x) < \min \left\{ \frac{N}{2}, \frac{Np(x)}{N-2p(x)} \right\}$, *and the following holds:*

$$1 < \alpha^- \leq \alpha^+ < p^- \leq p^+ < \beta^-, \qquad \text{on } \overline{\Omega}. \tag{7}$$

Then there exists $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$ *the problem* (1) *has at least one nontrivial weak solution.*

Under the condition (7), we cannot show (in a straightforward fashion) that any Palais-Smale (PS) sequence is bounded in X. Thus, we will look for a weak solution of (1) as a local minimizer of the functional I_λ using Ekeland's variational principle (see [6]). We need the following auxiliary results.

LEMMA 2.6. *There exists* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$ *there exist* $\rho, \delta > 0$ *such that* $I_\lambda (u) \geq \delta$ *for any* $u \in X$ *with* $\|u\|_a = \rho$.

Proof. By using the condition (7) and the compact embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, we have

$$|u|_{\alpha(x)} \leq C_3 \|u\|_a, \; C_3 > 0, \tag{8}$$

Let $\|u\|_a = \rho < 1$. Then by (8)

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{p^+} \int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx - \frac{\lambda b^+}{\alpha^-} \int_\Omega |u|^{\alpha(x)} \, dx + \frac{\lambda c^-}{\beta^+} \int_\Omega |u|^{\beta(x)} \, dx \\ &\geq \frac{1}{p^+} \|u\|_a^{p^+} - \frac{\lambda b^+ C_3^{\alpha^-}}{\alpha^-} \|u\|_a^{\alpha^-} \geq \left(\frac{1}{p^+} \|u\|_a^{p^+ - \alpha^-} - \frac{\lambda b^+ C_3^{\alpha^-}}{\alpha^-} \right) \|u\|_a^{\alpha^-} \\ &= \left(\frac{1}{p^+} \rho^{p^+ - \alpha^-} - \frac{\lambda b^+ C_3^{\alpha^-}}{\alpha^-} \right) \rho^{\alpha^-}. \end{aligned} \tag{9}$$

Let $\lambda^* = \frac{\alpha^-}{2b^+ C_3^{\alpha^-} p^+} \rho^{p^+ - \alpha^-}$. Then for any $u \in X$ with $\|u\|_a = \rho$, there exists $\delta = \frac{\rho^{p^+}}{2p^+}$ such that $I_\lambda(u) \geq \delta > 0$. □

LEMMA 2.7. *There exists $\varphi \in X$ such that $\varphi \geq 0$, $\varphi \neq 0$ and $I_\lambda(t\varphi) < 0$ for $t > 0$ small enough.*

Proof. Let $\varphi \in C_0^\infty(\Omega), \varphi \geq 0$, $\varphi \neq 0$ and $t \in (0,1)$. Since $\alpha^+ < p^- < \beta^-$, it reads

$$\begin{aligned} I_\lambda(t\varphi) &\leq \frac{t^{p^-}}{p^-} \int_\Omega \left(|\Delta \varphi|^{p(x)} + a(x) |\varphi|^{p(x)} \right) dx \\ &\quad - \frac{\lambda b^+ t^{\alpha^+}}{\alpha^-} \int_\Omega |\varphi|^{\alpha(x)} \, dx + \frac{\lambda c^- t^{\beta^-}}{\beta^+} \int_\Omega |\varphi|^{\beta(x)} \, dx \\ &\leq t^{p^-} \left(\frac{1}{p^-} \Lambda_{p(x),a}(\varphi) + \frac{\lambda c^-}{\beta^+} \Lambda_{\beta(x)}(\varphi) \right) - t^{\alpha^+} \left(\frac{\lambda b^+}{\alpha^-} \Lambda_{\alpha(x)}(\varphi) \right) < 0, \end{aligned}$$

for $t < \epsilon^{1/(p^- - \alpha^+)}$ with

$$0 < \epsilon < \min \left\{ 1, \frac{\frac{\lambda b^+}{\alpha^-} \Lambda_{\alpha(x)}(\varphi)}{\frac{1}{p^-} \Lambda_{p(x),a}(\varphi) + \frac{\lambda c^-}{\beta^+} \Lambda_{\beta(x)}(\varphi)} \right\},$$

from which we conclude that $I_\lambda(t\varphi) < 0$, where $\Lambda_{r(x)}(\cdot) := \int_\Omega |\cdot|^{r(x)} \, dx$. □

LEMMA 2.8. *Let $(u_n) \subset X$ be a bounded sequence such that $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n) \to 0$ in X^{-1}. Then, (u_n) is relatively compact.*

Thus, we will look for a weak solution of (1) as a local minimizer of the functional I_λ using Ekeland's variational principle. We begin by proving the following auxiliary results.

Proof. By Lemma 2.6 it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0)$, we have $\inf_{\partial B_\rho(0)} I_\lambda > 0$.

On the other hand, by Lemma 2.7 there exits $\varphi \in X$ such that $I_\lambda(t\varphi) < 0$ for all $t > 0$ small enough. Moreover, since relation (9) holds for all $u \in X$, i.e.

$$I_\lambda(u) \geq \frac{1}{p^+} \|u\|_a^{p^+} - \frac{\lambda b^+ C_3^{\alpha^-}}{\alpha^-} \|u\|_a^{\alpha^-},$$

it follows that $-\infty < \overline{c} := \inf_{\overline{B_\rho(0)}} I_\lambda < 0$. So, we have $0 < \varepsilon < \inf_{\partial B_\rho(0)} I_\lambda - \inf_{B_\rho(0)} I_\lambda$. Applying Ekeland's variational principle to the functional $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we can find $u_\varepsilon \in \overline{B_\rho(0)}$ such that $u_\varepsilon \in B_\rho(0)$.

Now, let us define $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $J_\lambda(u) := I_\lambda(u) + \varepsilon \|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of J_λ, and this implies that $\|I'_\lambda(u_\varepsilon)\|_{X^{-1}} \leq \varepsilon$. So, we deduce that there exists a (PS)-sequence $(u_n) \subset B_\rho(0)$ such that

$$I_\lambda(u_n) \to \overline{c} \text{ and } I'_\lambda(u_n) \to 0 \text{ in } X^{-1}. \tag{10}$$

Since the sequence $(u_n) \subset X$ is bounded and X is reflexive, up to a subsequence, we get $u_n \rightharpoonup \overline{u}$ in X. So, by (10) we have $\langle I'_\lambda(u_n), u_n - \overline{u}\rangle \to 0$. Therefore, we have

$$\langle I'_\lambda(u_n), u_n - \overline{u}\rangle = \int_\Omega \Big(|\Delta u_n|^{p(x)-2} \Delta u_n \Delta(u_n - \overline{u}) + a(x)|u_n|^{p(x)-2} u_n (u_n - \overline{u})\Big)\, dx$$
$$-\lambda \int_\Omega \Big(b(x)|u_n|^{\alpha(x)-2} u_n (u_n - \overline{u}) - c(x)|u_n|^{\beta(x)-2} u_n (u_n - \overline{u})\Big)\, dx \to 0.$$

Since $u_n \rightharpoonup \overline{u}$ in X, by compact embedding, we have $u_n \to \overline{u}$ in $L^{\alpha(x)}(\Omega)$ and $u_n \to \overline{u}$ in $L^{\beta(x)}(\Omega)$. Therefore,

$$\int_\Omega \Big(b(x)|u_n|^{\alpha(x)-2} u_n (u_n - \overline{u}) - c(x)|u_n|^{\beta(x)-2} u_n (u_n - \overline{u})\Big)\, dx \to 0.$$

So, we conclude that

$$\langle \Phi'(u_n), u_n - \overline{u}\rangle = \int_\Omega \Big(|\Delta u_n|^{p(x)-2} \Delta u_n \Delta(u_n - \overline{u}) + a(x)|u_n|^{p(x)-2} u_n (u_n - \overline{u})\Big)\, dx \to 0.$$

Since the functional Φ is of (S_+) type (see [1, Proposition 2.5]), we obtain that $u_n \to u$ in X. The proof is completed. □

Proof (of Theorem 2.5). Since $I_\lambda \in C^1(X, \mathbb{R})$, by the relation (10) it follows that $I_\lambda(\overline{u}) = \overline{c}$ and $I'_\lambda(\overline{u}) = 0$. Thus, $\overline{u} \in X$ is a nontrivial weak solution for (1). □

THEOREM 2.9. *Suppose that* $\alpha(x) < \min\left\{\frac{N}{2}, \frac{Np(x)}{N-2p(x)}\right\}$ *and the following holds:*

$$1 < \beta^- \leq \beta^+ < p^- \leq p^+ < q < \alpha^- \text{ on } \overline{\Omega}. \tag{11}$$

Then for any $\lambda \in (0, \infty)$ *the problem* (1) *has at least one nontrivial weak solution.*

We will apply Mountain Pass Theorem (see, e.g. [21, 31]). To this end, we need the following lemma.

LEMMA 2.10. *i) There exist* $\gamma > 0$, $\delta > 0$ *such that* $I_\lambda(u) \geq \delta$ *for any* $u \in X$ *with* $\|u\|_a = \gamma$.

ii) There exists $u \in X$ *such that* $\|u\|_a > \gamma$, $I_\lambda(u) < 0$.

Proof. i) By using the condition (11) and the compact embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, we have $|u|_{\alpha(x)} \leq C_4 \|u\|_a$, $C_4 > 0$.

Let $\|u\|_a = \gamma < 1$. Then we have

$$I_\lambda(u) \geq \frac{1}{p^+}\int_\Omega \Big(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\Big)\, dx - \frac{\lambda b^+}{\alpha^-}\int_\Omega |u|^{\alpha(x)}\, dx + \frac{\lambda c^-}{\beta^+}\int_\Omega |u|^{\beta(x)}\, dx$$
$$\geq \frac{1}{p^+}\|u\|_a^{p^+} - \frac{\lambda b^+ C_4^{\alpha^-}}{\alpha^-}\|u\|_a^{\alpha^-}.$$

Then for any $u \in X$ with $\|u\|_a = \gamma < 1$ small enough, there exists $\delta > 0$ such that $I_\lambda(u) \geq \delta > 0$, for every $\lambda \in (0, \infty)$.

ii) Let $u \in X$ with $\|u\|_a = \gamma > 1$, and $t > 1$. Then

$$\begin{aligned} I_\lambda(tu) &\leq \frac{1}{p^-}\int_\Omega \left(|\Delta tu|^{p(x)} + a(x)\,|tu|^{p(x)}\right)\,dx \\ &\quad - \lambda\left(\frac{b^+}{\alpha^-}\int_\Omega |tu|^{\alpha(x)}\,dx - \frac{c^-}{\beta^+}\int_\Omega |tu|^{\beta(x)}\,dx\right) \\ &\leq \frac{t^{p^+}}{p^-}\int_\Omega \left(|\Delta u|^{p(x)} + a(x)\,|u|^{p(x)}\right)\,dx \\ &\quad - t^{\alpha^-}\frac{\lambda b^+}{\alpha^-}\int_\Omega |u|^{\alpha(x)}\,dx + t^{\beta^-}\frac{\lambda c^-}{\beta^+}\int_\Omega |u|^{\beta(x)}\,dx. \end{aligned}$$

So, we conclude that $I_\lambda\,(tu) \to -\infty$ as $t \to +\infty$. □

Finally, we will show that under the condition (11), Lemma 2.8 holds for functional I_λ as well for all $\lambda \in (0, \infty)$. To this end, using Lemma 2.10 and the Mountain Pass Theorem, we deduce that there exists a (PS)-sequence, defined as in (10), $\{u_n\} \subset X$ for I_λ. We prove that $\{u_n\}$ is bounded in X. Assume the contrary. Then, passing to a subsequence, still denoted by $\{u_n\}$, we may assume that $\|u_n\|_a \to \infty$ as $n \to \infty$. Thus, we may consider that $\|u_n\|_a > 1$, for any integer n. Moreover, by condition (C), for any real number t we have

$$\begin{aligned} \Theta\,(x,t) &\geq b(x)\left(\frac{1}{q} - \frac{1}{\alpha\,(x)}\right)|t|^{\alpha(x)} + c\,(x)\left(\frac{1}{\beta\,(x)} - \frac{1}{q}\right)|t|^{\beta(x)} \\ &\geq b^-\left(\frac{1}{q} - \frac{1}{\alpha^-}\right)|t|^{\alpha(x)} + c^-\left(\frac{1}{\beta^+} - \frac{1}{q}\right)|t|^{\beta(x)} \geq M > 0, \end{aligned} \tag{12}$$

where $\Theta\,(x,t) := \frac{1}{q}\left(b(x)\,|t|^{\alpha(x)} - c(x)\,|t|^{\beta(x)}\right) - \left(\frac{b(x)}{\alpha(x)}\,|t|^{\alpha(x)} - \frac{c(x)}{\beta(x)}\,|t|^{\beta(x)}\right)$.

Then, using (10) and (12) for n large enough, we have

$$\begin{aligned} C &\geq I_\lambda\,(u_n) - \frac{1}{q}\,|\langle I_\lambda'(u_n), u_n\rangle| \\ &\geq \int_\Omega \frac{1}{p(x)}\left(|\Delta u_n|^{p(x)} + a(x)\,|u_n|^{p(x)}\right)\,dx \\ &\quad - \lambda\int_\Omega\left(\frac{b(x)}{\alpha\,(x)}\,|u_n|^{\alpha(x)} - \frac{c(x)}{\beta\,(x)}\,|u_n|^{\beta(x)}\right)\,dx \\ &\quad - \frac{1}{q}\left[\int_\Omega \left(|\Delta u_n|^{p(x)} + a(x)\,|u_n|^{p(x)}\right)\,dx - \lambda\int_\Omega \left(b(x)\,|u_n|^{\alpha(x)} - c(x)\,|u_n|^{\beta(x)}\right)\,dx\right] \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q}\right)\|u_n\|_a^{p^-} + \lambda\int_\Omega \Theta\,(x,u_n)\,dx \geq \left(\frac{1}{p^+} - \frac{1}{q}\right)\|u_n\|_a^{p^-} + \lambda M\,|\Omega|\,. \end{aligned}$$

Since $p^- > 1$, we get a contradiction. So, $\|u_n\|_a$ must be bounded. The rest of the proof is similar to the proof of Lemma 2.8, so we omit it. Therefore we obtain that $u_n \to u$ in X.

Proof (of Theorem 2.9). From Lemmas 2.8 and 2.10, and the fact that $I_\lambda\,(0) = 0$, I_λ satisfies the Mountain Pass Theorem. So I_λ has a nontrivial critical point, i.e. (1) has at least one nontrivial weak solution. □

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Faculty of Education, Bayburt University, Turkey

E-mail: rayazoglu@bayburt.edu.tr; rabilmashiyev@gmail.com

Faculty of Science and Arts, Namik Kemal University, Turkey

E-mail: galisoy@nku.edu.tr

Faculty of Sciences and Arts, Dumlupinar University, Turkey

E-mail: ismail.ekincioglu@dpu.edu.tr