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LOCAL CONVERGENCE OF BILINEAR OPERATOR FREE METHODS UNDER WEAK CONDITIONS

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Abstract. We study third-order Newton-type methods free of bilinear operators for solving nonlinear equations in Banach spaces. Our convergence conditions are weaker than the conditions used in earlier studies. Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

1. Introduction

Let $\mathcal{B}_1, \mathcal{B}_2$ denote Banach spaces and D be a nonempty, open and convex subset of $\mathcal{B}_1$. Due to a plethora of applications, finding solutions x^* of the nonlinear equation

$$F(x) = 0, \tag{1}$$

where $F : D \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is an important problem in applied mathematics [2, 3, 6, 11, 12, 15, 20, 22]. Newton-type methods are commonly used for approximating a solution x^* of (1). Order of convergence is an important issue in the study of iterative methods. In general, convergence analysis of higher order iterative methods requires assumptions on higher order Fréchet-derivatives of the operator F. This restricts the applicability of these methods.

In this paper we study three higher order methods [1, 3, 6, 8] defined for each $n = 0, 1, 2, \dots$ by:

$$\begin{aligned} y_n &= x_n + F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - F'(x_n)^{-1}F(y_n), \end{aligned} \tag{2}$$

$$\begin{aligned} y_n &= x_n + F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= x_n - F'(x_n)^{-1}[x_n, y_n; F]F'(x_n)^{-1}F(y_n), \end{aligned} \tag{3}$$

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and

$$\begin{aligned} y_n &= x_n + A_n^{-1}F(x_n) \\ x_{n+1} &= y_n - A_n^{-1}F(y_n), \end{aligned} \tag{4}$$

where $[.,.;F]$ denotes the divided difference of order one, $A_n = [x_n - \alpha_n F(x_n), x_n + \alpha_n F(x_n); F]$ and $\{\alpha_n\}$ is suitable sequence of operators in $L(\mathcal{B}_2, \mathcal{B}_1)$. Here, $L(\mathcal{B}_2, \mathcal{B}_1)$ denotes the set of bounded linear operators between $\mathcal{B}_1$ and $\mathcal{B}_2$.

In our convergence analysis we use assumptions only on the first Fréchet derivative of the operator F. This way the methods (2), (3) and (4) can be applied to solve equations but the earlier results cannot be applied [1–22] (see Example 3.2).

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis of methods (2), (3) and (4). We also provide a radius of convergence, computable error bounds and a uniqueness result. Special cases and numerical examples are given in the last section.

2. Local convergence

We present the local convergence of method (2), method (3) and method (4), respectively in this section using some scalar functions and parameters. Define parameter ρ_0 by

$$\rho_0 = \sup\{t \geq 0 : w_0(t) < 1\}, \tag{5}$$

where $w_0 : [0, +\infty) \longrightarrow \mathbb{R}$ is a continuous and non-decreasing function with $w_0(0) = 0$. Let $w, v : [0, \rho_0) \to \mathbb{R}$ be continuous and non-decreasing functions with $w(0) = 0$. Define function g_1 on the interval $[0, \rho_0)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)\,d\theta + 2\int_0^1 v(\theta t)\,d\theta}{1 - w_0(t)}.$$

Define parameter $\bar{\rho}_0$ by

$$\bar{\rho}_0 = \max\{t \in [0, \rho_0] : w_0(g_1(t)t) < 1\}. \tag{6}$$

Moreover, define functions g_2, and h_2 on the interval $[0, \rho_0)$ by

$$\begin{aligned} g_2(t) = \Bigg[& \frac{\int_0^1 w((1-\theta)g_1(t)t)\,d\theta}{1 - w_0(g_1(t)t)} \\ & + \frac{(w_0(t) + w_0(g_1(t)t))\int_0^1 v(\theta g_1(t)t)\,d\theta}{(1 - w_0(g_1(t)t))(1 - w_0(t))} \Bigg] g_1(t) \end{aligned}$$

and $h_2(t) = g_2(t) - 1$. We get that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to \bar{\rho}_0^-$. It then follows from the intermediate value theorem that function h_2 has zeros in the interval $(0, \bar{\rho}_0)$. Denote by ρ_2 the smallest such zero. Then, we have that for each $t \in [0, \rho_2)$, $0 \leq g_2(t) < 1$. Let

$$\rho_1 = g_1(\rho_2)\rho_2 \tag{7}$$

and

$$R^* = \max\{\rho_1, \rho_2\}. \tag{8}$$

Let $B(z,\rho), \bar{B}(z,\rho)$ stand, respectively for the open and closed balls in $\mathcal{B}_1$ with center $z \in \mathcal{B}_1$ and of radius $\rho > 0$. We present the local convergence of method (2).

THEOREM 2.1. *Suppose:*

(i) $F : D \subset \mathcal{B}_1 \to \mathcal{B}_2$ *is a continuously Fréchet differentiable operator and there exists* $x^* \in D$ *such that*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1) \tag{9}$$

(ii) There exists a function $w_0 : [0, +\infty) \to \mathbb{R}$ *continuous and nondecreasing with* $w_0(0) = 0$, *such that for each* $x \in D$,

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le w_0(\|x - x^*\|). \tag{10}$$

(iii) There exist functions $w : [0, \bar{\rho}_0) \to \mathbb{R}$ *with* $w(0) = 0$ *and* $v : [0, \bar{\rho}_0) \to \mathbb{R}$ *such that for each* $x, y \in D_0 = D \cap B(x^*, \bar{\rho}_0)$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le w(\|x - y\|) \tag{11}$$

$$\|F'(x^*)^{-1}F'(x)\| \le v(\|x - x^*\|), \tag{12}$$

and

(iv) $\overline{B}(x^*, R^*) \subseteq D$,

where radii $\bar{\rho}_0, R^*$ *are given by* (6) *and* (8), *respectively. Then, the sequence* $\{x_n\}$ *generated for* $x_0 \in U(x^*, \rho_2) - \{x^*\}$ *by method* (2) *is well defined, remains in* $U(x^*, R^*)$ *and converges to* x^*. *Moreover, the following estimates hold*

$$\|y_n - x^*\| \le g_1(\rho_2)\, \rho_2 \le \rho_1 \tag{13}$$

and

$$\|x_{n+1} - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < \rho_2. \tag{14}$$

Furthermore, if there exists for $\rho^* \ge \rho_2$ *such that*

$$\int_0^1 w_0(\theta \rho^*)\, d\theta < 1, \tag{15}$$

then the limit point x^* *is the only solution of equation* $F(x) = 0$ *in* $D_1 := D \cap \overline{B}(x^*, \rho^*)$.

Proof. We shall use induction on k. Let $x \in U(x^*, \rho_2)$. Using (5) and (10), we get that

$$\begin{aligned} \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| &\le w_0(\|x - x^*\|) \\ &\le w_0(\rho_2) \le w_0(\rho_0) < 1. \end{aligned} \tag{16}$$

In view of (16) and the Banach lemma on invertible operators [3, 6, 12], we get that $F'(x)$ is invertible and

$$\|F'(x)^{-1}F'(x^*)\| \le \frac{1}{1 - w_0(\|x - x^*\|)}. \tag{17}$$

We have, in particular that y_0 and x_1 are well defined by method (2) for $n = 0$. By (9) we can write

$$F(x) = F(x) - F(x^*) = \int_0^1 F'(x^* + \theta(x - x^*))\,d\theta. \tag{18}$$

Note that $\|x^* + \theta(x - x^*) - x^*\| = \theta\|x - x^*\|$, so $x^* + \theta(x - x^*) \in B(x^*, \rho_2)$ for each $\theta \in [0, 1]$. Using (12) and (18), we get that

$$\|F'(x^*)^{-1}F(x)\| \le \int_0^1 v(\theta\|x - x^*\|)\,d\theta\|x_0 - x^*\|. \tag{19}$$

The first substep of method (2) can be written as

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0) + 2F'(x_0)^{-1}F(x_0).$$

By (7), (8), (11), (17) and (19) we have in turn that

$$\begin{aligned}
\|y_0 - x^*\| &\le \|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \times \|\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)\,d\theta\| \\
&\quad + \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\
&\le \frac{[\int_0^1 w((1-\theta)\|x_0 - x^*\|)\,d\theta + 2\int_0^1 v(\theta\|x_0 - x^*\|)\,d\theta]\|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\
&\le g_1(\rho_2)\rho_2 = \rho_1,
\end{aligned} \tag{20}$$

which shows (13) for $n = 0$, and $y_0 \in U(x^*, R^*)$. Then, (17) and (19) hold with y_0 replacing x_0 and x, respectively. We can write by the second substep of method (2) that

$$\begin{aligned}
x_1 - x^* &= y_0 - x^* - F'(y_0)^{-1}F(y_0) \\
&\qquad + F'(y_0)^{-1}(F'(x_0) - F'(y_0))F'(x_0)^{-1}F(y_0).
\end{aligned}$$

Then, as in (20) we obtain in turn that

$$\begin{aligned}
\|x_1 - x_0\| &\le \frac{\int_0^1 w((1-\theta)\|y_0 - x^*\|)\,d\theta\|y_0 - x^*\|}{1 - w_0(\|y_0 - x^*\|)} \\
&\quad + \|F'(y_0)^{-1}F'(x^*)\|\left(\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\|\right. \\
&\quad \left.+\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\|\right) \\
&\quad \times \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(y_0)\| \\
&\le \left[\frac{\int_0^1 w((1-\theta)\|y_0 - x^*\|)\,d\theta}{1 - w_0(\|y_0 - x^*\|)}\right. \\
&\quad \left. + \frac{(w_0(\|x_0 - x^*\|) + w_0(\|y_0 - x^*\|))\int_0^1 v(\theta\|y_0 - x^*\|)\,d\theta}{(1 - w_0(\|y_0 - x^*\|)(1 - w_0(\|x_0 - x^*\|))}\right] \\
&\quad \times \|y_0 - x^*\| \\
&\le g_2(\|x_0 x -^* \|)\|x_0 - x^*\| \le \|x_0 - x^*\| < \rho_2,
\end{aligned} \tag{21}$$

which shows (14) for $n = 0$ and $y_0 \in U(x^*, \rho_2)$. The induction for (13) and (14) is completed, if we replace x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the preceding estimates. Moreover, from the estimate

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < \rho_2,$$

where $c = g_2(\|x_0 - x*\|) \in [0, 1)$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, \rho_2)$. Let $y^* \in D_1$ with $F(y^*) = 0$. Define Q by $Q := \int_0^1 F'(x^* + \theta(y^* - x^*))\, d\theta$. Using (10) and (15), we get that

$$\|F'(x^*)^{-1}(Q - F'(x^*))\| \leq \int_0^1 w_0(\theta\|y^* - x^*\|)\, d\theta \leq \int_0^1 w_0(\theta\rho^*) < 1.$$

Hence, Q is invertible. Then, from the identity $0 = F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$. □

REMARK 2.2. (a) In the case when $w_0(t) = L_0 t, w(t) = Lt$ and $D_0 = D$, the radius $r_A = \frac{2}{2L_0+L}$ was obtained by Argyros in [3,5,6] as the convergence radius for Newton's method under condition (9)-(11). Notice that the convergence radius for Newton's method given independently by Rheinboldt [19] and Traub [22] is given by

$$\rho_{TR} = \frac{2}{3L} < r_A.$$

As an example, let us consider the function $H(x) = e^x - 1$. Then $x^* = 0$. Set $D = B(0, 1)$. Then, we have that $L_0 = e - 1 < L = e^{\frac{1}{L_0}}$, so $\rho_{TR} = 0.24252961 < r_A = 0.382691912232$. Moreover, the new error bounds [3, 5, 6] are:

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones [12, 19, 22]

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, we do not expect the radius of convergence of method (2) given by ρ_2 to be larger than r_A.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [3–6].

(c) The results can be also be used to solve equations where the operator F' satisfies the autonomous differential equation [3–6]:

$$F'(x) = P(F(x)),$$

where $P : \mathcal{B}_2 \longrightarrow \mathcal{B}_2$ is a known continuous operator and $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let us present an example, when $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (2) or method (3) or method (4) are not changing, if we use the new instead of the old conditions [1]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC) [16]

$$\xi = \frac{\ln \frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{\ln \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \quad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{\ln \frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \quad \text{for each } \ n = 0, 1, 2, \ldots$$

(e) In view of (10) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + w_0(\|x - x^*\|),\end{aligned}$$

condition (12) can be dropped and can be replaced by

$$v(t) = 1 + w_0(t) \quad \text{or} \quad v(t) = 1 + w_0(r_0), \ \text{since } t \in [0, r_0).$$

Next, we present the local convergence of method (3) in an analogous way. But first, we define functions g_2 and h_2 on the interval $[0, \rho_0)$ by

$$\begin{aligned} g_2(t) &= \frac{\int_0^1 w((1-\theta)t)\,d\theta}{1 - w_0(t)} + \frac{(w_0(t) + u(t, g_1(t)t))\int_0^1 v(\theta t)\,d\theta}{(1 - w_0(t))^2}, \\ h_2(t) &= g_2(t) - 1, \end{aligned}$$

where $u : [0, \rho_0)^2 \longrightarrow \mathbb{R}$ is a continuous and nondecreasing function with $u(0, 0) = 0$ and g_1 is as in Theorem 2.1. We have that $h_2(0) = -1 < 0$ and $h_2(t) \longrightarrow +\infty$ as $t \longrightarrow \rho_0^-$. Denote by ρ_2 the smallest zero of function h_2 on the interval $(0, \rho_0)$. Using the second substep of method (3) and the estimate

$$\begin{aligned}\|x_{k+1} - x^*\| &\leq \|x_k - x^* - F'(x_k)^{-1}F(x_k)\| \\ &\quad + \|F'(x_k)^{-1}F'(x_k)\| \big[\|F'(x^*)^{-1}(F'(x_k) - F'(x^*))\| \\ &\quad + \|F'(x^*)^{-1}([x_k, y_k; F] - F'(x^*))\| \\ &\quad + \|F'(x_k)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_k)\|\big] \\ &\leq \Bigg[\frac{\int_0^1 w((1-\theta)\|x_k - x^*\|)\,d\theta}{1 - w_0(\|x_k - x^*\|)} \\ &\quad \frac{(w_0(\|x_k - x^*\|) + u(\|x_k - x^*\|, \|y_k - x^*\|)\int_0^1 v(\theta\|x_k - x^*\|)\,d\theta}{(1 - w_0(\|x_k - x^*\|))^2}\Bigg] \\ &\quad \times \|x_k - x^*\| \\ &\leq g_2(\|x_k - x^*\|)\|x_k - x^*\| \leq \|x_k - x^*\| < \rho_2,\end{aligned}$$

instead of (21) as well as the rest of the proof of Theorem 2.1, we arrive at the following local convergence result for method (3):

THEOREM 2.3. *Suppose that:*

(i) $F, x^*, \rho_0, w_0, w, v, R^*, D_0$ *are as in Theorem 2.1 (with* $\bar{\rho}_0 = \rho_0$*).*

(ii) There exists function $u : [0, \rho_0)^2 \to \mathbb{R}$ *continuous and nondecreasing with* $u(0,0) = 0$*, such that for each* $x, y \in D_0$*,*

$$\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \leq u(\|x - x^*\|, \|y - x^*\|). \tag{22}$$

Then, the conclusions of Theorem 2.1 hold for the sequence $\{x_n\}$ *generated by method* (3).

Let $\{\alpha_n\} \in L(\mathcal{B}_2, \mathcal{B}_1)$ be a given sequence of linear operators. Suppose that there exist $\beta \geq 0$ and $\gamma \geq 0$ such that for each $x \in D_0$

$$\|I - \alpha_n[x, x^*; F]\| \leq \beta \tag{23}$$

and

$$\|I + \alpha_n[x, x^*; F]\| \leq \gamma \tag{24}$$

Define parameter $\bar{\rho}_0$ by

$$\bar{\rho}_0 = \max\{t \in [0, \rho_0] : p(t) < 1\},$$

where $p(t) = u(\beta t, \gamma t)$. Moreover, define function g_1 on the interval $[0, \bar{\rho}_0)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)\, d\theta}{1 - w_0(t)} + \int_0^1 v(\theta t)\, d\theta \left(\frac{1}{1 - w_0(t)} + \frac{1}{1 - p(t)} \right).$$

Furthermore, define parameters $\bar{\bar{\rho}}_0$ by

$$\bar{\bar{\rho}}_0 = \max\{t \in [0, \rho_0] : w_0(g_1(t)t) < 1\}.$$

Set $q = \min\{\bar{\rho}_0, \bar{\bar{\rho}}_0\}$. Finally, define functions g_2 and h_2 on the interval $[0, q)$ by

$$g_2(t) = \left[\frac{\int_0^1 w((1-\theta)g_1(t)t)\, d\theta}{1 - w_0(t)} + \frac{(p(t) + w_0(g_1(t)t)) \int_0^1 v(\theta g_1(t)t)\, d\theta}{(1 - w_0(g_1(t)t))(1 - p(t))} \right] g_1(t)$$

and $h_2(t) = g_2(t) - 1$.

We have that $h_2(0) = -1 < 0$ and $h_2(t) \longrightarrow +\infty$ as $t \longrightarrow q^-$. Denote by ρ_2 the smallest zero of function h_2 on the interval $(0, q)$. Then, we have that for each $t \in [0, \rho_2)$

$$\begin{aligned} & 0 \leq g_2(t) < 1, \\ & 0 \leq w_0(t) < 1, \\ & 0 \leq w_0(g_1(t)t) < 1 \\ \text{and} \quad & 0 \leq p(t) < 1. \end{aligned}$$

Set $\rho_1 = g_1(\rho_2)\rho_2$ and

$$R^* = \max\{\rho_1, \rho_2, \beta\rho_2, \gamma\rho_2\}. \tag{25}$$

As before, we have the estimates

$$\|F'(x^*)^{-1}(A_k - F'(x^*))\| \leq u(\|x_k - x^* - \alpha_k F(x_k)\|, \|x_k - x^* + \alpha_k F(x_k)\|)$$

but

$$\|x_k - x^* - \alpha_k F(x_k)\| = \|(I - \alpha_k[x_k, x^*; F])(x_k - x^*)\| \leq \beta \|x_k - x^*\|$$

and

$$\|x_k - x^* + \alpha_k F(x_k)\| = \|(I + \alpha_k [x_k, x^*; F])(x_k - x^*)\| \le \gamma \|x_k - x^*\|,$$

so

$$\begin{aligned}\|F'(x^*)^{-1}(A_k - F'(x^*))\| &\le u(\beta\|x_k - x^*\|, \gamma\|x_k - x^*\|)\\ &= p(\|x_k - x^*\|) \le p(\rho_2) < 1.\end{aligned}$$

Hence, A_k is invertible and

$$\|A_k^{-1}F'(x_k)\| \le \frac{1}{1 - p(\|x_k - x^*\|)}.$$

We also have the estimate from the first substep of method (4):

$$\begin{aligned}\|y_k - x^*\| &\le \|x_k - x^* - F'(x_k)^{-1}F(x_k)\|\\ &\quad + (\|F'(x_k)^{-1}F'(x^*)\| + \|A_k^{-1}F'(x^*)\|)\|F'(x^*)^{-1}F(x_k)\|\\ &\le \frac{\int_0^1 w((1-\theta)\|x_k - x^*\|)\,d\theta\|x_k - x^*\|}{1 - w_0(\|x_k - x^*\|)}\\ &\quad + \int_0^1 v(\theta\|x_k - x^*\|)\,d\theta\|x_k - x^*\|\\ &\quad \times \left(\frac{1}{1 - w_0(\|x_k - x^*\|)} + \frac{1}{1 - p(\|x_k - x^*\|}\right)\\ &\le g_1(\rho_2)\rho_2 = \rho_1.\end{aligned}$$

Then, from the second substep of method (4), we similarly obtain in turn that:

$$\begin{aligned}\|x_{k+1} - x^*\| &= \|y_k - x^* - F'(y_k)^{-1}F(y_k) + (F'(y_k)^{-1} - A_k^{-1})F(y_k)\|\\ &\le \|y_k - x^* - F'(y_k)^{-1}F(y_k)\| + \|F'(y_k)^{-1}F'(x^*)\|\\ &\quad \times \big[\|F'(x^*)^{-1}(A_k - F'(x^*))\|\\ &\quad + \|F'(x^*)^{-1}(F'(y_k) - F'(x^*))\|\big]\\ &\quad \times \|A_k^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(y_k)\|\\ &\le \Bigg[\frac{w((1-\theta)\|y_k - x^*\|)\,d\theta}{1 - w_0(\|y_k - x^*\|)}\\ &\quad + \frac{(p(\|x_k - x^*\|) + w_0(\|y_k - x^*\|))\int_0^1 v(\theta\|y_k - x^*\|)\,d\theta}{(1 - w_0(\|y_k - x^*\|))(1 - p(\|x_k - x^*\|))}\Bigg]\\ &\quad \times \|y_k - x^*\|\\ &\le g_2(\|x_k - x^*\|)\|x_k - x^*\| \le \|x_k - x^*\| < \rho_2.\end{aligned}$$

Hence, we arrive at the following local convergence result for method (4).

THEOREM 2.4. *Suppose that:*

(i) $F, x^*, \rho_0, w_0, w, v, p^*, D_0$ *are as in Theorem 2.3 and* R^* *be given by* (25) *and*

(ii) $\beta, \gamma, \{\alpha_n\}$ *be satisfy* (23) *and* (24).

Then, the conclusions of Theorem 2.1 hold for the sequence $\{x_n\}$ *generated by method* (4).

3. Numerical examples

The numerical examples are presented in this section. We choose:

$$[x, y; F] = \int_0^1 F'(y + \theta(x - y))\, d\theta.$$

EXAMPLE 3.1. Let $X = \mathbb{R}^3, D = \bar{U}(0, 1), x^* = (0, 0, 0)^T$. Define function F on D for $w = (x, y, z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2} y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We choose $w_0(t) = L_0 t, w(t) = e^{\frac{1}{L_0}} t, v(t) = e^{\frac{1}{L_0}}$, $L_0 = e - 1$ for methods (2), (3) and (4). Moreover, for methods (3) and (4), we let $u(s, t) = \frac{1}{2}(L_0 s + L_0 t)$. Furthermore, we set $\alpha_n = 0$, $\beta = \gamma = 1$ in method (4). The parameters are given in Table 1.

Table 1: parameters for method (2), (3) and (4)

Method	ρ_1	ρ_2	R^*
(3)	4.1107	0.0671	0.0671
(4)	7.2233	00.2746	0.2746

EXAMPLE 3.2. Let $X = C[0, 1], D = \bar{U}(x^*, 1)$. We study the nonlinear integral equation of the mixed Hammerstein-type [3, 6, 18] defined by

$$x(s) = \int_0^1 K(s, t) \frac{x(t)^2}{2}\, dt,$$

where the kernel K is the Green's function given on the interval $[0, 1] \times [0, 1]$ by

$$K(s, t) = \begin{cases} (1 - s)t, & t \le s \\ s(1 - t), & s \le t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of the equation (1), where $F : C[0, 1] \longrightarrow C[0, 1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 K(s, t) \frac{x(t)^2}{2}\, dt.$$

Notice that

$$\| \int_0^1 K(s,t)\,dt \| \leq \frac{1}{8}.$$

Then, we have that the Fréchet-derivative is given by

$$F'(x)y(s) = y(s) - \int_0^1 K(s,t)x(t)\,dt.$$

Hence, since $F'(x^*(s)) = I$, we obtain that

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}\|x - y\|.$$

One can see that, higher order derivatives of F do not exist in this example. That is, the earlier results cannot be applied as already stated in the introduction.

We choose $w_0(t) = w(t) = \frac{1}{8}t, v(t) = 1 + w_0(t)$ for methods (2), (3) and (4). Moreover, in addition for methods (3) and (4), we take $u(s,t) = \frac{1}{16}(s+t)$. Furthermore, let $\alpha_n = 0, \beta = \gamma = 1$ in method (4).

The parameters are given in Table 2.

Table 2: parameters for method (2), (3) and (4)

Method	ρ_1	ρ_2	R^*
(2)	2.0786	0.1373	0.1373
(3)	2.3554	0.5855	0.5855
(4)	2.2295	0.3882	0.3882

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