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TIME-LIKE HAMILTONIAN DYNAMICAL SYSTEMS IN MINKOWSKI SPACE $\mathbb{R}^3_1$ AND THE NONLINEAR EVOLUTION EQUATIONS

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Abstract. We show that all of the curve motions specified in the Frenet-Serret frame are described by the time evolution of an integral curve of a time-like Hamiltonian dynamical system in Minkowski space such that the integral curve under consideration is a geodesic curve on a leaf of the foliation determined by the Poisson structure. Accordingly, any nonlinear soliton equation related to curve dynamics is obtained as the time evolution of an integral curve of a Hamiltonian system. As an expository example, we define Hashimoto function in the Darboux frame which is reduced to the classical Hashimoto function provided that the Poisson vector corresponds to principal normal of an integral curve and show that the defocusing version of the nonlinear Schrödinger equation and the mKdV equation are obtained by the time evolution of this function.

1. Introduction

The study of nonlinear soliton equations as a moving curve dates back to the pioneering work of Hashimoto, which is manifesting the nonlinear Schrödinger equation as a time evolution of a vortex filament in an incompressible inviscid fluid [12], and has gained more attraction in recent years. A considerable number of works in literature are devoted to study of the nonlinear soliton equations emanating from the curve dynamics in various geometries [4, 5, 8, 10, 14, 15, 19, 22].

Theory of moving curves and its relation to nonlinear evolution equations in three dimensions is somehow interesting in the breadth of Hamiltonian dynamical systems. In literature, all of the non-stretching curve motions describing the time evolution of a curve with respect to an independent parameter, are specified (in general) in the Frenet-Serret frame and many integrable models are obtained from the time evolution of the curvature quantities in Minkowski spaces [6, 7, 9, 18]. On the other hand

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owing to the existence of an isomorphism between bi-vectors and vector fields, form of the Hamilton's equations tell us that Poisson vector corresponding to a Poisson structure is an ingredient of normal bundle of a trajectory of the flow determined by the Hamiltonian vector field in three dimensions. Due to the Jacobi identity, Poisson structure itself defines a codimension one foliation and accordingly, the time evolution of the normal curvature, geodesic curvature and geodesic torsion of an integral curve in three space are determined by the compatibility condition for s and t derivatives of the Hamiltonian vector field and Poisson vector. In this sense, according to us representation of the evolution equations in the frame composed of the Hamiltonian vector field, Poisson vector and the gradient of Hamiltonian function serves richer information for the time evolution of a curve under consideration.

By virtue of this point of view in this work we consider the time evolution of a trajectory of the flow determined by the Hamiltonian vector field. In order to prove the major result of this work we firstly define the notion of a time-like Hamiltonian dynamical system in accordance with the pseudo-Riemannian metric in Minkowski space $\mathbb{R}^3_1$. Afterwards we define an orthonormal frame field composed of the Hamiltonian vector field, Poisson vector and the gradient of Hamiltonian function and we show that this frame defines Darboux frame along an integral curve on a leaf of the foliation determined by the Poisson vector. We show as the main result of this paper that all of the non-stretching curve motions specified in the Frenet-Serret frame are described by the time evolution of an integral curve of a time-like Hamiltonian dynamical system in Minkowski space $\mathbb{R}^3_1$ such that corresponding integral curve is a geodesic curve on a leaf of the foliation determined by the Poisson structure. This result manifests the form of the evolution equations is depend on the choice of the Poisson vector and the all existing results related to the time evolution of a time-like curve in literature are obtained from the time evolution of an integral curve of a Hamiltonian dynamical system with Poisson vector $\mathbf{n}$. In the final part of this paper we define the Hasimoto function in Darboux frame and we show as an illustrative example that the defocusing version of the nonlinear Schrödinger equation and the mKdV equation are represented as the time evolution of this function.

2. Basic notions

2.1 Minkowski space $\mathbb{R}^3_1$

Minkowski space $\mathbb{R}^3_1$ is a three dimensional manifold endowed with pseudo-Riemannian metric

$$\eta(\ ,\ ) = \eta_{ij} dx^i dx^j, \tag{1}$$

where η_{ij} is given in coordinate basis ∂_{x^i} as

$$(\eta_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The norm of a vector field X is defined by $||X|| = \sqrt{|\eta(X,X)|}$. A vector field X on $\mathbb{R}^3_1$ is called *space-like* if $\eta(X,X) > 0$ or $X = 0$, *time-like* if $\eta(X,X) < 0$ and *light-like* if $\eta(X,X) = 0$ and $X \neq 0$. Accordingly, a parametrized curve $\gamma = \gamma(t)$ on $\mathbb{R}^3_1$ is called space-like (respectively time-like, light-like) if its velocity field has the corresponding causality. Also for a regular space-like or time-like curve $\gamma : I \mapsto \mathbb{R}^3_1$ there exist an arc length parametrization which is given by

$$s = \int_a^t \sqrt{|\eta(\dot{\gamma}, \dot{\gamma})|}\, du.$$

The Lorentzian cross product $\times_{\mathrm{L}}$ of two vector fields $X, Y \in T\mathbb{R}^3_1$ is defined by

$$\eta(X \times_{\mathrm{L}} Y, Z) := \mathrm{vol}_\eta(X,Y,Z), \quad \forall Z \in T\mathbb{R}^3_1, \tag{2}$$

here vol_η denotes the volume form. As a direct consequence of this definition cross product can be also represented by

$$X \times_{\mathrm{L}} Y = \begin{vmatrix} i & j & -k \\ X^1 & X^2 & X^3 \\ Y^1 & Y^2 & Y^3 \end{vmatrix}. \tag{3}$$

The Frenet-Serret frame for a space-like or time-like curve with non-lightlike normals with respect to arc length parametrization and its structure equations along curve are given respectively by

$$\mathbf{t} = \gamma', \ \mathbf{n} = \frac{1}{\sqrt{|\eta(\gamma'', \gamma'')|}}\gamma'', \ \mathbf{b} = \mathbf{t} \times_{\mathrm{L}} \mathbf{n}$$

and

$$\frac{d}{ds}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_2\kappa & 0 \\ -\epsilon_1\kappa & 0 & -\epsilon_1\epsilon_2\tau \\ 0 & -\epsilon_2\tau & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \tag{4}$$

where the curvature κ and the torsion τ are defined by $\kappa = \eta(\mathbf{t}', \mathbf{n})$, $\tau = \eta(\mathbf{n}', \mathbf{b})$ (see [13]). Here over prime denotes the derivative with respect to arc length parameter s. Since $\eta(\mathbf{t}, \mathbf{t}) = \epsilon_1$ and $\eta(\mathbf{n}, \mathbf{n}) = \epsilon_2$ for $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ by definition we have $\eta(\mathbf{b}, \mathbf{b}) = -\epsilon_1\epsilon_2$. Due to the pseudo-Riemannian metric (1) there are three cases for such a Frenet-Serret frame in Minkowski space $\mathbb{R}^3_1$:

$$\begin{array}{lll} \epsilon_1 = -1, & \epsilon_2 = 1, & \eta(\mathbf{b}, \mathbf{b}) = 1 \\ \epsilon_1 = 1, & \epsilon_2 = \pm 1, & \eta(\mathbf{b}, \mathbf{b}) = \mp 1 \end{array} \tag{5}$$

From (2) or (3) the multiplication rule for Frenet-Serret frame follows as

$$\mathbf{t} \times_{\mathrm{L}} \mathbf{n} = \mathbf{b}, \ \mathbf{n} \times_{\mathrm{L}} \mathbf{b} = -\epsilon_2\mathbf{t}, \ \mathbf{b} \times_{\mathrm{L}} \mathbf{t} = -\epsilon_1\mathbf{n}$$

3. Hamiltonian systems in Minkowski space $\mathbb{R}^3_1$

A Hamiltonian dynamical system is described by a system of ordinary differential equations and completely determined by Hamiltonian function and the Poisson structure and it is formulated geometrically within a Poisson manifold [16, 20]. Poisson structure on a manifold can be interpreted as a skew-symmetric contravariant rank two tensor subject to the Jacobi identity which is identified with a holonomic vector field, so called Poisson vector, in Minkowski space $\mathbb{R}^3_1$ and therefore it defines a codimension one foliation.

In this section we define the notion of a Hamiltonian dynamical system in Minkowski space and we consider time-like Hamiltonian systems locally with non-vanishing velocity vector field and we also restrict ourselves to non-lightlike Poisson vector and non-lightlike gradient of Hamiltonian function. All constructions/objects will be considered on the whole $\mathbb{R}^3_1$ or on some domain of $\mathbb{R}^3_1$.

Let (x^1, x^2, x^3) be a local coordinates on the Minkowski space $\mathbb{R}^3_1$, an autonomous dynamical system associated to a vector field $v = v^i \partial_{x^i}$ is given by the system of autonomous ordinary differential equations:

$$\dot{x}^i = v^i(x), \quad x = (x^1, x^2, x^3).$$

Here overdot denotes the derivative with respect to time t.

We define a Hamiltonian system with respect to a Poisson structure on $\mathbb{R}^3_1$ by

$$\dot{x} = v(x) := -\Omega(x)(dH(x), .), \tag{6}$$

where $H(x)$ is the Hamiltonian function and $\Omega(x)$ is the Poisson bi-vector. Poisson bi-vector is a skew-symmetric contravariant two-tensor satisfying the Jacobi identity $[\Omega(x), \Omega(x)] = 0$, which is given by the Schouten-Nijenhuis bracket. System of equations (6) are called Hamilton's equations. In terms of the local coordinate basis $\partial_i = \partial_{x^i}$, the Poisson bi-vector is interpreted as $\Omega = \Omega^{ij} \partial_i \wedge \partial_j$. Bi-vector Ω in fact determines a Poisson structure on manifold by $\{f, g\} = \Omega(df, dg)$.

A manifold endowed with a Poisson structure is called a Poisson manifold. For a Hamiltonian system on a Poisson manifold constants of motion or the conserved quantities are determined by the equation $\dot{f} = \{f, H\}$.

A Hamiltonian dynamical system (6) is said to be space-like if $\eta(v, v) > 0$ and time-like if $\eta(v, v) < 0$. Since $\imath_\Omega \mathrm{vol}_\eta$ defines a 1-form, by means of the Minkowski metric (1) we can define pairing between bi-vectors and vector fields on $\mathbb{R}^3_1$ as

$$J^j := \eta^{ij} \left(\imath_\Omega \mathrm{vol}_\eta\right)_i, \tag{7}$$

where the vector field $J = (\Omega^{23}, \Omega^{31}, -\Omega^{12})$ is called the Poisson vector associated to the Poisson structure [1, 11]. Here $\imath_\Omega \mathrm{vol}_\eta$ stands for the contraction of the volume form with Poisson bi-vector. By favour of η the gradient of a differentiable function H is defined by

$$\nabla H = \eta^{ij} \frac{\partial H}{\partial x^j} \frac{\partial}{\partial x^i}, \tag{8}$$

and therefore Hamilton's equations (6) and the Jacobi identity are interpreted respectively by

$$v = J \times_{\mathrm{L}} \nabla H, \tag{9}$$

and

$$\eta(J, \mathrm{curl}\ J) = 0, \tag{10}$$

where the curl J is defined as

$$\text{curl } J = \begin{vmatrix} i & j & -k \\ \partial_{x^1} & \partial_{x^2} & -\partial_{x^3} \\ J^1 & J^2 & J^3 \end{vmatrix}.$$

The holonomicity condition (10) implies the existence of a family of surfaces orthogonal to the vector field J or a codimension one foliation [3, 21] in Minkowski space $\mathbb{R}^3_1$. The leaves of this foliation determined by a level surfaces of a function, say $\psi(x^1, x^2, x^3)$, and Poisson vector is of the form $\phi\nabla\psi$. In this sense, any Hamiltonian dynamical system is described by the Hamiltonian function and a holonomic vector field. An integral curve of (9) lies on a symplectic leaf $\psi(x^1, x^2, x^3) = c$. Since J is determined by the principal and binormal vector fields $\mathbf{n}$ and $\mathbf{b}$ according to (5) at a given point, tangent planes to these leaves can have space-like and time-like casuality.

Let $\delta/\delta s^1$, $\delta/\delta s^2$ and $\delta/\delta s^3$ denote the local directional derivative operators in the direction of tangent, normal and binormal of a curve and commutation relations of these vector fields are determined by structure functions of the Frenet-Serret triad. For the explicit values of these structure functions see for example [17, 22]. In terms of these we can write the gradient operator (8) as

$$\nabla H = \eta_s^{kl} \frac{\delta H}{\delta s^k} \frac{\delta}{\delta s^l},$$

where

$$\eta_s^{kl} = \eta^{ij} \frac{\partial s^k}{\partial x^j} \frac{\partial s^l}{\partial x^i}. \qquad (11)$$

The functions $s^i = s^i(x^1, x^2, x^3)$ can be seen as (non-commutative) coordinates associated to the Frenet-Serret frame and they represent the arc-lengths along the flows of the vector fields $(\mathbf{t}, \mathbf{n}, \mathbf{b})$. In [17], authors uses the term *anholonomic coordinates* for the functions s^i. The columns (rows) of the Jacobian matrix

$$\left(\frac{\partial s^k}{\partial x^j}\right) = \begin{pmatrix} \frac{\partial s^1}{\partial x^1} & \frac{\partial s^1}{\partial x^2} & \frac{\partial s^1}{\partial x^3} \\ \frac{\partial s^2}{\partial x^1} & \frac{\partial s^2}{\partial x^2} & \frac{\partial s^2}{\partial x^3} \\ \frac{\partial s^3}{\partial x^1} & \frac{\partial s^3}{\partial x^2} & \frac{\partial s^3}{\partial x^3} \end{pmatrix},$$

represented by the Frenet-Serret frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$. Accordingly we see from (11) that

$$\eta_s^{11} = \epsilon_1, \ \eta_s^{22} = \epsilon_2, \ \eta_s^{33} = -\epsilon_1\epsilon_2,$$

and hence the gradinet operator in the Frenet-Serret basis takes the form

$$\nabla = \epsilon_1 \mathbf{t} \frac{\delta}{\delta s^1} + \epsilon_2 \mathbf{n} \frac{\delta}{\delta s^2} - \epsilon_1\epsilon_2 \mathbf{b} \frac{\delta}{\delta s^3}.$$

This implies that the gradient operator is depend on the casual characters of the frame elements.

Since $\eta(\nabla H, v) = 0$, i.e. $\delta_{s^1} H = 0$, the Hamiltonian function is of the form $H = H(s^2, s^3)$ and its gradient is therefore determined by

$$\nabla H = \epsilon_2 \frac{\delta H}{\delta s^2} \mathbf{n} - \epsilon_1\epsilon_2 \frac{\delta H}{\delta s^3} \mathbf{b}.$$

On the other hand, from (9) J is orthogonal to v and hence it can be written as $J = \alpha\mathbf{n} + \beta\mathbf{b}$. It is also important to note that transformation is the Poisson structure is invariant under the transformation $J \mapsto fJ$ due to the dimensional reason, and that in general, the multiplication of the Poisson bi-vector by a function is not a Poisson structure. Accordingly, without loss of generality we can take the Poisson vector as $J = \mathbf{n} + \mu\mathbf{b}$. The Jacobi identity for $J = \mathbf{n} + \mu\mathbf{b}$ in Frenet-Serret frame reads

$$\eta(J, \text{curl } J) = \Omega_n + \mu\Omega_{nb} + \mu^2\Omega_b + \eta(\mathbf{n}, \nabla\mu \times_{\mathrm{L}} \mathbf{b}) = 0 \tag{12}$$

where the scalar quantities Ω_n, Ω_{nb} and Ω_b are defined by

$$\begin{aligned} \Omega_n &= \eta(\mathbf{n}, \nabla \times_{\mathrm{L}} \mathbf{n}) \\ \Omega_{nb} &= \eta(\mathbf{n}, \nabla \times_{\mathrm{L}} \mathbf{b}) + \eta(\mathbf{b}, \nabla \times_{\mathrm{L}} \mathbf{n}) \\ \Omega_b &= \eta(\mathbf{b}, \nabla \times_{\mathrm{L}} \mathbf{b}). \end{aligned}$$

These scalar quantities completely determine the value of non-holonomicity of the complexified vector field $\xi = \mathbf{n} + i\mathbf{b}$ (see [3]).

Since $\eta(\mathbf{n}, \nabla\mu \times_{\mathrm{L}} \mathbf{b}) = -\eta(\nabla\mu, \mathbf{n} \times_{\mathrm{L}} \mathbf{b}) = \epsilon_2\eta(\nabla\mu, \mathbf{t})$ from the definition of the gradient operator in the Frenet-Serret basis we have

$$\eta(\mathbf{n}, \nabla\mu \times_{\mathrm{L}} \mathbf{b}) = \epsilon_2\epsilon_1^2 \frac{\delta\mu}{\delta s^1}. \tag{13}$$

As a consequence the following is immediate.

PROPOSITION 3.1. *The Jacobi identity for non-lightlike Poisson structure* $J = \boldsymbol{n} + \mu\boldsymbol{b}$ *associated to Hamiltonian dynamical system in* $\mathbb{R}^3_1$ *is described by the Riccati type pde*

$$\Omega_n + \mu\Omega_{nb} + \mu^2\Omega_b = -\epsilon_2 \frac{\delta\mu}{\delta s^1}. \tag{14}$$

Proof. Substituting (13) into (12) results in (14). □

This expression manifests that the Jacobi identity for a Poisson structure can be expressible as a partial differential equation involving arc length coordinate only and this equation does not depend on the casual character of the velocity vector field. That is (14) describes the Jacobi identity for the Poisson structure for both of the space-like and time-like Hamiltonian dynamical systems, and can be seen as the Minkowski space analogue of the Riccati equation obtained in [1].

4. Darboux Frame associated to a Hamiltonian system

Since a Hamiltonian dynamical system is completely determined by the Hamiltonian function H and Poisson structure $J = \mathbf{n} + \mu\mathbf{b}$, $(v, J, \nabla H)$ forms a natural frame field associated to the Hamiltonian dynamical system in $\mathbb{R}^3_1$. ∇H and J need not be orthogonal with respect to the metric (1) in general. Any vector field X in $T\mathbb{R}^3_1$ can be (locally) uniquely decomposed into three components for an orthonormal frame (e_1, e_2, e_3) with $e_3 = e_1 \times_{\mathrm{L}} e_2$ as $X = \epsilon_1\eta(X, e_1)e_1 + \epsilon_2\eta(X, e_2)e_2 - \epsilon_1\epsilon_2\eta(X, e_3)e_3$, and thereby we can define an orthonormal frame on an integral curve of a given

Hamiltonian system by applying Gram-Schmidt process as follows:

$$e_1 = \frac{v}{\|v\|}, \quad e_2 = \frac{1}{\|J\|}J,$$
$$e_3 = \frac{1}{\|\nabla H - \epsilon_2 \eta(\nabla H, e_2)e_2\|}\left(\nabla H - \epsilon_2 \eta(\nabla H, e_2)e_2\right).$$

Since $\|J\| = \sqrt{|1-\epsilon_1\mu^2|}$, for convenience we assume $1-\epsilon_1\mu^2 > 0$. This is readily satisfied for a time-like dynamical system. By a direct calculation we find

$$\nabla H - \epsilon_2 \eta(\nabla H, e_2)e_2 = -\frac{\epsilon_2}{1-\epsilon_1\mu^2}\left(\mu\frac{\delta H}{\delta s^2} + \epsilon_1\frac{\delta H}{\delta s^3}\right)(\epsilon_1\mu\mathbf{n} + \mathbf{b}).$$

In this work we only focus on the time-like case $\epsilon_1 = -1$. This case implies $\epsilon_2 = 1$ and $\eta(\mathbf{b}, \mathbf{b}) = 1$. Accordingly for $\mu\frac{\delta H}{\delta s^2} - \frac{\delta H}{\delta s^3} > 0$ we obtain

$$e_1 = \frac{v}{\|v\|}, \quad e_2 = \frac{1}{\sqrt{1+\mu^2}}J, \quad e_3 = e_2 \times_{\mathrm{L}} e_1 = \frac{1}{\sqrt{1+\mu^2}}J^{\perp}, \tag{15}$$

where $J^{\perp} = \mu\mathbf{n} - \mathbf{b}$. In terms of the Frenet-Serret basis, this frame is given in the matrix form as

$$e_1 = \mathbf{t}, \ \begin{pmatrix} e_2 \\ e_3 \end{pmatrix} = \frac{1}{\sqrt{1+\mu^2}}\begin{pmatrix} 1 & \mu \\ \mu & -1 \end{pmatrix}\begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix}.$$

As we mentioned above, due to the conformal invariance in three dimensions the vector field e_2 is the Poisson vector associated to the given Poisson structure and readily satisfies the Jacobi identity $\eta(e_2, \mathrm{curl} e_2) = 0$. Since e_2 defines a family of surfaces, let us restrict ourselves to a member of this family and denote it by $\mathcal{S}$. Clearly an integral curve of the time-like Hamiltonian system is contained in S and we assume that integral curve under consideration is re-parametrized by its arc-length.

PROPOSITION 4.1. *Let $v = e_2 \times_{\mathrm{L}} \nabla H$ be a time-like Hamiltonian dynamical system with Poisson structure $e_2 = \frac{1}{\sqrt{1+\mu^2}}(\boldsymbol{n} + \mu\boldsymbol{b})$. Then the frame* (15) *defines Darboux frame along an integral curve on the surface $\mathcal{S}$ with unit normal e_2. Moreover, derivative of (e_1, e_2, e_3) along the trajectory of the flow determined by the Hamiltonian vector field v described by the following system of equations*

$$\frac{d}{ds}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_n & \kappa_g \\ \kappa_n & 0 & \tau_\tau \\ \kappa_g & -\tau_\tau & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{16}$$

where κ_n, κ_g and τ_τ are defined by

$$\kappa_n = \frac{\kappa}{\sqrt{1+\mu^2}}, \ \kappa_g = \frac{\kappa\mu}{\sqrt{1+\mu^2}}, \ \tau_\tau = -\left(\tau + \frac{\dot{\mu}}{1+\mu^2}\right), \ \dot{\mu} = \delta_s\mu. \tag{17}$$

Proof. By definition (e_1, e_2, e_3) defines Darboux frame on $\mathcal{S}$. Equations in (16) are obtained by a straightforward calculation. □

The quantities κ_n, κ_g and τ_τ in (16) describes the normal curvature, geodesic curvature and geodesic torsion of an integral curve of the Hamiltonian dynamical system (9). The appearance of μ and $\dot{\mu}$ in these equations determines the values of non-holonomicity Ω_n and Ω_b and hence the vanishing of the curvature quantities have a direct geometrical meaning for the determination of Poisson structures with the normal legs of Frenet-Serret frame for underlying Hamiltonian system. It follows directly from (17) that an integral curve of the Hamiltonian dynamical system is an asymptotic curve iff it is a straight line, it is a geodesic for $\kappa > 0$ iff $\mu = 0$, and it is a line of curvature iff $\tau = -\frac{\dot{\mu}}{1+\mu^2}$.

It is also suitable to note here that setting $\mu = 0$ in (21) gives

$$e_1 = \mathbf{t},\ e_2 = \mathbf{n},\ e_3 = -\mathbf{b} \tag{18}$$

Using these and setting $\mu = 0$ in (16) and (17) we obtain the Frenet-Serret equations for a time-like unit speed curve as it is stated in (4):

$$\frac{d}{ds}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}.$$

We should also emphasis that the existence of a Poisson structure J with $\mu = 0$ is determined by the existence of trivial solution of the Riccati equation and this holds if and only if $\Omega_n = 0$ and can be seen more transparently for the dynamical systems admitting two functionally independent conserved quantities in three dimensions. Consider the dynamical system on the domain $D : x^2 + y^2 < 2$ of $\mathbb{R}^3_1$ given by the set of equations

$$\dot{x} = -y,\ \dot{y} = x,\ \dot{z} = \sqrt{2}, \tag{19}$$

which integrates (up to position) to $x = r\cos t,\ y = r\sin t,\ z = \sqrt{2}t$.

Clearly the vector field $v = (\dot{x}, \dot{y}, \dot{z})$ is time-like. The domain D is foliated by the constant values of function $H_1(x, y, z) = \frac{1}{2}(x^2 + y^2)$ and an integral curve of v lies on a leaf of this foliation. Frenet-Serret frame associated to an arbitrary speed time-like curve $t \mapsto \gamma(t)$ is constructed in some domain of $\mathbb{R}^3_1$ as follows:

$$\mathbf{t} = \frac{v}{\|v\|},\ \mathbf{n} = \frac{v \times_{\mathrm{L}} (\nabla \times_{\mathrm{L}} v)}{\|v \times_{\mathrm{L}} (\nabla \times_{\mathrm{L}} v)\|},\ \mathbf{b} = \frac{v \times_{\mathrm{L}} \mathbf{n}}{\|v \times_{\mathrm{L}} \mathbf{n}\|}. \tag{20}$$

For $v = \dot{\gamma} = (-y, x, \sqrt{2})$ and $\ddot{\gamma} = (-x, -y, 0)$, (20) can be equivalently given by

$$\mathbf{t} = \frac{v}{\|v\|},\ \mathbf{b} = \frac{\dot{\gamma} \times_{\mathrm{L}} \ddot{\gamma}}{\|\dot{\gamma} \times_{\mathrm{L}} \ddot{\gamma}\|},\ \mathbf{n} = \mathbf{b} \times_{\mathrm{L}} \mathbf{t}. \tag{21}$$

It follows from (20) or (21) that

$$\mathbf{t} = \frac{1}{\|v\|}(-y, x, \sqrt{2}),\ \|v\| = \sqrt{2 - (x^2 + y^2)}$$

$$\mathbf{b} = \frac{1}{\phi}(\sqrt{2}y, -\sqrt{2}x, -(x^2 + y^2)),\ \phi = \sqrt{(x^2 + y^2)(2 - (x^2 + y^2))}$$

$$\mathbf{n} = -\frac{(2 - (x^2 + y^2))}{\phi\|v\|}(x, y, 0) = -\frac{1}{\sqrt{x^2 + y^2}}\nabla H_1.$$

Another conserved quantity or the Hamiltonian function can be found by inspection as $H_2 = z - \sqrt{2}\arctan\left(\frac{y}{x}\right),\ x \neq 0$, whose gradient is found as $(x^2 + y^2)\nabla H_2 =$

$\left(\sqrt{2}y, -\sqrt{2}x, -(x^2+y^2)\right)$. Thus we have

$$\mathbf{b} = \sqrt{\frac{x^2+y^2}{2-(x^2+y^2)}}\nabla H_2,$$

Clearly H_1 and H_2 are functionally independent since ∇H_1 and ∇H_2 are orthogonal vector fields with respect to the metric η. As a consequence, the dynamical system (19) is written for the Poisson vector $J = \mathbf{n}$ and the Hamiltonian function H_2 by $v = \psi J \times_{\mathrm{L}} \nabla H_2$, and this implies $\mu = 0$.

5. Main Result

Now we state and prove the theorem exhibiting the main result of this paper.

THEOREM 5.1. *All of the non-stretching time evolution of a time-like curve specified in the form* $\dot{\mathbf{r}} = U\mathbf{n} + V\mathbf{b} + W\mathbf{t}$ *are described by the time evolution of an integral curve of a time-like Hamiltonian dynamical system in Minkowski space* $\mathbb{R}^3_1$ *such that the integral curve under consideration is a geodesic curve on a leaf of the foliation determined by the Poisson structure.*

Proof. Consider one-parameter family of smooth curves $\mathbf{r}(s,t)$ such that for each fixed time t, a point particle moves along a trajectory $\mathbf{r}(s,t)$ of the flow determined by the Hamiltonian vector field (9) and re-parameterized by a natural parameter s. Time evolution of a point particle is specified in the Darboux frame as

$$\dot{\mathbf{r}} = Ue_2 + Ve_3 + We_1, \quad \dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial t}. \tag{22}$$

We assume here that motion is non-stretching, that is, the arc-length $s = \int_a^b \|\partial_{s'}\mathbf{r}\|\, ds'$ is independent from the parameter t i.e. $\partial_t s = 0$. In this case we have the compatibility condition $(\partial_s\partial_t - \partial_t\partial_s)\mathbf{r} = 0$. Motion is said to be local if the functions U, V and W are depend only on the local values of κ_n, κ_g and τ_τ and their invariant derivatives with respect to arc-length s. Taking the s derivative of (22) and using the compatibility condition $\mathbf{r}_{st} = \mathbf{r}_{ts}$, we find $\dot{e}_1 = (U_s - V\tau_\tau + W\kappa_n)e_2 + (V_s + U\tau_\tau + W\kappa_g)e_3$ and $W_s + U\kappa_n + V\kappa_g = 0$. This condition is equivalent to say that the motion is non-stretching. By a direct calculation we find $\dot{e}_{1,s} = (a\kappa_n + b\kappa_g)e_1 + (a_s - b\tau_\tau)e_2 + (b_s + a\tau_\tau)e_3$, where a and b are defined by $a = U_s - V\tau_\tau + W\kappa_n,\ b = V_s + U\tau_\tau + W\kappa_g$.

A real matrix A is called pseudo-orthogonal with respect to the metric η if $A^T\eta A = \eta$, and the Lie algebra $\mathfrak{so}(2,1)$ of the pseudo-orthogonal group $SO(2,1)$ consists of matrices of the form $X^T = -\eta X \eta^{-1}$. Therefore, the time-derivative of the frame (e_1, e_2, e_3) is determined by

$$\dot{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & -c & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{23}$$

where a and b are as above.

Using $e_{1,st} = (a\kappa_n + b\kappa_g)e_1 + (\dot{\kappa}_n - c\kappa_g)e_2 + (\dot{\kappa}_g + c\kappa_n)e_3$, from the compatibility condition $e_{1,st} = e_{1,ts}$ we find

$$\dot{\kappa}_n = a_s - b\tau_\tau + c\kappa_g, \qquad \dot{\kappa}_g = b_s + a\tau_\tau - c\kappa_n. \tag{24}$$

Here c is determined by these equations.

Using the identity $\kappa^2 = \kappa_n^2 + \kappa_g^2$ from (24), we get

$$\frac{1}{2}\frac{\partial}{\partial t}\kappa^2 = \kappa_n(a_s - b\tau_\tau) + \kappa_g(b_s + a\tau_\tau). \tag{25}$$

From the compatibility condition $e_{2,st} = e_{2,ts}$ we also find

$$\dot{\tau}_\tau = c_s + a\kappa_g - b\kappa_n. \tag{26}$$

We see from (17) that $\kappa_g = 0$ with $\kappa > 0$ if and only if $\mu = 0$. In this case we have $\kappa_n = \kappa$ and $\tau_\tau = -\tau$. Also in the case of $\mu = 0$ Poisson vector e_2 transforms to $\mathbf{n}$ and the Jacobi identity implies $\Omega_n = 0$. This is equivalent to say that integral curve of a time-like Hamiltonian system is geodesic curve on the surface $\mathcal{S}$ with unit normal $\mathbf{n}$. If we set $\mu = 0$ in (25) and (26) using the expressions a, b and c in terms of U, V, W we find $\dot{\kappa}$ and $\dot{\tau}$ as follows:

$$\begin{aligned}
\partial_t\kappa &= U_{ss} - (\kappa^2 + \tau^2)U - \kappa_s \int^s \kappa U\, ds' + 2\tau V_s + \tau_s V \\
\partial_t\tau &= -\frac{\partial}{\partial s}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}(V_s - \tau U) - \frac{\tau}{\kappa}(U_s + \tau V) + \tau \int^s \kappa U\, ds'\right] + \kappa V_s - \kappa\tau U
\end{aligned} \tag{27}$$

If we set $\mu = 0$ in (23) and use (18) we obtain

$$\begin{aligned}
\dot{\mathbf{t}} &= (U_s + V\tau + W\kappa)\mathbf{n} + (U\tau - V_s)\mathbf{b}, \\
\dot{\mathbf{n}} &= (U_s + V\tau + W\kappa)\mathbf{t} - \left[\frac{1}{\kappa}\frac{\partial}{\partial s}(V_s - U\tau) - \frac{\tau}{\kappa}(U_s + \tau V + \kappa W)\right]\mathbf{b}, \\
\dot{\mathbf{b}} &= -(V_s - U\tau)\mathbf{t} + \left[\frac{1}{\kappa}\frac{\partial}{\partial s}(V_s - U\tau) - \frac{\tau}{\kappa}(U_s + \tau V + \kappa W)\right]\mathbf{n}.
\end{aligned} \tag{28}$$

Corresponding point evolution of these equations is obtained by setting $\mu = 0$ in (22) and using again (18) as follows:

$$\dot{\mathbf{r}} = U\mathbf{n} - V\mathbf{b} + W\mathbf{t}. \tag{29}$$

If we replace V for equations in (27), (28) and (29) by $-\tilde{V}$, then the proof follows. □

These equations are the Minkowski space analogues of the corresponding equations for Euclidean space derived in [19] and any nonlinear soliton equation obtained from the time evolution of the curvature and the torsion of a time-like curve in literature is covered by the case of $\mu = 0$, see for instance, the nonlinear Schrödinger equation as the binormal motion of a time-like curve or as the Heisenberg spin chain in Minkowski space [7,9,18]. We are going to obtain nonlinear Schrödinger equation and the mKdV equation as the time evolution of the Hashimoto function, which is composed of the geodesic curvature, normal curvature and geodesic torsion in the following section.

6. Hashimoto function in Darboux frame

As an illustrative example for the main result of this paper, we shall construct Hashimoto function in Darboux frame and we obtain the nonlinear Schrödinger equation and the mKdV equation by the time evolution of this function.

Equations in (16) are written in complexified form as

$$(e_2 + ie_3)_s + i\tau_\tau(e_2 + ie_3) = (\kappa_n + i\kappa_g)e_1.$$

If we define $\xi = (e_2 + ie_3)\varepsilon, \quad \varepsilon = e^{i\int^s \tau_\tau \, ds'}$, then we can write (16) as $\frac{\partial \xi}{\partial s} = \psi e_1$, $\frac{\partial e_1}{\partial s} = \frac{1}{2}(\psi\xi^* + \psi^*\xi)$, where ψ is defined by

$$\psi = (\kappa_n + i\kappa_g)\varepsilon, \tag{30}$$

and asterisk denotes the complex conjugation. We call this function as Hashimoto function in Darboux frame. In terms of ψ and ε, κ_n and κ_g are written as

$$\kappa_n = \frac{1}{2}(\psi\varepsilon^* + \psi^*\varepsilon), \quad \kappa_g = \frac{1}{2i}(\psi\varepsilon^* - \psi^*\varepsilon). \tag{31}$$

As one can easily see, if $\mu = 0$ then $\kappa_g = 0$ and from (21) it follows that

$$\xi = (\mathbf{n} - i\mathbf{b})e^{-i\int^s \tau \, ds'} \quad \text{and} \quad \psi = \kappa e^{-i\int^s \tau \, ds'}.$$

In this case the Poisson vector is determined by the principal normal $\mathbf{n}$. Since the Hashimoto function (30) have complete description for an integral curve of a given time-like Hamiltonian system, its time evolution determines the time evolution of an integral curve. Accordingly we have the following:

PROPOSITION 6.1. *The defocusing version of the nonlinear Schrödinger equation*

$$i\partial_t\psi + \partial_{ss}^2\psi - \frac{1}{2}|\psi|^2\psi = 0,$$

and the complex mKdV equation

$$\partial_t\psi + \psi_{sss} - \frac{3}{2}\psi_s|\psi|^2 = 0,$$

are obtained as the time evolution of an integral curve of a given time-like Hamiltonian system.

Proof. The proof of this proposition is based on straightforward calculation. Using the equations in (24) and (26) we can compute the time derivative of (30) as

$$\partial_t\psi = [(a + ib)_s + i\tau_\tau(a + ib)]\varepsilon + i\psi \int^s (a\kappa_g - b\kappa_n) \, ds', \tag{32}$$

where $a\kappa_g - b\kappa_n = U_s\kappa_g - V_s\kappa_n - \tau_\tau(U\kappa_n + V\kappa_g)$. Substituting a and b into equation (32) results in

$$\partial_t\psi = (U\varepsilon)_{ss} + i(V\varepsilon)_{ss} + [W_s(\kappa_n + i\kappa_g) + W(\kappa_{n,s} + i\kappa_{g,s}) \\ + i\tau_\tau W(\kappa_n + i\kappa_g)]\varepsilon + i\psi \int^s [U_{s'}\kappa_g - V_{s'}\kappa_n - \tau_\tau(U\kappa_n + V\kappa_g)] \, ds'.$$

Due to the identities $|\psi|^2 = \kappa_n^2 + \kappa_g^2$, $W_s = -(U\kappa_n + V\kappa_g)$, we have

$$\partial_t\psi = (U\varepsilon)_{ss} + i(V\varepsilon)_{ss} - i\psi(U\kappa_g - V\kappa_n) - (U+iV)|\psi|^2\varepsilon \\ - \psi_s \int^s (U\kappa_n + V\kappa_g)\,ds' + i\psi \int^s [U_{s'}\kappa_g - V_{s'}\kappa_n - \tau_\tau(U\kappa_n + V\kappa_g)]\,ds'.$$

Using the identities

$$\int^s V_{s'}\kappa_n\,ds' = V\kappa_n - \int^s V\kappa_{n,s'}\,ds',$$
$$\int^s U_{s'}\kappa_g\,ds' = U\kappa_g - \int^s U\kappa_{g,s'}\,ds',$$

we obtain by a direct calculation that

$$i\psi \int^s (U_{s'}\kappa_g - V_{s'}\kappa_n)\,ds' - i\psi(U\kappa_g - V\kappa_n) = i\psi \int^s (V\kappa_{n,s'} - U\kappa_{g,s'})\,ds'.$$

Therefore we have

$$\partial_t\psi = (U\varepsilon)_{ss} + i(V\varepsilon)_{ss} - (U+iV)|\psi|^2\varepsilon - \psi_s \int^s (U\kappa_n + V\kappa_g)\,ds' \\ + i\psi \int^s (V\kappa_{n,s'} - U\kappa_{g,s'})\,ds' - i\psi \int^s \tau_\tau(U\kappa_n + V\kappa_g)\,ds'. \quad (33)$$

From (31) it follows that

$$U\kappa_n + V\kappa_g = \frac{1}{2}[\psi^*\varepsilon(U+iV) + (\psi^*\varepsilon(U+iV))^*] = \mathrm{Re}(\psi^*\varepsilon(U+iV))$$
$$\kappa_{n,s} = \frac{1}{2}(\psi_s\varepsilon^* + \psi_s^*\varepsilon) + \kappa_g\tau_\tau$$
$$\kappa_{g,s} = \frac{1}{2i}(\psi_s\varepsilon^* - \psi_s^*\varepsilon) - \kappa_n\tau_\tau.$$

Using the fact $\mathrm{Re}(\psi_s\varepsilon^*(V+iU)) = \mathrm{Im}(\psi_s^*\varepsilon(U+iV))$ we obtain

$$i\psi \int^s (V\kappa_{n,s'} - U\kappa_{g,s'})\,ds' - i\psi \int^s \tau_\tau(U\kappa_n + V\kappa_g)\,ds' = i\psi \int^s \mathrm{Im}[(\psi_{s'}^*\varepsilon)(U+iV)]\,ds'.$$

If we put these into (33) we finally get

$$\partial_t\psi = (U\varepsilon)_{ss} + i(V\varepsilon)_{ss} - (U+iV)|\psi|^2\varepsilon \\ - \psi_s \int^s \mathrm{Re}(\psi^*\varepsilon(U+iV))\,ds' + i\psi \int^s \mathrm{Im}[(\psi_{s'}^*\varepsilon)(U+iV)]\,ds'.$$

Therefore the time evolution of the function ψ is written by the integro-differential operator as

$$\partial_t\psi = \left[\partial_{ss}^2 - |\psi|^2 - \psi_s \int^s ds'\mathrm{Re}\psi^* + i\psi \int^s ds'\mathrm{Im}\psi_{s'}^*\right](U+iV)\varepsilon. \quad (34)$$

Choosing $U = 0$ and $V = \kappa_n + i\kappa_g$ in (34) results in the defocusing version of the non-linear Schrödinger equation: $i\partial_t\psi + \partial_{ss}^2\psi - \frac{1}{2}|\psi|^2\psi = 0$. If we set $U = -(\kappa_{n,s} + i\kappa_{g,s})$ and $V = -(\kappa_n + i\kappa_g)\tau_\tau$ we obtain $(U+iV)\varepsilon = -\partial_s\psi$. Substituting this into (34) results in the defocusing version of the modified KdV equation: $\partial_t\psi + \psi_{sss} - \frac{3}{2}\psi_s|\psi|^2 = 0$. □

As it is indicated in [19], no choice of the pair $\{U, V\}$ could give the defocusing version of the non-linear Schrödinger equation for non-strecthing curve motions in $\mathbb{R}^3$. However, as we have shown and as it is stated in [7], the defocusing version can be obtained by the moving curve in Minkowski space $\mathbb{R}^3_1$. We should also emphasis that since U and V are depends on the curvature quantities κ_n, κ_g and τ_τ, they have also μ dependence and hence the choice of U and V is determined by the choice of Poisson structure associated to Hamiltonian system. Besides, since the equations in (16) for a time-like curve in $\mathbb{R}^3_1$ reduce to Frenet-Serret equations for $\mu = 0$, they can be seen as the (AKNS) scattering problem [2] at zero eigenvalue for $r = +q^*$ [19] provided that the integral curve of a given time-like Hamiltonian dynamical system is a geodesic line on the surface with unit normal $\mathbf{n}$. This is the circumstance that we have $\kappa_g = 0$, $\kappa_n = \kappa$, $\tau_\tau = -\tau$ and all the existing results related to the motion of a time-like curve in an ambient space in literature are covered within this special case by setting $\mu = 0$ in (30) and (34).

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