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ON THE L^p-BOUNDEDNESS OF A CLASS OF SEMICLASSICAL FOURIER INTEGRAL OPERATORS

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Abstract. In this paper, we investigate the L^p-boundedness of semiclassical Fourier integral operators defined by symbols $a(x,\xi)$ which behave in the spatial variable x like L^p functions and are smooth in the ξ variable.

1. Introduction

A Fourier integral operator (FIO for short) is a singular integral defined by

$$A(a,\varphi)f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}\, a(x,\xi)\, \hat{f}(\xi)\,\mathrm{d}\xi, \tag{1}$$

where φ is the phase function, a is called the symbol of the FIO $A(a,\varphi)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing functions.

From the beginning of the theory of FIOs, many efforts have been made to study the regularity of these operators in functional spaces. The analysis of the local L^2 boundedness of FIOs goes back to Eskin [10] and Hörmander [12] for zeroth order symbols and homogeneous phases of order 1 in the frequency variable ξ which satisfy the non-degeneracy condition, that is the Hessian matrix $[\frac{\partial^2\varphi}{\partial_x\partial_\xi}]$ has non-vanishing determinant.

Local L^p boundedness of Fourier integral operators was proved by Beals [3] for symbols in $S^{-m}_{1,0}$ while the optimal results for Hörmander's symbol classes were obtained by Seeger, Sogge and Stein [21] where $a(x,\xi) \in S^m_{\rho,1-\rho}$, $\frac{1}{2} \le \rho \le 1$ and the order $m = (\rho - n)|\frac{1}{p} - \frac{1}{2}|$.

Since 1970s, motivated by applications in microlocal analysis and hyperbolic partial differential equations, many authors extended local L^2 boundedness results to global $L^2(\mathbb{R}^n)$ regularity, see for instance Asada and Fujiwara [2], Ruzhansky and Sugimoto [19] and Caldéron and Vaillaincourt [4] for pseudodifferential operators.

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Recently the authors of this article [9] proved $L^2(\mathbb{R}^n)$-boundedness of FIOs with weighted symbols. For general $1 < p < \infty$, Coriasco and Ruzhansky [5,6] established $L^p(\mathbb{R}^n)$-continuity of FIOs with amplitudes in a suitable subclass of the Hörmander class, where certain decay of the amplitudes in the spatial variables are assumed.

Applications to several problems in nonlinear partial differential equations and problems on non-smooth domains require non-regular symbols, i.e. symbols which are smooth in the frequency variable ξ but less/ non-regular in the spatial variable x. Pseudodifferential operators with non-smooth symbols have attracted much interest in the literature. Marshall [14,15] and Taylor [23,24] proved regularity of pseudodifferential operators in Sobolev and Besov spaces with symbols in Sobolev spaces.

In [13], Kenig and Staubach defined a class of pseudodifferential operators with symbols $a(x, \xi)$ that are smooth in ξ and L^∞ in the spatial variable x and they explored their L^p-boundedness properties. Motivated by this investigations, Dos Santos Ferreira and Staubach [8] generalized these results for Fourier integral operators with smooth and rough phases and with the aforementioned family of symbols on weighted and unweighted spaces. Continuing this investigations, Rodriguez-López and Staubach [18] established L^p-boundedness of FIOs with amplitudes that are non-smooth in x and exhibit an L^p-type behaviour, $1 \leq p \leq \infty$, instead of L^∞-behaviour showed in [8,13].

In the semiclassical case, an h-FIO $T_h(a, \varphi)$ has the following form

$$T_h(a,\varphi)f(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\varphi(x,\xi)} a(x,\xi;h) \hat{f}_h(\xi)\, \mathrm{d}\xi, \quad f \in \mathcal{S}(\mathbb{R}^n). \tag{2}$$

For the readers interested in the basics of semiclassical analysis, [7, 16, 17, 25] contain a survey on this theory. Martinez [16], Dimassi and Sjöstrand [7] and Zworski [25] established $L^2(\mathbb{R}^n)$-boundedness of h-pseudodifferential operators with symbols defined by tempered weights. The norm is uniform in h if the symbols are rapidly decreasing. For h-FIOs there has been, comparatively, a smaller amount of activity concerning the investigation of the corresponding L^p-boundedness properties. Harrat and Senoussaoui [11] proved L^2-boundedness and L^2-compactness of a class of h-FIOs. Namely, they showed that if the weight of the amplitude is bounded (respectively tends to 0) then the h-FIO is bounded (respectively compact) on $L^2(\mathbb{R}^n)$. Recently, Aitemrar and Senoussaoui [1] proved analogous results for a suitable class of h-FIOs.

Motivated by the lack of L^p-boundedness results for h-FIOs in the literature of semiclassical analysis, the purpose of this work is to extend the aforementioned works in the semiclassical analysis. Following the ideas of [18], we consider non-regular symbols $a \in L^p S^m_\rho(\mathbb{R}^n)$, such that $a(x, \xi; h)$ are smooth in the frequency variable ξ and behave in the spatial variable x like an L^p function uniformly in h. For the phase function, we consider the class Φ^k defined in [8], which consists of homogeneous phase functions of degree 1 in the frequency variable ξ, with a specific control over the derivatives of orders greater than or equal to k. We also assume that they are strongly non-degenerate, see Definition 2.8 below. This kind of phase function appears in the study of wave equations.

The main result of this paper is the following

Theorem 1.1. *Let $0 \leq \rho \leq 1$, $0 < h \leq 1$, $0 < r \leq \infty$, $1 \leq p, q \leq \infty$ verify $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Suppose that $\varphi \in \Phi^2(\mathbb{R}^n)$ satisfies the SND condition and assume $a \in L^p S^m_\rho(\mathbb{R}^n)$ such that, for some $\varepsilon > 0$,*

$$m(\xi) \leq C_0 \,\langle\xi\rangle^{\frac{\rho n}{s} - 2M - \frac{n-1}{2}\left(\frac{1}{s} + \frac{1}{\min(p,s')}\right) - \varepsilon}, \quad C_0 > 0, \quad \xi \in \mathbb{R}^n, \tag{3}$$

with $s = \min(2, p, q)$, $\frac{1}{s} + \frac{1}{s'} = 1$, and some $M > \frac{n}{2s}$. Then the h-FIO $T_h(a, \varphi)$ defined in (2) is bounded from $L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ and

$$\|T_h(a, \varphi)\|_{L^q \to L^r} \leq C\, h^{-2M}, \quad C > 0.$$

When $T_h(a, \varphi)$ is a pseudodifferential operator, i.e. when the phase function takes the special form $\langle x, \xi\rangle$, Theorem 1.1 can be improved. We will show that the decay of the weight is less then showed in the case of Fourier integral operators. To our knowledge, the present paper exhibits for the first time L^p-regularity of h-Fourier integral operators with rough symbols.

To summarize, the paper is organized as follows. In the second section we give definitions of symbols and phase functions that appear in the h-FIO treated here. Tools for proving L^p boundedness of h-FIO are mentioned in the third section. In the fourth section we prove our main result.

2. Preliminaries

We here give the definitions of the symbol and phase function classes that we will use in the sequel, and fix some notations. We will, in particular, denote by $\langle\xi\rangle$ the *smoothed absolute value (or Japanese bracket)* given by $(1 + |\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}^n$.

Recall that a real-valued function f belongs to $L^p(\mathbb{R}^n)$ space, with $0 < p \leq \infty$, if f is measurable on $\mathbb{R}^n$ and

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty, \text{ if } 0 < p < \infty,$$

and
$$\|f\|_{L^\infty} = \inf\{M \in \mathbb{R}, |f(x)| \leq M \text{ a.e. } x \in \mathbb{R}^n\}, \text{ if } p = \infty.$$

Definition 2.1. A continuous function $m : \mathbb{R}^n \to [0, +\infty[$ is called a tempered weight on $\mathbb{R}^n$ if

$$\exists C_0 > 0\, \exists l \in \mathbb{R} \quad m(\xi) \leq C_0\, m(\xi^*)\, (1 + |\xi^* - \xi|)^l\,, \ \xi, \xi^* \in \mathbb{R}^n.$$

Example 2.2. Functions of the form $\lambda_t(\xi) = (1 + |\xi|)^t$, $t \in \mathbb{R}$, define tempered weights.

Definition 2.3. Let $0 \leq \rho \leq 1$, $0 < h \leq 1$, $1 \leq p \leq \infty$ and let m be a tempered weight on $\mathbb{R}^n$. A function $a(x, \xi; h)$, $x, \xi \in \mathbb{R}^n$, which is measurable in $x \in \mathbb{R}^n$ and smooth in ξ a.e. $x \in \mathbb{R}^n$, belongs to the symbol class $L^p S^m_\rho(\mathbb{R}^n)$, if for each multi-index $\alpha \in \mathbb{Z}^n_+$ there exists a constant $C_\alpha > 0$ such that

$$\left\|\partial^\alpha_\xi a(\cdot, \xi; h)\right\|_{L^p(\mathbb{R}^n)} \leq C_\alpha\, m(\xi) \langle\xi\rangle^{-\rho|\alpha|}, \quad 0 < h \leq 1.$$

We define the associated seminorms

$$|a|_{p,m,d} = \sum_{|\alpha|\leq d} \sup_{\xi\in\mathbb{R}^n} m^{-1}(\xi)\langle\xi\rangle^{\rho|\alpha|} \left\|\partial_\xi^\alpha a(\cdot,\xi;h)\right\|_{L^p(\mathbb{R}^n)}.$$

Example 2.4. The symbols introduced in [18] belong to $L^pS^m_\rho(\mathbb{R}^n)$ with weight $m(\xi) = \langle\xi\rangle^t$, $t\in \mathbb{R}$.

Example 2.5. The symbol of the semiclassical Schrödinger operator $-h^2\Delta + V(x)$ is given by $a_S(x,\xi;h) = |\xi|^2 + V(x)$. It is easily proved that $a_S \in L^\infty S^m_1(\mathbb{R}^n)$ with $m(\xi) = \langle\xi\rangle^2$, if $V \in L^\infty(\mathbb{R}^n)$.

We have the following result for the product of rough symbols. The proof is a straightforward application of Leibniz's rule and Holder's inequality.

Lemma 2.6. *Let $1 \leq p,q \leq \infty$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $a \in L^pS^{m_1}_\rho(\mathbb{R}^n)$ and $b \in L^qS^{m_2}_\rho(\mathbb{R}^n)$ then $ab \in L^rS^{m_1m_2}_\rho(\mathbb{R}^n)$. Moreover, if $\eta(\xi) \in C_0^\infty(\mathbb{R}^n)$ and $a_\varepsilon(x,\xi;h) = a(x,\xi;h)\eta(\varepsilon\xi)$ with $\varepsilon \in (0,1]$, then*

$$\sup_{0<\varepsilon\leq 1}\sup_{\xi\in\mathbb{R}^n} m^{-1}(\xi)\langle\xi\rangle^{\rho|\alpha|} \left\|\partial_\xi^\alpha a_\varepsilon(\cdot,\xi;h)\right\|_{L^p(\mathbb{R}^n)} \leq C_{\eta,|\alpha|,\rho}\, |a|_{p,m,|\alpha|}.$$

In our investigation, we deal with the following phase functions.

Definition 2.7. A real valued function $\varphi(x,\xi)$ belongs to the class $\Phi^k(\mathbb{R}^n)$, $k\in\mathbb{N}$ if $\varphi(x,\xi) \in C^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$, is positively homogeneous of degree 1 in the frequency variable ξ, and satisfies the following condition: For any pair of multi-indices α and β; $|\alpha|+|\beta|\geq k$, there exists a positive constant $C_{\alpha,\beta}$ such that

$$\sup_{(x,\,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}} |\xi|^{-1+|\alpha|}|\partial_\xi^\alpha\partial_x^\beta\varphi(x,\xi)| \leq C_{\alpha,\beta}. \tag{4}$$

This restriction is motivated by the elementary example $\varphi(x,\xi) = |\xi| + \langle x,\xi\rangle$, for which the first order ξ-derivatives are not bounded but all the derivatives of order equal or higher than 2 are bounded.

Definition 2.8. A real valued phase $\varphi \in C^2(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$ satisfies the strong non-degeneracy condition (or SND condition for short), if there exists a constant $C>0$ such that

$$\left|\det\frac{\partial^2\varphi(x,\xi)}{\partial x_j\partial\xi_k}\right| \geq C, \quad \text{for all} \quad (x,\,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}. \tag{5}$$

Example 2.9. The phase function $\varphi(x,\xi) = |\xi|+\langle x,\xi\rangle$, which appears in the solution of the wave equation, belongs to $\Phi^2(\mathbb{R}^n)$ and is well strongly non-degenerate.

Remark 2.10. The strong non-degeneracy condition is necessary to prove the global L^p-boundedness of Fourier integral operators as to be shown in the following example. In fact, suppose that $g:\mathbb{R}^n\to\mathbb{R}^n$ is a smooth diffeomorphism such that $\det g'(x)\neq 0$

for all $x \in \mathbb{R}^n$ and let $a(x,\xi) = 1 \in L^\infty S_0^m$ with $m(\xi) = 1$, $\xi \in \mathbb{R}^n$. Then, the h-FIO $T_h(1,\varphi)$, with $\varphi(x,\xi) = \langle g(x), \xi\rangle$, is merely the composition operator $f \circ g(x)$. Hence

$$\|T_h(1,\varphi)f\|_{L^p} = \|f \circ g\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(y)|^p \left|\det g'(g^{-1}(y))\right|^{-1} \mathrm{d}y\right)^{\frac{1}{p}}$$

is bounded if and only if there exists $C > 0$ such that $|\det g'(x)|^{-1} \leq C$ for all $x \in \mathbb{R}^n$. But by assumption $|\det g'(x)| = |\det \frac{\partial^2\varphi}{\partial x \partial \xi}| \geq C > 0$, which ensures the global L^p boundedness of $T_h(1,\varphi)$.

Constants in this paper will be denoted by the letter C. They do not depend on the semiclassical parameter h and their values may vary from line to line. We sometimes write $a \lesssim b$ as shorthand for $a \leq Cb$ and $[\cdot]$ designs the integral part of a real number.

3. Tools in proving L^p-boundedness

We introduce first the semiclassical Fourier transform depending on the parameter h.

Definition 3.1. The semiclassical Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ for $h > 0$ is

$$\hat{f}_h(\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle y,\xi\rangle} f(y)\,\mathrm{d}y, \quad \xi \in \mathbb{R}^n. \tag{6}$$

We have the following elementary estimates:

$$\|\hat{f}_h\|_{L^2(\mathbb{R}^n)} = (2\pi h)^{\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)}$$

and

$$\|\hat{f}_h\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)}.$$

Interpolating between these two bounds we obtain the semiclassical version of Hausdorff-Young's inequality.

Lemma 3.2 (Hausdorff-Young inequality). *Let $p, q \in \mathbb{R}$ be such that $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f \in L^p(\mathbb{R}^n)$. Then*

$$\left\|\hat{f}_h\right\|_{L^q(\mathbb{R}^n)} \lesssim h^{\frac{n}{q}} \|f_h\|_{L^p(\mathbb{R}^n)}.$$

We now recall Minkowsky's inequality, that will often be needed in the sequel.

Lemma 3.3 (Minkowsky's inequality for integrals). *Let $1 \leq p < \infty$ and f a measurable function on $\mathbb{R}_x^n \times \mathbb{R}_y^n$. Then*

$$\left\{\int_{\mathbb{R}_y^n} \left|\int_{\mathbb{R}_x^n} f(x,y)\,\mathrm{d}x\right|^p \mathrm{d}y\right\}^{\frac{1}{p}} \leq \int_{\mathbb{R}_x^n} \left\{\int_{\mathbb{R}_y^n} |f(x,y)|^p\,\mathrm{d}y\right\}^{\frac{1}{p}} \mathrm{d}x.$$

The following Schur's Lemma provides sufficient conditions for linear operators to be bounded on L^p spaces.

Lemma 3.4 (Schur's Lemma). *Suppose that a locally integrable function K in $\mathbb{R}^n \times \mathbb{R}^n$ satisfies*

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)|\,\mathrm{d}x \leq C \quad \textit{and} \quad \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)|\,\mathrm{d}y \leq C.$$

Then the integral operator with kernel K extends to a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, for $1 \leq p \leq \infty$.

We will also need the so-called Seeger-Sogge-Stein partition of unity, see [21, 22]. First, let $\Psi_0 \in C_0^\infty(B(0,2))$, where $B(0,2)$ is the closed ball centered at the origin with radius 2, and let $\Psi \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function such that $\operatorname{supp}\Psi \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. We recall the Littlewood-Paley partition of unity, that is, for suitable Ψ_0 and Ψ with the stated properties, $\Psi_0(\xi) + \sum_{j=1}^{\infty} \Psi_j(\xi) = 1$, where $\Psi_j(\xi) = \Psi(2^{-j}\xi)$.

To get useful estimates for the symbol and the phase function, one imposes a second decomposition, superimposed on the first. Roughly speaking, each dyadic shell $2^{j-1} \leq |\xi| \leq 2^{j+1}$ is partitioned into thin truncated cones of thickness roughly $2^{\frac{j}{2}}$. Each such truncated cone is essentially an elongated rectangle, whose major side has length $\sim 2^j$ while all the other sides have length $\sim 2^{\frac{j}{2}}$.

More precisely, for each positive integer j, consider a (roughly) equally spaced set of collections $\{\xi_j^\nu\}_{\nu=1}^{J_j}$ of unit vectors with grid length $2^{-\frac{j}{2}}$ on the unit sphere $\mathbb{S}^{n-1}$. That is, we fix a collection $\{\xi_j^\nu\}_{\nu=1}^{J_j}$ of unit vectors that satisfy:

1. $|\xi_j^\nu - \xi_j^\mu| \geq 2^{-\frac{j}{2}}$, if $\nu \neq \mu$, $\nu, \mu = 1, \cdots, J_j$;

2. If $\xi \in \mathbb{S}^{n-1}$, then there exists a ξ_j^ν such that $|\xi - \xi_j^\nu| < 2^{-\frac{j}{2}}$.

To do this, take a maximal collection $\{\xi_j^\nu\}_{\nu=1}^{J_j}$ for which (1) holds. Note that there are roughly $2^{\frac{(n-1)j}{2}}$ such elements in the collection $\{\xi_j^\nu\}_{\nu=1}^{J_j}$.

Let Γ_j^ν denote the corresponding cone in the ξ space whose central direction is ξ_j^ν i.e.

$$\Gamma_j^\nu = \left\{ \xi : \left| \frac{\xi}{|\xi|} - \xi_j^\nu \right| \leq 2 \cdot 2^{-\frac{j}{2}} \right\}.$$

One can construct an associated partition of unity given by functions χ_j^ν, $\nu = 1, \cdots, J_j$, each one homogeneous of degree 0 in ξ and supported in Γ_j^ν, with

$$\sum_{\nu=1}^{J_j} \chi_j^\nu(\xi) = 1 \text{ for all } \xi \neq 0 \text{ and all } j,$$

and
$$|\partial_\xi^\alpha \chi_j^\nu(\xi)| \leq C_\alpha 2^{\frac{|\alpha| j}{2}} |\xi|^{-|\alpha|}. \tag{7}$$

If one chooses axis in the ξ space so that $\mathbb{R}^n = \mathbb{R}\xi_j^\nu \oplus {\xi_j^\nu}^\perp$, that is, $\xi = \xi_1 \xi_j^\nu + \xi'$, $\xi' = (\xi_2, \cdots, \xi_n)$ is perpendicular to ξ_j^ν, we have an improvement if we differentiate in the ξ_1 direction,

$$|\partial_{\xi_1}^N \chi_j^\nu(\xi)| \leq C_N |\xi|^{-N}, \quad N \geq 1. \tag{8}$$

Using Ψ_j's and χ_j^ν's, we can construct a Littlewood-Paley partition of unity $\Psi_0(\xi)+\sum_{j=1}^{\infty}\sum_{\nu=1}^{J_j}\chi_j^\nu(\xi)\Psi_j(\xi)=1$. Therefore, the h-FIO $T_h(a,\varphi)$, defined in (2), can be decomposed into a low-frequency part T_0, and high frequency parts $T_h^{j,\nu}$, $j=1,2,\cdots,\nu=1,\cdots,J_j$, as follows:

$$\begin{aligned} T_h(a,\varphi)f(x) &= T_h^0 f(x)+\sum_{j=1}^{\infty}\sum_{\nu=1}^{J_j}T_h^{j,\nu}f(x) \qquad (9)\\ &= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\varphi(x,\xi)}a(x,\xi;h)\Psi_0(\xi)\hat{f}_h(\xi)\,\mathrm{d}\xi \\ &+\ \frac{1}{(2\pi h)^n}\sum_{j=1}^{\infty}\sum_{\nu=1}^{J_j}\int_{\mathbb{R}^n} e^{\frac{i}{h}\varphi(x,\xi)}a(x,\xi;h)\Psi_j(\xi)\chi_j^\nu(\xi)\hat{f}_h(\xi)\,\mathrm{d}\xi. \end{aligned}$$

Recalling the identity $\varphi(x,\xi)-\langle y,\xi\rangle=\langle\nabla_\xi\varphi(x,\xi_j^\nu)-y,\xi\rangle+[\varphi(x,\xi)-\langle\nabla_\xi\varphi(x,\xi_j^\nu),\xi\rangle]$, define $\Phi(x,\xi)=\varphi(x,\xi)-\langle\nabla_\xi\varphi(x,\xi_j^\nu),\xi\rangle$. By Euler's homogeneity formula, we have $\Phi(x,\xi)=\langle\nabla_\xi\varphi(x,\xi)-\nabla_\xi\varphi(x,\xi_j^\nu),\xi\rangle$.

It is proved in [22] that, for $N\geq 1$, Φ satisfies

$$|\partial_{\xi_1}^N\Phi(x,\xi)|\leq C_N\,2^{-Nj},\qquad\text{and}\qquad|(\nabla_{\xi'})^N\Phi(x,\xi)|\leq C_N\,2^{-\frac{Ni}{2}}. \qquad (10)$$

Now let us consider

$$A_j^\nu(x,\xi;h)=e^{\frac{i}{h}\Phi(x,\xi)}a(x,\xi;h)\chi_j^\nu(\xi)\Psi_j(\xi). \qquad (11)$$

Using these, we can write $T_h^{j,\nu}$ as an h-FIO with a linear phase function in ξ. Let $f\in\mathcal{S}(\mathbb{R}^n)$ and $0<h\leq 1$. Then

$$T_h^{j,\nu}f(x)=\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}e^{\frac{i}{h}\langle(\nabla_\xi\varphi)(x,\xi_j^\nu),\xi\rangle}A_j^\nu(x,\xi;h)\hat{f}_h(\xi)\,\mathrm{d}\xi.$$

LEMMA 3.5. *Let $a\in L^pS_\rho^m(\mathbb{R}^n)$ and $\varphi\in\Phi^2(\mathbb{R}^n)$ and suppose that m is a weight function. Then for every multi-index $\alpha=(\alpha_1,\alpha')\in\mathbb{N}^n$ the symbol A_j^ν satisfies the estimate*

$$\left\|\partial_\xi^\alpha A_j^\nu(\cdot,\xi;h)\right\|_{L^p(\mathbb{R}^n)}\leq C_\alpha\,h^{-|\alpha|}\left(\sup_{\mathrm{supp}_\xi A_j^\nu}m\right)\,2^{(-|\alpha|\rho+|\alpha'|/2)j},\ \xi\in\mathbb{R}^n,\ 0<h\leq 1.$$

Proof. The proof is a direct application of Leibniz's rule, the fact that $a\in L^pS_\rho^m(\mathbb{R}^n)$ and equations (7), (8) and (10). □

The procedure described above proves Lemma 3.6, which shows that the analysis of the boundedness of $T_h(a,\varphi)$ is reduced to checking the continuity properties of a finite sum of h-FIOs with phase functions given by a linear term plus an element of $\Phi^1(\mathbb{R}^n)$. The proof is the same as of [8, Lemma 1.10].

LEMMA 3.6. *Let $T_h(a,\varphi)$ be the h-FIO defined in (2), $0<h\leq 1$, with symbol $a\in L^pS_\rho^m(\mathbb{R}^n)$, and phase function $\varphi\in\Phi^2(\mathbb{R}^n)$. Then $T_h(a,\varphi)$ can be written as a finite sum of h-FIOs of the form*

$$\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}e^{\frac{i}{h}(\psi_k(x,\xi)+\langle\nabla_\xi\varphi(x,\tau_k),\xi\rangle)}a_k(x,\xi;h)\,\hat{f}_h(\xi)\,\mathrm{d}\xi,\quad f\in\mathcal{S}(\mathbb{R}^n), \qquad (12)$$

where τ_k *is a point on the unit sphere* $\mathbb{S}^{n-1}$, $a_k \in L^pS^m_\rho(\mathbb{R}^n)$, $\psi_k \in \Phi^1(\mathbb{R}^n)$ *and* $k = 1, \cdots, J_k$.

In fact, the authors of [8] localized the symbol $a(x,\xi)$ on the frequency variable ξ around a point on the unit sphere $\mathbb{S}^{n-1}$. Namely, in view of the compactness of $\mathbb{S}^{n-1}$, they introduce a finite covering of the unit sphere. Then they showed, using properties of φ, that the phase φ can be reduced to a linear term plus a phase in $\Phi^1(\mathbb{R}^n)$. Hence the claimed result.

The next lemma is an application of [8, Lemma 1.17].

Lemma 3.7. *Let* $\eta \in C_0^\infty(\mathbb{R}^n)$, $\psi \in \Phi^1(\mathbb{R}^n)$ *and set, for* $x, z \in \mathbb{R}^n$,

$$K(x,z) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} \eta(\xi) e^{\frac{i}{h}(\psi(x,\xi)+\langle z,\xi\rangle)} \mathrm{d}\xi.$$

Then, for any $\mu \in (0,1)$, *there exists* $C > 0$ *such that*

$$|K(x,z)| \le C\, h^{-n}\left(1+\frac{|z|}{h}\right)^{-n-\mu}, \quad x, z \in \mathbb{R}^n,\ h \in (0,1].$$

Proof. Let $b(x,\xi) = \eta(\xi)e^{\frac{i}{h}\psi(x,\xi)}$. Since $\psi \in \Phi^1(\mathbb{R}^n)$, we can verify by Leibniz's rule that b satisfies the hypothesis of [8, Lemma 1.17], i.e.

$$\sup_{\xi \in \mathbb{R}^n\setminus 0} h^{|\alpha|}|\xi|^{|\alpha|-1}\left\|\partial_\xi^\alpha b(\cdot,\xi)\right\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

So, for any $\mu \in (0,1)$, there exists $C > 0$ such that

$$|K(x,z)| = \frac{1}{(2\pi h)^n}\left|\int_{\mathbb{R}^n} b(x,\xi)e^{\frac{i}{h}\langle z,\xi\rangle}\,\mathrm{d}\xi\right| \le C\,h^{-n}\left(1+\frac{|z|}{h}\right)^{-n-\mu},\ x,z\in\mathbb{R}^n. \qquad \square$$

4. L^p-boundedness of h-Fourier integral operators

In this section we give the detailed proof of Theorem 1.1. We decompose $T_h = T_h(a,\varphi)$ as in (9), in the form $T_h = T_h^0 + \sum_{j=1}^\infty T_h^j = T_h^0 + \sum_{j=1}^\infty \sum_{\nu=1}^{J_j} T_h^{j,\nu}$. In the next theorem we first establish the boundedness of the low frequency term T_h^0. To this aim, it is, of course, enough to consider a h-FIO $T_h(a_0,\varphi)$ of the type defined in (2), with a symbol $a_0(x,\xi;h)$ compactly supported in the frequency variable ξ. We then conclude the proof of Theorem 1.1 focusing on the high frequency terms.

Theorem 4.1. *Let* $0 \le \rho \le 1$, $0 < r \le \infty$, $1 \le p, q \le \infty$ *verify* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *Suppose that* $\varphi \in \Phi^2(\mathbb{R}^n)$ *satisfies the SND condition,* $a_0 \in L^pS^m_\rho(\mathbb{R}^n)$ *such that* m *is bounded on* $\mathbb{R}^n$ *and* $\mathrm{supp}_\xi\, a_0(x,\xi;h)$ *is compact. Then the h-FIO (2), with* a_0 *in place of* a, *is bounded from* $L^q(\mathbb{R}^n)$ *to* $L^r(\mathbb{R}^n)$ *uniformly with respect to* $h \in (0,1]$.

Proof. Consider a closed cube Q of side length L such that $\mathrm{supp}_\xi\, a_0(x,\xi;h) \subset Int(Q)$. We extend $a_0(x,\cdot;h)_{|Q}$ periodically with period L into $\tilde{a}_0(x,\xi;h) \in C^\infty(\mathbb{R}^n_\xi)$. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}\,\eta \subset Q$ and $\eta \equiv 1$ on $\mathrm{supp}_\xi\, a_0(x,\xi;h)$, so we have

$a_0(x,\xi;h) = \tilde{a}_0(x,\xi;h)\eta(\xi)$. Expanding $\tilde{a}_0(x,\xi;h)$ in a Fourier series, we have

$$T_h(a_0,\varphi)f(x) = \sum_{k\in\mathbb{Z}^n} a_k(x;h)T_h(\eta,\varphi)f_k(x), \tag{13}$$

where
$$a_k(x;h) = \frac{1}{L^n}\int_{\mathbb{R}^n} e^{-i\frac{2\pi}{L}\langle k,\xi\rangle} a_0(x,\xi;h)\,\mathrm{d}\xi,$$

and
$$T_h(\eta,\varphi)f_k(x) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\varphi(x,\xi)}\eta(\xi)\widehat{(f_k)}_h(\xi)\,\mathrm{d}\xi,$$

with $f_k(x) = f(x - \frac{2\pi k}{L})$ for any $k \in \mathbb{Z}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. We start by proving that the operator $T_h(\eta,\varphi)$ is bounded on $L^q(\mathbb{R}^n)$, for $1 \le q \le \infty$. By Lemma 3.6 we can assume that $\varphi(x,\xi) = \psi(x,\xi) + \langle t(x),\xi\rangle$, with a smooth map $t : \mathbb{R}^n \to \mathbb{R}^n$.

For $f \in C_0^\infty(\mathbb{R}^n)$, in view of the compactness of the supports of both f and η, and of the definition of $\hat{f}_h$, we have

$$T_h(\eta,\varphi)f_k(x) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle\xi,t(x)\rangle}e^{\frac{i}{h}\psi(x,\xi)}\eta(\xi)\widehat{(f_k)}_h(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} K(x,t(x)-y)f(y)\,\mathrm{d}y,$$

where
$$K(x,z) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\eta(\xi)e^{\frac{i}{h}\langle\xi,z\rangle}e^{\frac{i}{h}\psi(x,\xi)}\mathrm{d}\xi. \tag{14}$$

From Lemma 3.7 follows that for any $\mu \in (0,1)$, there exists a constant $C > 0$ such that

$$|K(x,z)| \le C\,h^{-n}\left(1 + \frac{|z|}{h}\right)^{-n-\mu}, \quad x,z \in \mathbb{R}^n.$$

Hence $\sup_{x\in\mathbb{R}^n}\int |K(x,t(x)-y)|\,\mathrm{d}y < \infty$. Furthermore using the change of variables $z = t(x)$, the SND condition yields that $|\det t'(x)| \ge C > 0$. Then global inverse function Theorem [20, Theorem 1.22] implies that t is a global diffeomorphism on $\mathbb{R}^n$ and the Jacobian of the change of variables $J(z)$ satisfies $|\det J(z)| \le 1/C$. Therefore using also (14), with $\mu \in (0,1)$ we get

$$\begin{aligned}\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,t(x)-y)|\,\mathrm{d}x &= \sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}\left|K(t^{-1}(z),z-y)\right|\,|\det J(z)|\,\mathrm{d}z \\ &\le \frac{1}{C}\,h^{-n}\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}\left(1+\frac{|z-y|}{h}\right)^{-n-\mu}\mathrm{d}z < \infty.\end{aligned}$$

Thus Schur's Lemma yields that $T_h(\eta,\varphi)$ is bounded on L^q for all $1 \le q \le \infty$.

Now, taking $l = 1,\cdots,n$ such that $|k_l| \ne 0$ and integrating by parts we have

$$a_k(x;h) = \frac{c_{n,N,L}}{|k_l|^N}\int_{\mathbb{R}^n}\partial_{\xi_l}^N a_0(x,\xi;h)e^{-i\frac{2\pi}{L}\langle k,\xi\rangle}\mathrm{d}\xi.$$

The hypothesis on the symbol $a_0(x,\xi;h)$ and Lemma 2.6 yield

$$\max_{s=0,\cdots,N}\int_{\mathbb{R}^n}\left\|\partial_{\xi_l}^s a_0(\cdot,\xi;h)\right\|_{L^p(\mathbb{R}^n)}\mathrm{d}\xi \le c_{n,N,L,\rho}\,|a_0|_{p,m,N}\int_{\mathbb{R}^n} m(\xi)\langle\xi\rangle^{-\rho|\alpha|}\mathrm{d}\xi.$$

Using the definition of the weight m, there exists $l \in \mathbb{R}$ such that

$$\max_{s=0,\cdots,N}\int_{\mathbb{R}^n}\left\|\partial_{\xi_l}^s a_0(\cdot,\xi;h)\right\|_{L^p(\mathbb{R}^n)}\mathrm{d}\xi \le c_{n,N,L,\rho}\,|a_0|_{p,m,N}\,m(0)\int_{\mathbb{R}^n}\lambda_l(\xi)\langle\xi\rangle^{-\rho|\alpha|}\mathrm{d}\xi.$$

The last integral is finite if $N > n + l$. Thus

$$\|a_k(\cdot;h)\|_{L^p(\mathbb{R}^n)} \le c_{n,N,L,\rho}\, |a_0|_{p,m,N}\, (1+|k|)^{-N}. \tag{15}$$

It remains to prove the boundedness of the h-FIO (2). So, assume first that $r \ge 1$. Then by (13), using Minkowsky and Hölder inequalities, one has

$$\begin{aligned}\|T_h(a_0,\varphi)f\|_{L^r(\mathbb{R}^n)} &\le \sum_{k\in\mathbb{Z}^n} \|a_k(\cdot;h)T_h(\eta,\varphi)f_k\|_{L^r(\mathbb{R}^n)}\\ &\le \sum_{k\in\mathbb{Z}^n} \|a_k(\cdot;h)\|_{L^p(\mathbb{R}^n)}\, \|T_h(\eta,\varphi)f_k\|_{L^q(\mathbb{R}^n)}\,. \end{aligned} \tag{16}$$

Since the translations are isometries on $L^q(\mathbb{R}^n)$, we have $\|T_h(\eta,\varphi)f_k\|_{L^q(\mathbb{R}^n)} \le c_{\eta,\varphi}\, \|f\|_{L^q(\mathbb{R}^n)}$. Thus using (15) we obtain

$$\|T_h(a_0,\varphi)f\|_{L^r(\mathbb{R}^n)} \lesssim |a_0|_{p,m,N}\, \|f\|_{L^q(\mathbb{R}^n)} \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-N} \lesssim \|f\|_{L^q(\mathbb{R}^n)}\,.$$

Assume now that $0 < r < 1$. If $N > \frac{n}{r}$, the equation (13) and Hölder's inequality yield

$$\begin{aligned}\int_{\mathbb{R}^n} |T_h(a_0,\varphi)f(x)|^r \mathrm{d}x &\le \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |T_h(\eta,\varphi)f_k(x)|^r\, |a_k(x;h)|^r \mathrm{d}x\\ &\le \sum_{k\in\mathbb{Z}^n} \|a_k(\cdot;h)\|^r_{L^p(\mathbb{R}^n)}\, \|T_h(\eta,\varphi)f_k(x)\|^r_{L^q(\mathbb{R}^n)}\\ &\lesssim |a_0|^r_{p,m,N} \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-Nr}\, \|f\|^r_{L^q(\mathbb{R}^n)} \lesssim ||f||^r_{L^q(\mathbb{R}^n)}. \end{aligned}$$

Finally, for $N = [\max\{n+l, \frac{n}{r}\}]+1$, $\|T_h(a_0,\varphi)f\|_{L^r(\mathbb{R}^n)} \lesssim ||f||_{L^q(\mathbb{R}^n)}$, completing the proof. □

We prove now the boundedness of the high frequency part.

Proof (Proof of Theorem 1.1). We shall consider that $q < \infty$. The case $q = \infty$ is proved with minor modifications of the first one. By Theorem 4.1, T_0 satisfies the desired boundedness, so, we confine ourselves to the analysis of the high frequency part. We recall that

$$\begin{aligned} T_h^{j,\nu} f(x) =& \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle(\nabla_\xi\varphi)(x,\xi_j^\nu),\xi\rangle} A_j^\nu(x,\xi;h)\hat{f}(\xi)\,\mathrm{d}\xi, \quad f\in\mathcal{S}(\mathbb{R}^n)\\ =& \int_{\mathbb{R}^n} K_j^\nu(x,(\nabla_\xi\varphi)(x,\xi_j^\nu)-y;h) f(y)\,\mathrm{d}y, \end{aligned}$$

where $\quad K_j^\nu(x,z;h) = \dfrac{1}{(2\pi h)^n} \displaystyle\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle z,\xi\rangle} A_j^\nu(x,\xi;h)\,\mathrm{d}\xi.$

Consider the differential operator L defined by $L = I - 2^{2j}\frac{\partial^2}{\partial\xi_1^2} - 2^j\Delta_{\xi'}$. According to Lemma 3.5, for every $N\in\mathbb{N}$ and for all $0 < h \le 1$, we have

$$\left\|L^N A_j^\nu(\cdot,\xi)\right\|_{L^p(\mathbb{R}^n)} \le C_N\, h^{-2N}\, 2^{2Nj(1-\rho)} \left(\sup_{\mathrm{supp}_\xi A_j^\nu} m\right).$$

Set $t_j^\nu(x) = (\nabla_\xi\varphi)(x,\xi_j^\nu)$. As before, the SND condition on the phase function yields that $|\det Dt_j^\nu(x)| \geq C > 0$. Let $g(y) = h^{-2}2^{2j}y_1^2 + h^{-2}2^j|y'|^2$. It follows $L^N e^{\frac{i}{h}\langle y,\xi\rangle} = (1+g(y))^N e^{\frac{i}{h}\langle y,\xi\rangle}$ for all integers N. Now, we write

$$\begin{aligned}\mathbf{I}_j^1 + \mathbf{I}_j^2 &= \sum_{\nu=1}^{J_j}\left(\int_{\sqrt{g(y)}\leq 2^{-j\rho}} + \int_{\sqrt{g(y)}>2^{-j\rho}}\right)|K_j^\nu(x,y;h)f(t_j^\nu(x)-y)|\,\mathrm{d}y \\ &= \sum_{\nu=1}^{J_j}\int_{\mathbb{R}^n}|K_j^\nu(x,y;h)f(t_j^\nu(x)-y)|\,\mathrm{d}y.\end{aligned}$$

Hölder's inequality in ν and y simultaneously and thereafter, since $1 \leq s \leq 2$, the Hausdorff-Young inequality in the y variable of the second integral yield

$$\begin{aligned}\mathbf{I}_j^1 &\leq \left[\sum_{\nu=1}^{J_j}\int_{\sqrt{g(y)}\leq 2^{-j\rho}}|f(t_j^\nu(x)-y)|^s\mathrm{d}y\right]^{\frac{1}{s}}\left[\sum_{\nu=1}^{J_j}\int|K_j^\nu(x,y;h)|^{s'}\mathrm{d}y\right]^{\frac{1}{s'}} \\ &\lesssim h^{-\frac{n}{s}}\left[\sum_{\nu=1}^{J_j}\int_{\sqrt{g(y)}\leq 2^{-j\varrho}}\left|f(t_j^\nu(x)-y)\right|^s\mathrm{d}y\right]^{\frac{1}{s}}\left[\sum_{\nu=1}^{J_j}\left(\int|A_j^\nu(x,\xi)|^s\,\mathrm{d}\xi\right)^{\frac{s'}{s}}\right]^{\frac{1}{s'}}.\end{aligned}$$

Set $F_j^\nu(x,y) = f(t_j^\nu(x)-y)$, raise the expression in the estimate of $\mathbf{I}_j^1$ to the r-th power and integrate in x. Hence, Hölder's inequality implies that

$$\begin{aligned}\left\|\mathbf{I}_j^1\right\|_{L^r(\mathbb{R}^n)} &\leq h^{-\frac{n}{s}}\left[\int_{\mathbb{R}^n}\left(\sum_{\nu=1}^{J_j}\int_{\sqrt{g(y)}\leq 2^{-j\rho}}\left|F_j^\nu(x,y)\right|^s\mathrm{d}y\right)^{\frac{q}{s}}\mathrm{d}x\right]^{\frac{1}{q}} \\ &\times\left\{\int_{\mathbb{R}^n}\left[\sum_{\nu=1}^{J_j}\left(\int|A_j^\nu(x,\xi;h)|^s\,\mathrm{d}\xi\right)^{\frac{s'}{s}}\right]^{\frac{p}{s'}}\mathrm{d}x\right\}^{\frac{1}{p}}. \qquad (17)\end{aligned}$$

We shall estimate the two factors of the right hand side of (17) separately. Minkowsky's integral inequality in y and ν yields

$$\begin{aligned}\left[\sum_{\nu=1}^{J_j}\int_{\sqrt{g(y)}\leq 2^{-j\rho}}\left(\int\left|F_j^\nu(x,y)\right|^q\mathrm{d}x\right)^{\frac{s}{q}}\mathrm{d}y\right]^{\frac{1}{s}} &\leq \left[\sum_{\nu=1}^{J_j}\int_{\sqrt{g(y)}\leq 2^{-j\rho}}\mathrm{d}y\right]^{\frac{1}{s}}\|f\|_{L^q(\mathbb{R}^n)} \\ &\lesssim h^{\frac{n}{s}}2^{\frac{n-1}{2s}j}2^{-\frac{n+1}{2s}j}\left[\int_{|y|\leq 2^{-j\rho}}\mathrm{d}y\right]^{\frac{1}{s}}\|f\|_{L^q(\mathbb{R}^n)} \qquad (18) \\ &\lesssim h^{\frac{n}{s}}2^{\frac{n-1}{2s}j}2^{-\frac{n+1}{2s}j}2^{-\frac{n\rho}{s}j}\|f\|_{L^q(\mathbb{R}^n)} \lesssim h^{\frac{n}{s}}2^{-\frac{n\rho+1}{s}j}\|f\|_{L^q(\mathbb{R}^n)}.\end{aligned}$$

To estimate the second factor in the right hand side of (17), we set
$W_j = \max_{v=1,\cdots,J_j}\sup_{\mathrm{supp}_\xi A_j^\nu} m$ and consider separately two cases.

Suppose first that $p \geq s'$. By Minkowsky's inequality and taking into account that the measure of the support of A_j^ν in the frequency variable ξ is $O(2^{\frac{n+1}{2}j})$, the second factor in the right hand side of (17) is bounded by

$$\left\{\sum_{\nu=1}^{J_j}\left[\int\left(\int |A_j^\nu(x,\xi;h)|^s \mathrm{d}\xi\right)^{\frac{p}{s}}\mathrm{d}x\right]^{\frac{s'}{p}}\right\}^{\frac{1}{s'}} \leq \left\{\sum_{\nu=1}^{J_j}\left[\int\left(\int |A_j^\nu(x,\xi;h)|^p \mathrm{d}x\right)^{\frac{s}{p}}\mathrm{d}\xi\right]^{\frac{s'}{s}}\right\}^{\frac{1}{s'}}$$

$$\lesssim W_j\left(\sum_{\nu=1}^{J_j}|\mathrm{supp}_\xi A_j^\nu|^{\frac{s'}{s}}\right)^{\frac{1}{s'}} \lesssim W_j\, 2^{\frac{n+1}{2s}j}2^{\frac{n-1}{2s'}j}.$$

Consider now the case $p < s'$. The second factor on the right hand side of (17) is bounded by

$$\left\{\sum_{\nu=1}^{J_j}\int\left(\int |A_j^\nu(x,\xi;h)|^s \mathrm{d}\xi\right)^{\frac{p}{s}}\mathrm{d}x\right\}^{\frac{1}{p}} \leq \left\{\sum_{\nu=1}^{J_j}\left[\int\left(\int |A_j^\nu(x,\xi;h)|^p \mathrm{d}x\right)^{\frac{s}{p}}\mathrm{d}\xi\right]^{\frac{p}{s}}\right\}^{\frac{1}{p}}$$

$$\lesssim W_j\left(\sum_{\nu=1}^{J_j}|\mathrm{supp}_\xi A_j^\nu|^{\frac{p}{s}}\right)^{\frac{1}{p}} \lesssim W_j\, 2^{\frac{n+1}{2s}j}2^{\frac{n-1}{2p}j}.$$

Therefore (18) and the previous estimates give

$$\|\mathbf{I}_j^1\|_{L^r(\mathbb{R}^n)} \lesssim W_j\, 2^{j\left(-\frac{n\rho}{s}+\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)\right)}\|f\|_{L^q(\mathbb{R}^n)}, \text{ for all } h\in(0,1].$$

To deal with $\mathbf{I}_j^2$ let us take $M > \frac{n}{2s}$. By Hölder's inequality,

$$\|\mathbf{I}_j^2\|_{L^r(\mathbb{R}^n)} \leq \left[\int\left(\sum_{\nu=1}^{J_j}\int_{\sqrt{g(y)}>2^{-j\rho}}\left|F_j^\nu(x,y)\right|^s(1+g(y))^{-sM}\mathrm{d}y\right)^{\frac{q}{s}}\mathrm{d}x\right]^{\frac{1}{q}}$$

$$\times\left[\int\left(\sum_{\nu=1}^{J_j}\int|K_j^\nu(x,y;h)\,(1+g(y))^M|^{s'}\mathrm{d}y\right)^{\frac{p}{s'}}\mathrm{d}x\right]^{\frac{1}{p}}. \qquad (19)$$

Minkowsky's integral inequality yields that the first term of the right hand side of (19) is bounded by a constant times

$$\|f\|_{L^q(\mathbb{R}^n)}\left[\sum_{\nu=1}^{J_j}\int_{\sqrt{g(y)}>2^{-j\rho}}(1+g(y))^{-sM}\mathrm{d}y\right]^{\frac{1}{s}} \qquad (20)$$

$$\lesssim \|f\|_{L^q(\mathbb{R}^n)}\,h^{\frac{n}{s}}2^{\frac{n-1}{2s}j}2^{-\frac{n+1}{2s}j}\left[\int_{|y|>2^{-j\rho}}|y|^{-2sM}\mathrm{d}y\right]^{\frac{1}{s}}$$

$$\lesssim \|f\|_{L^q(\mathbb{R}^n)}\,h^{\frac{n}{s}}2^{\frac{n-1}{2s}j}2^{-\frac{n+1}{2s}j}2^{\rho(2M-\frac{n}{s})j} \lesssim \|f\|_{L^q(\mathbb{R}^n)}\,h^{\frac{n}{s}}2^{(2M\rho-\frac{1}{s}(\rho n+1))j}.$$

To estimate the second factor in (19), we consider two cases also. Let us first assume that $M \in \mathbb{N}$. So we repeat the same steps used before, we have

$$\left\{ \int \left[\sum_{\nu=1}^{J_j} \int |K_j^\nu(x,y;h)\,(1+g(y))^M|^{s'}\,\mathrm{d}y \right]^{\frac{p}{s'}} \mathrm{d}x \right\}^{\frac{1}{p}}$$

$$\leq \left\{ \int \left[\sum_{\nu=1}^{J_j} \left(\int \left|L^M A_j^\nu(x,\xi;h)\right|^s \right)^{\frac{s'}{s}} \mathrm{d}\xi \right]^{\frac{p}{s'}} \mathrm{d}x \right\}^{\frac{1}{p}}$$

$$\lesssim W_j\; h^{-\frac{n}{s}-2M} 2^{2M(1-\rho)j} 2^{\frac{n+1}{2s}j} 2^{\frac{n-1}{2\min(s',p)}j}$$

$$\lesssim W_j\; h^{-\frac{n}{s}-2M}\; 2^{(2M(1-\rho)+\frac{n+1}{2s}+\frac{n-1}{2\min(s',p)})j}. \tag{21}$$

If now M is non-integer, write M as $[M]+\{M\}$, where $\{M\}$ is the fractional part of M. Therefore Hölder's inequality with conjugate exponents $\frac{1}{\{M\}}$, $\frac{1}{1-\{M\}}$ and (21) give the same result as for the integer case.

Thus, for every $2M > \frac{n}{s}$, for every $0 < h \leq 1$, (20) and (21) yield

$$\left\|\mathbf{I}_j^2\right\|_{L^r} \lesssim W_j\; h^{-2M} 2^{j\left(2M(1-\rho)+\rho(2M-\frac{n}{s})+\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)\right)} \|f\|_{L^q},$$

$$\left\|\mathbf{I}_j^2\right\|_{L^r} \lesssim W_j\; h^{-2M} 2^{j\left(2M-\frac{n\rho}{s}+\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)\right)} \|f\|_{L^q}.$$

Now putting the estimates of $\mathbf{I}_j^1$ and $\mathbf{I}_j^2$ together and summing yields

$$\left\|T_h^j f\right\|_{L^r} \lesssim W_j \left(2^{j\left(-\frac{n\rho}{s}+\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)\right)} + h^{-2M} 2^{j\left(2M-\frac{n\rho}{s}+\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)\right)} \right) \|f\|_{L^q}.$$

Then, we obtain

$$\left\|T_h^j f\right\|_{L^r} \lesssim W_j\, h^{-2M} 2^{j\left[2M-\frac{n\rho}{s}+\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)\right]} \|f\|_{L^q}, h \in (0,1].$$

If $R = \min(r,1)$, one has

$$\left\|\sum_{j=1}^{\infty} T_h^j f\right\|_{L^r}^R \leq \sum_{j=1}^{\infty} \left\|T_h^j f\right\|_{L^r}^R$$

$$\lesssim \sum_{j=1}^{\infty} W_j^R\, h^{-2MR} 2^{jR\left[-\frac{\rho n}{s}+2M+\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)\right]} \|f\|_{L^q}^R \lesssim h^{-2MR} \|f\|_{L^q}^R,$$

in view of the main hypothesis (3), the definition (11) of A_j^ν, Lemma 3.5, and the fact that, on $\operatorname{supp}_\xi A_j^\nu$, $\langle\xi\rangle \sim 2^j$.

This completes the proof of Theorem 1.1. □

Note that if we study pseudodifferential operators (PDOs)

$$a(x,hD)f(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\frac{i}{h}\langle x,\xi\rangle} a(x,\xi) \hat{f}_h(\xi)\,\mathrm{d}\xi,\; f \in \mathcal{S}(\mathbb{R}^n), \tag{22}$$

the phase $\langle x,\xi\rangle$ is linear in ξ. In this case the Seeger-Sogge-Stein decomposition is not necessary to prove the global $L^q \to L^r$ boundedness and it suffices to use just the

Littlewood-Paley decomposition. Thus a minor modification in the proof of Theorem 1.1 allows to achieve the following result.

THEOREM 4.2. *Let* $0 \leq \rho \leq 1$, $0 < h \leq 1$, $0 < r \leq \infty$ *and* $1 \leq p, q \leq \infty$ *verify* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *Assume that* $a \in L^p S^m_\rho(\mathbb{R}^n)$ *with* $m(\xi) < C_0 \langle\xi\rangle^{\frac{n(\rho-1)}{s}}$, $C_0 > 0$, *where* $s = \min(2, p, q)$. *Then the h-PDO* $a(x, hD)$ *is bounded from* $L^q(\mathbb{R}^n)$ *to* $L^r(\mathbb{R}^n)$ *uniformly with respect to* h.

Proof. Using Littlewood-Paley decomposition described in Section 3, the PDO $a(x, hD)$ can be written as

$$a(x, hD)f(x) = \sum_{j=0}^{\infty} a_j(x, hD)f(x) = \frac{1}{(2\pi h)^n} \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} \mathrm{e}^{\frac{i}{h}\langle x,\xi\rangle} a(x, \xi)\psi_j(\xi)\,\mathrm{d}\xi.$$

As before, $a_j(x, hD)$ can be rewritten as $a_j(x, hD)f(x) = \int_{\mathbb{R}^n} K(x, y)\, f(x-y)\,\mathrm{d}y$ with

$$K(x, y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\frac{i}{h}\langle y,\xi\rangle} a_j(x, \xi)\,\mathrm{d}\xi, \quad a_j(x, \xi) = a(x, \xi)\Psi_j(\xi).$$

Consider the differential operator L defined by $L = I - 2^{2j}\Delta_\xi$, then it is easy to check that

$$L\,\mathrm{e}^{\frac{i}{h}\langle y,\xi\rangle} = (1 + h^{-2}2^{2j}|y|^2)\,\mathrm{e}^{\frac{i}{h}\langle y,\xi\rangle}.$$

Hence,

$$a_j(x, hD)f(x) = \int_{\mathbb{R}^n} K(x, y)\,(1 + h^{-2}2^{2j}|y|^2)^M\,(1 + h^{-2}2^{2j}|y|^2)^{-M}\,f(x - y)\,\mathrm{d}y.$$

To obtain the desired result, we use the same technics showed in the proof of Theorem 1.1. □

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