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ΘΓ *N*-GROUP

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Abstract. In this paper, we introduce the notion of ΘΓ N-group as a generalization of algebraic structures of N-group and gamma nearring. We present motivating examples of ΘΓ N-groups and prove classical isomorphism theorems.

1. Introduction

A nearring $(N, +, \cdot)$ is an algebraic system with binary operations addition and multiplication satisfying the axioms of a ring, except commutativity of addition and one of the distributive laws. A natural example of nearring is the set of all mappings from a group $(G, +)$ to itself under addition and composition of mappings. Groenewald [18], Veldsman [36] introduced different types of prime ideals of nearrings such as completely prime, 3-prime and equiprime ideals. Equiprime ideals gave rise to a Kurosh-Amitsur prime radical for nearrings (see [13]). Veljko [37, 38] gave definitions of nilpotency, nilty, nil-radical, nilpotent-radical and nearring homomorphism of a general (non associative and non distributive) nearring and studied its affine endomorphism. N-groups are modules over nearrings (see [33]). Juglal, Groenewald and Lee [22] introduced characterizations of prime modules of zero symmetric nearring. Groenewald, Juglal and Meyer [19] discussed relations between primeness of zero symmetric nearring and its group nearring. Nobusawa [32] introduced Γ-ring, a generalization of ring. Barnes [3] studied notions of Γ-homomorphism, prime and (right) primary ideals, m-systems, radical of an ideal in Γ-rings. Sapanci and Nakajimaz [35] gave the condition for commutative property in gamma rings. Bell and Argac [4] studied derivations, product of derivations in nearrings and obtained commutativity results under suitable conditions.

Bhavanari [7] introduced gamma nearrings, a generalization of both nearrings and gamma-rings. This concept was further studied in [5, 6, 11, 27] and several results were proved. Booth and Groenewald [12, 14] introduced equiprime gamma nearrings

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and radicals of gamma nearrings. Jun, Sapanci and Ozturk [23] studied fuzzy ideals of gamma nearrings. Bhavanari and Kuncham [10] introduced the notion of a fuzzy coset in gamma nearring and obtained related important fundamental isomorphism theorems. Booth, Groenewald and Olivier [15] defined general regularity for gamma rings and explored ways of generating such regularities. Kedukodi, Kuncham and Bhavanari [24, 25] studied equiprime, 3-prime and c-prime fuzzy ideals of nearrings. As an application of equiprime ideals, in [26] the notion of reference point in rough sets was introduced. In [8], the same authors also studied graph theoretic aspects nearrings. Jagadeesha, Kedukodi, Kuncham [21] defined interval valued L-fuzzy ideals of nearrings based on t-norms and t-conorms and in [28], they studied homomorphic images of interval valued L-fuzzy ideals and proved isomorphism theorems.

In this paper, we introduce the notion of $\Theta\Gamma$ N-group which is a generalization of N-group and gamma nearring. A $\Theta\Gamma$ N-group is an algebraic structure where the operations belonging to the set Θ satisfy the right distributive property and the quasi associative property. We place on record the starting step where the idea of $\Theta\Gamma$ N-group arose. In the real number system, we know that the operations of subtraction and division are not associative. This is unlike their respective counterparts of addition and multiplication. However, we note that the operations subtraction and division are near associative operations. Consider the abelian group $(\mathbb{R}, +)$ and take $a, b, c \in \mathbb{R}$. Corresponding to usual subtraction, we can define an operation "sub_c" by $a\ sub_c\ b = (a - 2c) - b$. Then we have $(a - b) - c = a\ sub_c(b - c)$. We name this near associativity as the quasi associative property. In Example 3.6 of this paper, we show that a similar quasi associative property is satisfied by usual division operation.

2. Preliminaries

We refer to [30, 31] for basic definitions, and for recent developments in nearrings, we refer to [29]. Computations in nearrings can be done using SONATA [1].

Definition 2.1. [33] Let $(G, +)$ be a group with additive identity 0. G is said to be an N-group if there exist a nearring $(N, +, \cdot)$, and a mapping $N \times \theta \times G \to G$ (the image $(n, g) \in N \times \theta \times G$ is denoted by $n\theta g$ where θ is an operation), satisfying $(n + m)\theta g = n\theta g + m\theta g$ and $(nm)\theta g = n\theta(m\theta g)$, for all $g \in G$ and $n, m \in N$. We denote this N-group by ${}_N G$.

Definition 2.2. [9] Let $(M, +)$ be a group (not necessarily abelian) and Γ be a non-empty set. Then M is said to be a Γ-nearring if there exists a mapping $M \times \Gamma \times M \to M$ (denote the image of (m_1, α_1, m_2) by $m_1\alpha_1 m_2$ for $m_1, m_2 \in M$ and $\alpha_1 \in \Gamma$) satisfying the following conditions:

$$(m_1 + m_2)\alpha_1 m_3 = m_1\alpha_1 m_3 + m_2\alpha_1 m_3 \quad \text{and} \quad (m_1\alpha_1 m_2)\alpha_2 m_3 = m_1\alpha_1(m_2\alpha_2 m_3),$$

for all $m_1, m_2, m_3 \in M$ and for all $\alpha_1, \alpha_2 \in \Gamma$.

Definition 2.3. [16] A nearring $(N, +, \cdot)$ is said to be non-associative if $(N, \cdot)$ is not a semigroup.

Definition 2.4. [33] Let N be a nearring and $a, b \in N$. $a \equiv b \Leftrightarrow \forall n \in N : na = nb$. N is said to be planar nearring if $|N/{\equiv}| \geq 3$ and if every equation $xa = xb + c$ $(a \neq b)$ has a unique solution (in N).

Definition 2.5. [39] A double planar nearring $(N, +, *, \cdot)$ is an ordered quadruple where each of the ordered triples $(N, +, *)$ and $(N, +, \cdot)$ is a nearring, and where $*$ and $\cdot$ are each left distributive over the other. That is, $a * (b \cdot c) = (a * b) \cdot (a * c)$ and $a \cdot (b * c) = (a \cdot b) * (a \cdot c)$, for all $a, b, c \in N$. If each of the nearrings $(N, +, *)$ and $(N, +, \cdot)$ is planar, then $(N, +, *, \cdot)$ is a double planar nearring.

For further concepts in planar nearrings we refer to [2, 40].

3. ΘΓ N-group

Definition 3.1. Let $(G, +_G)$ be a group. G is called a $\Theta\Gamma$ N-group if there exists a nearring $(N, +, \cdot)$ and there exist maps $\Theta(N \times \Theta \times G \to G)$, $\Gamma(N \times \Gamma \times N \to N)$ containing nearring multiplication $\cdot$, $\Delta_\Gamma(N \times \Delta_\Gamma \times G \to G)$ satisfying the following conditions.

1. θ is *right distributive*: $(n + m)\theta g = n\theta g +_G m\theta g$, for all $n, m \in N, g \in G, \theta \in \Theta$;

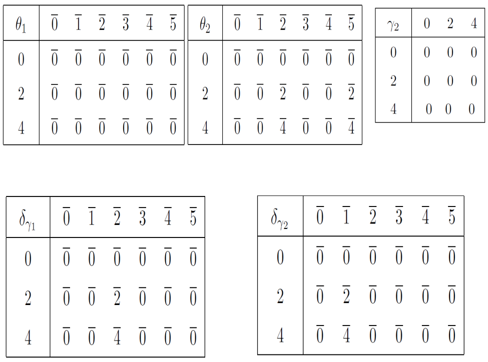
2. θ is *quasi associative*: for every $n, m \in N, \gamma \in \Gamma$, there exists $\delta_\gamma \in \Delta_\Gamma$ such that $(n\gamma m)\theta g = n\delta_\gamma(m\theta g)$, for all $g \in G$, $\theta \in \Theta$.

Example 3.2. Let $G = \mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$. Then $(\mathbb{Z}_6, +_6)$ is a group under addition modulo 6. Take a nearring $N = \{0, 2, 4\}$ with $+$ and $\cdot$ defined in Table 1. Let $\Theta = \{\theta_1, \theta_2\}$, $\Gamma = \{\gamma_1 = \cdot, \gamma_2\}$, $\Delta_\Gamma = \{\delta_{\gamma_1}, \delta_{\gamma_2}\}$ be given by the tables in Figure 1. It can be verified that $\mathbb{Z}_6$ is a $\Theta\Gamma$ N-group.

$+$	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

$\cdot$	0	2	4
0	0	0	0
2	0	2	0
4	0	4	0

Table 1: Binary operations $+$ and $\cdot$



θ_1	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
0	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
2	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
4	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$

θ_2	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
0	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
2	$\overline{0}$	$\overline{0}$	$\overline{2}$	$\overline{0}$	$\overline{0}$	$\overline{2}$
4	$\overline{0}$	$\overline{0}$	$\overline{4}$	$\overline{0}$	$\overline{0}$	$\overline{4}$

γ_2	0	2	4
0	0	0	0
2	0	0	0
4	0	0	0

δ_{γ_1}	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
0	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
2	$\overline{0}$	$\overline{0}$	$\overline{2}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
4	$\overline{0}$	$\overline{0}$	$\overline{4}$	$\overline{0}$	$\overline{0}$	$\overline{0}$

δ_{γ_2}	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
0	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
2	$\overline{0}$	$\overline{2}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
4	$\overline{0}$	$\overline{4}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$

Figure 1: $\Theta, \Gamma, \Delta_\Gamma$ from Example 3.2

Example 3.3 (Symmetries of a square). It is well known that symmetries of a square form a group known as the Dihedral group D_4. Consider a square as shown in Figure 2.

Let e denote no change in the square. Let R_1 be the rotation of the square by 90 degrees, R_2 be the rotation by 180 degrees, R_3 be the rotation by 270 degrees (all rotations in anti-clockwise direction based on the centroid). Let V be the vertical flip, H be the horizontal flip, D_1 and D_2 be the diagonal flips.

Take $G = \{e, R_1, R_2, R_3, V, H, D_1, D_2\}$. Then G is a group with the binary operation $+_G$ given in Figure 3.

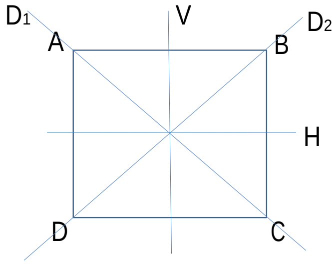


Figure 2: Symmetries of a square

$+_G$	e	R_1	R_2	R_3	V	H	D_1	D_2
e	e	R_1	R_2	R_3	V	H	D_1	D_2
R_1	R_1	R_2	R_3	e	D_2	D_1	V	H
R_2	R_2	R_3	e	R_1	H	V	D_2	D_1
R_3	R_3	e	R_1	R_2	D_1	D_2	H	V
V	V	D_1	H	D_2	e	R_2	R_1	R_3
H	H	D_2	V	D_1	R_2	e	R_3	R_1
D_1	D_1	H	D_2	V	R_3	R_1	e	R_2
D_2	D_2	V	D_1	H	R_1	R_3	R_2	e

Figure 3: Binary operation $+_G$

Let $N = (\mathbb{Z}_8, +_8, \cdot_8)$. Define $\Theta = \{\theta_1, \theta_2, \theta_3\}$, $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ and $\Delta_\Gamma = \{\delta_{\gamma_1}, \delta_{\gamma_2}, \delta_{\gamma_3}\}$ as in Figure 5. Take $\gamma_1 = \cdot_8$.

Using the tables from Figure 5, it can be verified that G is a $\Theta\Gamma$ N-group.

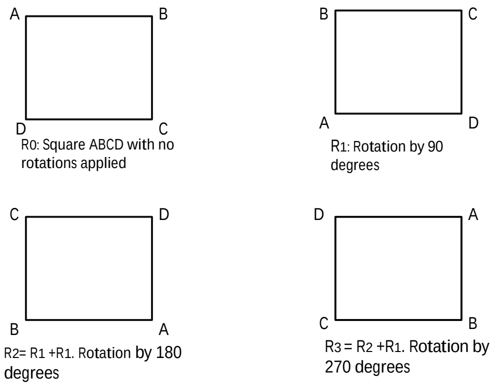


Figure 4: Some geometrical interpretations of computations

θ_1	e	R_1	R_2	R_3	V	H	D_1	D_2
$\overline{0}$	e	e	e	e	e	e	e	e
$\overline{1}$	e	R_1	R_2	R_3	V	H	D_1	D_2
$\overline{2}$	e	R_2	e	R_2	e	e	e	e
$\overline{3}$	e	R_3	R_2	R_1	V	H	D_1	D_2
$\overline{4}$	e	e	e	e	e	e	e	e
$\overline{5}$	e	R_1	R_2	R_3	V	H	D_1	D_2
$\overline{6}$	e	R_2	e	R_2	e	e	e	e
$\overline{7}$	e	R_3	R_2	R_1	V	H	D_1	D_2

θ_2	e	R_1	R_2	R_3	V	H	D_1	D_2
$\overline{0}$	e	e	e	e	e	e	e	e
$\overline{1}$	e	R_1	R_2	R_3	e	e	e	e
$\overline{2}$	e	R_2	e	R_2	e	e	e	e
$\overline{3}$	e	R_3	R_2	R_1	e	e	e	e
$\overline{4}$	e	e	e	e	e	e	e	e
$\overline{5}$	e	R_1	R_2	R_3	e	e	e	e
$\overline{6}$	e	R_2	e	R_2	e	e	e	e
$\overline{7}$	e	R_3	R_2	R_1	e	e	e	e

θ_3	e	R_1	R_2	R_3	V	H	D_1	D_2
$\overline{0}$	e	e	e	e	e	e	e	e
$\overline{1}$	e	e	e	e	V	H	D_1	D_2
$\overline{2}$	e	e	e	e	e	e	e	e
$\overline{3}$	e	e	e	e	V	H	D_1	D_2
$\overline{4}$	e	e	e	e	e	e	e	e
$\overline{5}$	e	e	e	e	V	H	D_1	D_2
$\overline{6}$	e	e	e	e	e	e	e	e
$\overline{7}$	e	e	e	e	V	H	D_1	D_2

γ_2	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$
$\overline{2}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{2}$	$\overline{5}$
$\overline{4}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{2}$	$\overline{7}$	$\overline{4}$	$\overline{1}$	$\overline{6}$	$\overline{3}$
$\overline{6}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{7}$	$\overline{0}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

γ_3	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$
$\overline{2}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{2}$	$\overline{5}$
$\overline{4}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$
$\overline{6}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{7}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{2}$	$\overline{5}$

$\delta_{\gamma_1} = \delta_{\gamma_3}$	e	R_1	R_2	R_3	V	H	D_1	D_2
$\overline{0}$	e	e	e	e	e	e	e	e
$\overline{1}$	e	R_1	R_2	R_3	V	H	D_1	D_2
$\overline{2}$	e	R_2	e	R_2	e	e	e	e
$\overline{3}$	e	R_3	R_2	R_1	V	H	D_1	D_2
$\overline{4}$	e	e	e	e	e	e	e	e
$\overline{5}$	e	R_1	R_2	R_3	V	H	D_1	D_2
$\overline{6}$	e	R_2	e	R_2	e	e	e	e
$\overline{7}$	e	R_3	R_2	R_1	V	H	D_1	D_2

δ_{γ_2}	e	R_1	R_2	R_3	V	H	D_1	D_2
$\overline{0}$	e	e	e	e	e	e	e	e
$\overline{1}$	e	R_1	R_2	R_3	V	H	D_1	D_2
$\overline{2}$	e	e	e	e	e	e	e	e
$\overline{3}$	e	R_3	R_2	R_1	V	H	D_1	D_2
$\overline{4}$	e	e	e	e	e	e	e	e
$\overline{5}$	e	R_1	R_2	R_3	V	H	D_1	D_2
$\overline{6}$	e	e	e	e	e	e	e	e
$\overline{7}$	e	R_3	R_2	R_1	V	H	D_1	D_2

Figure 5: $\Theta, \Gamma, \Delta_\Gamma$ from Example 3.3

Consider $(\overline{5}\gamma_1\overline{7})\theta_1 R_3 +_G (\overline{3}\gamma_1\overline{2})\theta_1 R_3$. This expression is equal to $\overline{3}\theta_1 R_3 +_G \overline{6}\theta_1 R_3$. Note that $\overline{3}\theta_1 R_3$ is the rotation of the square $ABCD$ by 270 degrees in anti-clockwise direction repeated three times, which yields R_1. Similarly, we obtain $\overline{6}\theta_1 R_3 = R_2$. Then $\overline{3}\theta_1 R_3 +_G \overline{6}\theta_1 R_3 = R_1 +_G R_2 = R_3$.

Now let us consider $(\overline{5}\gamma_1\overline{7} +_8 \overline{3}\gamma_1\overline{2})\theta_1 R_3 = (\overline{3} +_8 \overline{6})\theta_1 R_3 = \overline{1}\theta_1 R_3 = R_3$. Thus we get, $(\overline{5}\gamma_1\overline{7} +_8 \overline{3}\gamma_1\overline{2})\theta_1 R_3 = (\overline{5}\gamma_1\overline{7})\theta_1 R_3 +_G (\overline{3}\gamma_1\overline{2})\theta_1 R_3$.

Remark 3.4. Under similar operations, we can show that the group formed by symmetries of an equilateral triangle is a $\Theta\Gamma$ N-group.

To give the next example of ΘΓ N-group, we require some basic definitions and notations from lambda calculus. The lambda calculus is a theory of functions as formulas. In this system functions are written as expressions. Lambda calculus was introduced by Alonzo Church [17] in 1936 to formalize the concept of effective computability. We refer to [20] for the following definitions. The set of λ-terms (notation Λ) is built up from an infinite set of variables $V = \{v, v^{'}, v^{"}, \ldots\}$ using application and (function) abstraction:

$$x \in V \to x \in \Lambda, \quad M, N \in \Lambda \Rightarrow (MN) \in \Lambda, \quad M \in \Lambda, x \in V \Rightarrow (\lambda x M) \in \Lambda,$$

where M and N are expressions.

If f and x are lambda terms, and $n > 0$ a natural number, write $f^n x$ for the term $f(f(\ldots(fx)\ldots))$, where f occurs n times. For each natural number n, we define a lambda term $\overline{n}$, called the n-th Church numeral, as $\overline{n} = \lambda fx.f^n x$. Here are the first few Church numerals:

$$\overline{0} = \lambda fx.x, \quad \overline{1} = \lambda fx.fx, \quad \overline{2} = \lambda fx.f(fx), \quad \overline{3} = \lambda fx.f(f(fx)), \ \ldots$$

The successor function in [34] is defined as $+ \equiv \lambda wyx.y(wyx)$ and the product function is defined as $*_1 \equiv (\lambda xyz.x(yz))$.

Example 3.5. Let $G = \{\overline{0}, \overline{1}, \overline{2}, \ldots\}$. We will form a ΘΓ N-group from G. First, we prove that $(G, *_1)$ is a semigroup.

We show that, $(\overline{a} + \overline{b}) *_1 \overline{c} = \overline{a} *_1 \overline{c} + \overline{a} *_1 \overline{b}$.

Let $\overline{a}, \overline{b}, \overline{c} \in G$. We claim that $(\overline{a} + \overline{b}) = \lambda yx.y^{a+b}x$. We have

$$\begin{aligned}
\overline{a} + \overline{b} = \lambda sh.s^a hS\lambda sh.s^b h &= (\lambda sh.s^a h)(\lambda wyx.y(wyx))(\lambda sh.s^b h)\\
&= (a\lambda wyx.y(wyx))(\lambda sh.s^b h)\\
&= ((a-1)\lambda wyx.y(wyx))\lambda wyx.y(wyx)(\lambda sh.s^b h)\\
&= ((a-1)\lambda wyx.y(wyx))(\lambda yx.y(\lambda sh.s^b h)yx)\\
&= ((a-1)\lambda wyx.y(wyx))(\lambda yx.y(y^b(x)))\\
&= ((a-1)\lambda wyx.y(wyx))(\lambda yx.y^{b+1}(x)).
\end{aligned}$$

Continuing, we get $\overline{a} + \overline{b} = (\lambda wyx.y(wyx))(\lambda yx.y^{a+b-1}x)$, and operating once more $\overline{a} + \overline{b} = \lambda yx.y^{a+b}x$. Clearly, we have $\overline{a} + \overline{b} = \overline{b} + \overline{a}$.

Now we claim that $(\overline{a} + \overline{b}) *_1 \overline{c} = \lambda zh.z^{(a+b)c}h$. We have

$$\begin{aligned}
(\overline{a} + \overline{b}) *_1 \overline{c} &= (\lambda sh.s^{a+b}h).(\lambda sh.s^c h) = (\lambda z.(a+b)(cz))\\
&= \lambda z.(\lambda sh.s^{a+b}h)((\lambda sh.s^c h)z = \lambda z.(\lambda sh.s^{a+b}h)(\lambda h.z^c h)\\
&= \lambda z.(\lambda h.(a+b)\lambda h.z^c(h)h) = \lambda z.(\lambda h.z^{(a+b)c}(h)) = \lambda zh.z^{(a+b)c}h \qquad (1)
\end{aligned}$$

Now, consider

$$\begin{aligned}
\overline{a} *_1 \overline{c} + \overline{b} *_1 \overline{c} &= \lambda zh.z^{(ac)}h + \lambda zh.z^{(bc)}h\\
&= (\lambda zh.z^{(ac)}h)(\lambda wyx.y(wyx))(\lambda zh.z^{(bc)}h)\\
&= (ac)(\lambda wyx.y(wyx))(\lambda zh.z^{(bc)}h)\\
&= (ac-1)(\lambda wyx.y(wyx))(\lambda wyx.y(wyx))(\lambda zh.z^{(bc)}h)
\end{aligned}$$

$$
\begin{aligned}
&= (ac-1)(\lambda wyx.y(wyx))(\lambda yx.y(\lambda zh.z^{(bc)}h)yx) \\
&= (ac-1)(\lambda wyx.y(wyx))(\lambda yx.y(y^{(bc)}(x))) \\
&= (ac-1)(\lambda wyx.y(wyx))(\lambda yx.y^{(bc+1)}(x)).
\end{aligned}
$$

Continuing, we get $\overline{a} *_1 \overline{c} + \overline{b} *_1 \overline{c} = (\lambda wyx.y(wyx))(\lambda yx.y^{(bc+ac-1)}(x))$ and operating once more

$$\overline{a} *_1 \overline{c} + \overline{b} *_1 \overline{c} = (\lambda yx.y^{(bc+ac)}(x)) = (\lambda yx.y^{(a+b)c}(x)). \tag{2}$$

Thus by (1) and (2) we get $(\overline{a} + \overline{b}) *_1 \overline{c} = \overline{a} *_1 \overline{c} + \overline{b} *_1 \overline{c}$. Similarly, we get $\overline{a} *_1 (\overline{b} + \overline{c}) = \overline{a} *_1 \overline{b} + \overline{a} *_1 \overline{c}$.

Now we prove that $*_1$ is associative. We have, $(\overline{a} *_1 \overline{b}) *_1 \overline{c} = (\lambda zh.z^{(ab)}h) *_1 (\lambda zh.z^{(c)}h) = \lambda zh.z^{(abc)}h$ and $\overline{a} *_1 (\overline{b} *_1 \overline{c}) = (\lambda zh.z^{(a)}h) *_1 (\lambda zh.z^{(bc)}h) = \lambda zh.z^{(abc)}h$. Hence $(\overline{a} *_1 \overline{b}) *_1 \overline{c} = \overline{a} *_1 (\overline{b} *_1 \overline{c})$.

Now, we show that G can be extended to a $\Theta\Gamma$ N-group. To obtain this, Church pair can be used which is formed by extending Church Numerals to signed numbers. A Church pair contains Church numerals representing a positive and a negative value. Let $\hat{\mathrm{G}}$ denote the set of signed numbers. On the set $\hat{\mathrm{G}}$, addition and subtraction are naturally defined as follows:

$$
\begin{aligned}
x + y &= [x_p, x_n] + [y_p, y_n] = [x_p + y_p, x_n + y_n], \\
x - y &= [x_p, x_n] - [y_p, y_n] = [x_p + y_n, x_n + y_p].
\end{aligned}
$$

Define $\circledast_1$ as $x \circledast_1 y = [x_p, x_n] \circledast_1 [y_p, y_n] = [x_p *_1 y_p + x_p *_1 y_n, x_n *_1 y_p + x_n *_1 y_n]$, for all $x, y \in \hat{\mathrm{G}}$. Note that ($\hat{\mathrm{G}}$, $+$) is a group. We will show that N=($\hat{\mathrm{G}}$,+, $\circledast_1$) is a nearring. We prove that $(x + y) \circledast_1 z = x \circledast_1 z + y \circledast_1 z$. We have

$$
\begin{aligned}
(x + y) \circledast_1 z &= [x_p + y_p, x_n + y_n] \circledast_1 [z_p, z_n] \\
&= [(x_p + y_p) *_1 z_p + (x_p + y_p) *_1 z_n, \\
&\quad (x_n + y_n) *_1 z_p + (x_n + y_n) *_1 z_n], \qquad (3) \\
x \circledast_1 z + y \circledast_1 z &= [x_p *_1 z_p + x_p *_1 z_n, x_n *_1 z_p + x_n *_1 z_n] \\
&\quad + [y_p *_1 z_p + y_p *_1 z_n, y_n *_1 z_p + y_n *_1 z_n] \\
&= [x_p *_1 z_p + x_p *_1 z_n + y_p *_1 z_p + y_p *_1 z_n, \\
&\quad x_n *_1 z_p + x_n *_1 z_n + y_n *_1 z_p + y_n *_1 z_n] \\
&= [(x_p + y_p) *_1 z_p + (x_p + y_p) *_1 z_n, \\
&\quad (x_n + y_n) *_1 z_p + (x_n + y_n) *_1 z_n]. \qquad (4)
\end{aligned}
$$

From (3) and (4) we get $(x + y) \circledast_1 z = x \circledast_1 z + y \circledast_1 z$.

Now, we prove that $(x \circledast_1 y) \circledast_1 z = x \circledast_1 (y \circledast_1 z)$. We have

$$
\begin{aligned}
(x \circledast_1 y) \circledast_1 z &= [x_p *_1 y_p + x_p *_1 y_n, x_n *_1 y_p + x_n *_1 y_n] *_1 [z_p, z_n] \\
&= [(x_p *_1 y_p + x_p *_1 y_n) *_1 z_p + (x_p *_1 y_p + x_p *_1 y_n) *_1 z_n, \\
&\quad (x_n *_1 y_p + x_n *_1 y_n) *_1 z_p + (x_n *_1 y_p + x_n *_1 y_n) *_1 z_n] \\
&= [x_p *_1 y_p *_1 z_p + x_p *_1 y_n *_1 z_p + x_p *_1 y_p *_1 z_n + x_p *_1 y_n *_1 z_n, \\
&\quad x_n *_1 y_p *_1 z_p + x_n *_1 y_n *_1 z_p + x_n *_1 y_p *_1 z_n + x_n *_1 y_n *_1 z_n]
\end{aligned}
$$

$$\begin{aligned}&= [x_p *_1 (y_p *_1 z_p + y_p *_1 z_n) + x_p *_1 (y_n *_1 z_p + y_n *_1 z_n),\\ &\quad x_n *_1 (y_p *_1 z_p + y_p *_1 z_n) + x_n *_1 (y_n *_1 z_p + y_n *_1 z_n)]\\ &= [x_p, x_n] \circledast_1 [y_p *_1 z_p + y_p *_1 z_n, y_n *_1 z_p + y_n *_1 z_n]\\ &= x \circledast_1 (y \circledast_1 z).\end{aligned}$$

Now, define $*_2 \equiv (\lambda xyz.y(xz))$ and $\circledast_2$ as

$$x \circledast_2 y = [x_p, x_n] \circledast_2 [y_p, y_n] = [x_p *_2 y_p + x_p *_2 y_n, x_n *_2 y_p + x_n *_2 y_n].$$

Take $\Theta = \{\circledast_2\}$, $\Gamma = \{\circledast_1\}$ and $\Delta_\Gamma = \{\circledast_1\}$. We will prove that $\hat{\mathrm{G}}$ is a ΘΓ N-group.

Now, we claim that:

(i) $\circledast_2$ is right distributive.

$$\begin{aligned}(x+y) \circledast_2 z &= [x_p + y_p, x_n + y_n] \circledast_2 [z_p, z_n]\\ &= [(x_p + y_p) *_2 z_p + (x_p + y_p) *_2 z_n,\\ &\quad (x_n + y_n) *_2 z_p + (x_n + y_n) *_2 z_n], \qquad (5)\end{aligned}$$

$$\begin{aligned}x \circledast_2 z + y \circledast_2 z &= [x_p *_2 z_p + x_p *_2 z_n, x_n *_2 z_p + x_n *_2 z_n]\\ &\quad + [y_p *_2 z_p + y_p *_2 z_n, y_n *_2 z_p + y_n *_2 z_n]\\ &= [x_p *_2 z_p + x_p *_2 z_n + y_p *_2 z_p + y_p *_2 z_n,\\ &\quad x_n *_2 z_p + x_n *_2 z_n + y_n *_2 z_p + y_n *_2 z_n]\\ &= [(x_p + y_p) *_2 z_p + (x_p + y_p) *_2 z_n,\\ &\quad (x_n + y_n) *_2 z_p + (x_n + y_n) *_2 z_n]. \qquad (6)\end{aligned}$$

From (5) and (6) we have $(x+y) \circledast_2 z = x \circledast_2 z + y \circledast_2 z$.

(ii) $\circledast_2$ is quasi associative.

$$\begin{aligned}(x \circledast_1 y) \circledast_2 z &= [x_p *_1 y_p + x_p *_1 y_n, x_n *_1 y_p + x_n *_1 y_n] *_2 [z_p, z_n]\\ &= [(x_p *_1 y_p + x_p *_1 y_n) *_2 z_p + (x_p *_1 y_p + x_p *_1 y_n) *_2 z_n,\\ &\quad (x_n *_1 y_p + x_n *_1 y_n) *_2 z_p + (x_n *_1 y_p + x_n *_1 y_n) *_2 z_n]\\ &= [x_p *_1 y_p *_2 z_p + x_p *_1 y_n *_2 z_p + x_p *_1 y_p *_2 z_n + x_p *_1 y_n *_2 z_n,\\ &\quad x_n *_1 y_p *_2 z_p + x_n *_1 y_n *_2 z_p + x_n *_1 y_p *_2 z_n + x_n *_1 y_n *_2 z_n]\\ &= [x_p *_1 (y_p *_2 z_p + y_p *_2 z_n) + x_p *_1 (y_n *_2 z_p + y_n *_2 z_n),\\ &\quad x_n *_1 (y_p *_2 z_p + y_p *_2 z_n) + x_n *_1 (y_n *_2 z_p + y_n *_2 z_n)]\\ &= [x_p, x_n] \circledast_1 [y_p *_2 z_p + y_p *_2 z_n, y_n *_2 z_p + y_n *_2 z_n]\\ &= x \circledast_1 (y \circledast_2 z).\end{aligned}$$

Hence $(x \circledast_1 y) \circledast_2 z = x \circledast_1 (y \circledast_2 z)$. Thus $\hat{\mathrm{G}}$ is a ΘΓ N-group.

Example 3.6. Let $(\mathbb{R}, +)$ be the group of real numbers. Take $N = (\mathbb{R}, +, \cdot)$ and $a, b, c \in \mathbb{R}$. Define

$$a\ div\ b = \begin{cases} 0 & \text{if } b = 0,\\ \frac{a}{b} & \text{if } b \neq 0.\end{cases}$$

Corresponding to the operation div, define div_c by $a\ div_c b = a\ div\ (bc^2)$.

Let $\Theta = \{\cdot, div\}$, $\Gamma = \Theta$ and $\Delta_\Gamma = \{\cdot, div_c\}$. Clearly, the operations in Θ are right distributive. Note that the multiplication operation $\cdot$ in Θ is associative. We have

$$a\ div_c(b\ div\ c) = a\ div\ ((b\ div\ c)c^2) = \frac{a}{\frac{b}{c} \cdot c^2} = (a\ div\ b)\ div\ c.$$

This implies that div is quasi associative. Hence the operations in Θ are quasi associative. Therefore $\mathbb{R}$ is a $\Theta\Gamma$ N-group.

Proposition 3.7. *1. A non associative nearring induced by a nearring forms a $\Theta\Gamma$ N-group.*

2. A double planar nearring induced by a nearring forms a $\Theta\Gamma$ N-group.

Proof. 1. Let $(N, +, .)$ be a nearring and $k : N \to End(N, +)$ be a mapping. Define $* : N \times N \to N$ by $a * b = k(b)(a).b = f(a).b$, where $f = k(b)$ is an endomorphism for each b. Then $(N, +, *)$ is a non associative nearring. For,

$$(a + b) * c = k(c)(a + b).c = f(a + b).c = (f(a).c + f(b).c) = (a * c) + (b * c).$$

Let $f(k) \in End(N, +, .)$ be such that for $a, b, c \in N$ $a *_{f(k)} b = f(k)(a).b$. We will prove: $(a * b) * c = a *_{k(c)\circ k(b)} (b * c)$. We have

$$\begin{aligned}(a * b) * c &= (k(b)(a).b) * c = k(c)(k(b)(a).b).c = [k(c)k(b)(a).k(c)(b)].c\\ &= [k(c)k(b)(a)].[k(c)(b).c] = [k(c) \circ k(b)](a).(b * c) = a *_{k(c)\circ k(b)} (b * c).\end{aligned}$$

Hence N forms a $\Theta\Gamma$ N-group with $\Theta = \{*\}$, $\Gamma = \{*\}$, and $\Delta_\Gamma = \{*_{k(c)\circ k(b)}\}$.

2. Let N be a nearring. Define $a * b = a\ |b|$ and

$$a \circ b = \begin{cases} 0 & \text{if } b = 0, \\ a\frac{b}{|b|} & \text{if } b \neq 0. \end{cases}$$

Then $(N, +, *), (N, +, \circ)$ are planar nearrings. We have $(a*b)\circ c = (a\circ c)*(b\circ c)$, $(a \circ b) * c = (a * c) \circ (b * c)$. Now, $(N, +, *, \circ)$ is a double planar nearring. Define $a\delta^c_\circ b = (a*c)\circ(b)$ and $a\delta^c_* b = (a\circ c)*b$. Now, $(a\circ b)*c = (a*c)\circ(b*c) = a\delta^c_\circ(b*c)$, and $(a * b) \circ c = (a \circ c) * (b \circ c) = a\delta^c_*(b \circ c)$. Hence N forms a $\Theta\Gamma$ N-group with $\Theta = \{*, \circ\}$, $\Gamma = \{*, \circ\}$, and $\Delta_\Gamma = \{\delta^c_*, \delta^c_\circ\}$. □

Proposition 3.8. *1. Every N-group is a $\Theta\Gamma$ N-group.*

2. Every gamma nearring is a $\Theta\Gamma$ N-group.

Proof. 1. Take $\Theta = \{\theta\}$, $\Gamma = \{\cdot\}$ and $\Delta_\Gamma = \{\theta\}$.

2. Take $N = G$, $\Theta = \Gamma$ and $\Delta_\Gamma = \Gamma$. □

Proposition 3.9. *Let G be a group and N be nearring. Then for all $g \in G$, $n \in N$:*

1. $0_N\theta g = 0_G$, for all $\theta \in \Theta$.

2. $(-n)\theta g = -n\theta g$, for all $\theta \in \Theta$.

3. For $\gamma \in \Gamma$, $n\gamma 0_N = 0_N \Rightarrow n\delta_\gamma 0_G = 0_G$.

4. *Let $N = N_c$. Then for $\gamma \in \Gamma$, $\theta \in \Theta$, $(n\gamma m)\theta g = n\delta_\gamma 0_G$.*

Proof. 1. $(0_N + 0_N)\theta g = 0_N\theta g + 0_N\theta g$. Then $0_N\theta g = 0_N\theta g + 0_N\theta g$. Hence $0_N\theta g = 0_G$.

2. $0_G = 0_G\theta g = (-n+n)\theta g = (-n)\theta g + n\theta g$. This gives $(-n)\theta g = -n\theta g$.

3. $(n\gamma 0_N)\theta g = n\delta_\gamma(0_N\theta g) = n\delta_\gamma 0_G \implies n\delta_\gamma 0_G = (n\gamma 0_N)\theta g = 0_N\theta g = 0_G$.

4. $(n\gamma m)\theta g = (n\gamma 0_N\gamma m)\theta g = (n\gamma 0_N)\theta g = n\delta_\gamma(0_N\theta g) = n\delta_\gamma 0_G$. □

Definition 3.10. Let G be a $\Theta\Gamma$ N-group. A subgroup $(H,+)$ of $(G,+)$ is said to be a $\Theta\Gamma$ N-subgroup of G if $N\Theta H \subseteq H$.

Definition 3.11. Let N be a nearring and G, G' be $\Theta\Gamma$ N-groups. Then $h : G \to G'$ is called a ΘN-homomorphism if it satisfies

1. $h(x+y) = h(x) + h(y)$ and
2. $h(n\theta x) = n\theta h(x)$ for all $n \in N$, $x, y \in G$ and $\theta \in \Theta$.

The set of all ΘN-homomorphisms is denoted by $Hom_\Theta(G, G')$.

Definition 3.12. *$Ker\ h = \{x \in G | h(x) = 0'\}$.*

Definition 3.13. A normal subgroup H of a $\Theta\Gamma$ N-group $(G,+)$ is called a ΘN-ideal of G if $n\theta(x+a) - n\theta x \in H$ for all $n \in N$, $x \in G$, $a \in H$ and $\theta \in \Theta$.

Remark 3.14. Let H be a $\Theta\Gamma$ N-subgroup of $(G,+)$. Then the following two conditions are equivalent:

1. H is a Θ-N ideal of the $\Theta\Gamma$-N group G; and
2. $x \equiv y(\text{mod } H)$, $a \equiv b(\text{mod } H) \Rightarrow x+a \equiv y+b(\text{mod } H)$, and $n\theta x \equiv n\theta y(\text{mod } H)$.

Verification:

1 $\Rightarrow$ 2: Suppose that $x_1 \equiv x_1'(\text{mod } H)$ and $x_2 \equiv x_2'(\text{mod } H)$. This implies that $x_1 - x_1' \in H, x_2 - x_2' \in H$. Now we show that $x_1 + x_2 \equiv x_1' + x_2'(\text{mod } H)$ and $n\theta x_1 \equiv n\theta x_1'(\text{mod } H)$. Now $(x_1 + x_2) - (x_1' + x_2') = x_1 + (x_2 - x_2') - x_1' = x_1 + (x_2 - x_2') + x_1 - x_1 - x_1' = (x_1 + (x_2 - x_2') - x_1) - (x_1' - x_1) \in H$ (since H is normal, and $x_2 - x_2' \in H$). This implies $(x_1 + x_2) \equiv (x_1' + x_2')(\text{mod } H)$. Now $n\theta x_1 - n\theta x_1' = n\theta(x_1 - x_1' + x_1') - n\theta x_1' \in H$ (since $x_1 - x_1' \in H$ and H is an ideal of G). This means that $n\theta x_1 \equiv n\theta x_1'(\text{mod } H)$.

2 $\Rightarrow$ 1: First we show that H is a normal subgroup of G. Let $x \in G$ and $h \in H$. We know that $x \equiv x(\text{mod } H)$ and $h \equiv 0(\text{mod } H)$. By the assumed condition, $x + h \equiv x + 0(\text{mod } H)$. This implies $x + h \equiv x(\text{mod } H)$. Thus $x + h - x \in H$. Let $n \in N$. We know that $n \equiv n(\text{mod } H)$ and $x + h \equiv x(\text{mod } H)$. By the assumed condition, $n\theta(x+h) \equiv n\theta x(\text{mod } H)$. This implies that $n\theta(x+h) - n\theta x \in H$. Hence H is an ideal of G.

Remark 3.15. Let G be a $\Theta\Gamma$ N-group and H a normal subgroup of $(G,+)$.

Then the following two conditions are equivalent:

1. $n\theta(x+a)-n\theta x\in H$, for all $n\in N, x\in G, a\in H$ and $\theta\in\Theta$, and
2. $n\theta(b+x)-n\theta x\in H$, for all $n\in N, x\in G, b\in H$ and $\theta\in\Theta$.

Verification:

$1\Rightarrow 2: n\theta(b+x)-n\theta x=n\theta(x-x+b+x)-n\theta x=n\theta(x+a)-n\theta x\in H$ (by 1).

The proof of $2\Rightarrow 1$ is similar.

Proposition 3.16. *If* $I\trianglelefteq_{\Theta N} G$ *then* $G/I=\{g+I\mid g\in G\}$ *is a* $\Theta\Gamma$ N*-group.*

Proof. First, we define operations $+,\theta$ on G/I as follows:

$$(g_1+I)+(g_2+I)=(g_1+g_2)+I,\quad n\theta(g_1+I)=n\theta g_1+I.$$

It is easy to show that $+$ is well-defined.

We will prove that θ is well-defined. Let $n\theta(g_1+I)=n\theta(g_1'+I)$ and $x\in n\theta g_1+I$. Then $x=n\theta g_1+i=n\theta(g_1+0_G)+i=n\theta(g_1'+i_1')+i=i'+n\theta g_1'+i=0_G+i'+n\theta g_1'+i=(n\theta g_1'-n\theta g_1')+i'+n\theta g_1'+i=n\theta g_1'+(-n\theta g_1'+i'+n\theta g_1')+i\in n\theta g_1'+I$. Hence $n\theta g_1+I\subseteq n\theta g_1'+I$. Similarly, $n\theta g_1'+I\subseteq n\theta g_1+I$. Hence $n\theta g_1+I=n\theta g_1'+I$.

To show that G/I is a $\Theta\Gamma$ N-group, we will prove for $n,m\in N,\gamma\in\Gamma$, there exists $\delta_\gamma\in\Delta_\Gamma$ such that $(n\gamma m)\theta(g+I)=n\delta_\gamma(m\theta(g+I))$. Consider $(n\gamma m)\theta(g+I)=((n\gamma m)\theta g)+I$. Then there exists $\delta_\gamma\in\Delta_\Gamma$ such that $(n\gamma m)\theta g=n\delta_\gamma(m\theta g)$. Then $(n\delta_\gamma(m\theta g))+I=n\delta_\gamma(m\theta g+I)=n\delta_\gamma(m\theta(g+I))$. Therefore $(n\gamma m)\theta(g+I)=n\delta_\gamma(m\theta(g+I))$. Hence θ is quasi associative.

Clearly θ is right distributive. Hence G/I is a $\Theta\Gamma$ N-group. □

Definition 3.17. Let $I\trianglelefteq_{\Theta N} G$. Then $G/I=\{g+I\mid g\in G\}$ is called a factor $\Theta\Gamma$ N-group.

Proposition 3.18. *Let* $f:G\to G'$ *be a* ΘN*-homomorphism. Then* $Ker\ f$ *is a* ΘN*-ideal of* G*. Conversely, every* ΘN*-ideal is the kernel of a* ΘN*-homomorphism.*

Proof. We have $f(0)=0'$. Hence $0\in Ker\ f$. Let $g\in G,n\in N,a\in Ker\ f$. Then $f(a)=0'$. Now,

$$\begin{aligned}f(g+a-g)&=f(g)+f(a)-f(g)=0'\Rightarrow g+a-g\in Ker\ f,\\ f(n\theta(x+a)-n\theta x)&=f(n\theta(x+a))-f(n\theta x)=n\theta f(x+a)-n\theta f(x)\\ &=n\theta(f(x)+f(a))-n\theta f(x)=0'\Rightarrow n\theta(x+a)-n\theta x\in Ker\ f.\end{aligned}$$

Hence $Ker\ f$ is a ΘN-ideal of G. To prove the converse, define $\phi:G\to G/I$ by $\phi(g)=g+I$. We prove that ϕ is well defined and one-one. We have, $g_1=g_2\Leftrightarrow g_1+I=g_2+I\Leftrightarrow\phi(g_1)=\phi(g_2)$. Let $g+I\in G/I$. Then $\phi(g)=g+I$. Hence ϕ is onto. ϕ is a homomorphism because

$$\begin{aligned}\phi(g_1+g_2)&=(g_1+g_2)+I=g_1+I+g_2+I=\phi(g_1)+\phi(g_2),\\ \phi(n\theta g)&=n\theta g+I=n\theta(g+I)=n\theta\phi(g).\end{aligned}$$

Now, $Ker\ \phi=\{x\in G\mid\phi(x)=0+I\}=\{x\in G\mid g+I=0+I\}=I$. Hence ΘN-Ideal is the kernel of a ΘN-homomorphism. □

THEOREM 3.19. *Let* $f : G \to G'$ *be an onto* Θ N*-homomorphism and* $K = kerf$*. Then* K *is an ideal of* G *and* $G/K \cong G'$*.*

Proof. Define $\phi : G/K \to G'$ by $\phi(a+K) = f(a)$. We will show that ϕ is well defined and one-one. Let $a, b \in G$. We have $a+K = b+K \Leftrightarrow a-b \in K \Leftrightarrow f(a-b) = 0 \Leftrightarrow f(a) - f(b) = 0 \Leftrightarrow f(a) = f(b) \Leftrightarrow \phi(a+K) = \phi(b+K)$. Now, we prove that ϕ is onto. Let $y \in G'$. As f is onto, $y = f(a)$ for some $a \in G$. Now $a + K \in G/K$ and $\phi(a+K) = f(a) = y$. Now we prove that ϕ is homomorphism. We have

$$\begin{aligned}\phi((a+K)+(b+K)) &= \phi((a+b)+K) = f(a+b) = f(a)+f(b)\\ &= \phi(a+K)+\phi(b+K),\\ \phi(n\theta(a+K)) &= \phi(n\theta a+K) = f(n\theta a) = n\theta f(a) = n\theta\phi(a+K).\end{aligned}$$

Thus $G/K \cong G'$. □

THEOREM 3.20. 1. *Let* $f : G \to G'$ *be an onto* Θ N*-homomorphism and* $H = Ker\ f$*. If* K' *is* $\Theta\Gamma$ N*-subgroup (resp.* ΘN*-ideal) of* G *and* $K = \{x \in G : f(x) \in K'\} = f^{-1}(K')$*, then* K *is a* $\Theta\Gamma$ N*-subgroup(resp.* ΘN*-ideal) of* G *by* $H \subseteq K$ *and* $G/K \cong G'/K'$*.*

2. *Let* H *and* K *be* ΘN*-ideals of* $\Theta\Gamma$ N*-group* G *by* $H \subseteq K$*. Then* $G/K \cong (G/H)/(K/H)$*.*

Proof. 1. Define $\phi : G \to G'/K'$ by $\phi(x) = f(x) + K'$. First, we show that ϕ is a homomorphism. Let $a, b \in G$. We have $\phi(a+b) = f(a+b) + K' = (f(a)+f(b)) + K'$. We claim $(f(a)+f(b)) + K' = (f(a)+K') + (f(b)+K')$. Note that $x \in (f(a)+f(b))+K' \Rightarrow x = f(a)+f(b)+k'_1 = f(a)+0+f(b)+k'_1 \Rightarrow x \in f(a)+K'+f(b)+K' \Rightarrow (f(a)+f(b))+K' \subseteq (f(a)+K')+(f(b)+K')$. Let $y \in (f(a)+K')+(f(b)+K')$. Hence there exist $k'_1, k'_2 \in K'$ such that $y = f(a)+k'_1+f(b)+k'_2$. Then $y = f(a)+f(b)-f(b)+k'_1+f(b)+k'_2 \in f(a)+f(b)+K' \Rightarrow (f(a)+K')+(f(b)+K') \subseteq (f(a)+f(b))+K'$. Hence $\phi(a+b) = (f(a)+f(b))+K' = (f(a)+K')+(f(b)+K') = \phi(a)+\phi(b)$. Now, $\phi(n\theta a) = n\theta f(a)+K' = n\theta(f(a)+K') = n\theta\phi(a)$. Now we prove that ϕ is onto. Let $a'+K' \in G'/K'$, $a' \in G'$. Since f is onto, there exists $a \in G$ such that $f(a) = a'$. Hence $\phi(a) = f(a)+K' = a'+K'$. By Theorem 3.18, we get $G/Ker\ \phi \cong G'/K'$. Now, we have

$$Ker\ \phi = \{x \mid \phi(x) = e+K'\} = \{x \mid f(x)+K' = e+K'\} = \{x \mid f(x) \in K'\} = K.$$

Let $x \in G, k \in K, n \in N$. As $\phi(k) = 0$, we get

$$\begin{aligned}\phi(x+k-x) &= \phi(x)+\phi(k)-\phi(x) = 0 \Rightarrow x+k-x \in K;\\ \phi(n\theta k) &= n\theta\phi(k) = n\theta 0 = 0 \Rightarrow n\theta k \in K;\\ \phi(n\theta(x+k)-n\theta x) &= \phi(n\theta(x+k)) - \phi(n\theta x)\\ &= n\theta(\phi(x)+\phi(k)) - n\theta\phi(x) = 0 \Rightarrow n\theta(x+k)-n\theta x \in K.\end{aligned}$$

Hence K is $\Theta\Gamma$ N-subgroup (resp. ΘN-ideal) of G and $G/K \cong G'/K'$. Let $x \in H$. Then we have, $f(x) = e' \in K'$. This implies $\phi(x) = f(x)+K' = e'+K' = K'$. We get $x \in Ker\ \phi = K$. Hence $H \subseteq K$.

2. H is a normal subgroup of G and K is a normal subgroup of G containing H. Hence K/H and G/H are factor $\Theta\Gamma$ N-groups.

   First, we prove that K/H is a normal subgroup of G/H. Define $f : G/H \to G/K$ by $f(x+H) = x+K$. We prove that f is well-defined. We have $x+H = y+H \Rightarrow x-y \in H \Rightarrow x-y \in K \Rightarrow x+K = y+K \Rightarrow f(x+H) = f(y+H)$. We now show that f is an onto homomorphism with $Ker\ f = K/H$. We have

$$\begin{aligned} f[(x+H)+(y+H)] &= f[(x+y)+H] = (x+y)+K \\ &= (x+K)+(y+K) = f(x+H)+f(y+H), \quad \text{and} \\ f(n\theta(x+H)) &= f(n\theta x+H) = n\theta x+K = n\theta(x+K) = n\theta f(x+H). \end{aligned}$$

   Let $x+K \in G/K$. Then there exists $x+H \in G/H$ such that $f(x+H) = x+K$. We have, $Ker\ f = \{g+H | f(g+H) = e+K\} = \{g+H | g \in K\} = K/H$. Hence K/H is a normal subgroup of G/H. By Theorem 3.19, $(G/H)/Ker\ f \cong G/K$. Thus, $(G/H)/(K/H) \cong G/K$.

□

4. Conclusion

We have introduced the algebraic structure of $\Theta\Gamma$ N-group as a natural extension of N-group and gamma nearring. We have shown that the lambda calculus system induces a $\Theta\Gamma$ N-group. Other examples of $\Theta\Gamma$ N-group include non associative nearrings and double planar nearrings. We have defined the concept of ideal of $\Theta\Gamma$ N-group and proved isomorphism theorems. Different prime ideal notions and corresponding radicals of $\Theta\Gamma$ N-group can be studied as future work.

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