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МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 02.02.2018

research paper
оригинални научни рад

A NOTE ON IA-AUTOMORPHISMS OF A FINITE p-GROUP

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Abstract. Let G be a finite group. An automorphism α of G is called an IA-automorphism if $x^{-1}x^{\alpha} \in G'$ for all $x \in G$. The set of all IA-automorphisms of G is denoted by $\mathrm{Aut}^{G'}(G)$. A group G is called semicomplete if and only if $\mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G)$. In this paper, we obtain certain results on a finite p-group to be semicomplete.

1. Introduction

Let G be a finite group and N a characteristic subgroup of G. Let α be an automorphism of G. If $Ng^{\alpha} = Ng$ for all g in G, we shall say that α centralizes G/N. We let $\mathrm{Aut}^N(G) = \mathrm{Aut}(G, N)$ denote the centralizer in $\mathrm{Aut}(G)$ of G/N. Clearly $\mathrm{Aut}^N(G)$ is a normal subgroup of $\mathrm{Aut}(G)$, the automorphism group of G, and $\alpha \in \mathrm{Aut}^N(G)$ if and only if $x^{-1}x^{\alpha} \in N$ for all $x \in G$. The group $\mathrm{Aut}^{G'}(G)$ have been studied by several authors, where G' stands for the derived subgroup of G, see for example [3, 5, 6, 9, 10, 15–17]. Now let M be a normal subgroup of G. We let $\mathrm{Aut}_M(G)$ denote the group of all automorphisms of G centralizing M. Moreover, $\mathrm{Aut}_M^N(G) = \mathrm{Aut}_M(G, N) = \mathrm{Aut}^N(G) \cap \mathrm{Aut}_M(G)$. It is well-known that if G is a finite p-group, then so is the group $\mathrm{Aut}^{G'}(G)$.

In this paper, we study closely the group $\mathrm{Aut}^{G'}(G)$ for a finite p-group G. In Section 2 we give some basic results that are needed for the main results of the paper. In Sections 3 and 4 we prove the main results of the paper and give necessary and sufficient condition for a finite p-group G to be semicomplete when $(G, Z(G))$ is a Camina pair and G' is cyclic.

Throughout the paper all groups are assumed to be finite groups. We use standard notation in group theory. In particular, we use the notation $\mathrm{Hom}(G, A)$ to denote the group of homomorphisms of G into an abelian group A. A group G of order p^m is said to be of maximal class if $m > 2$ and the nilpotency class of G is $m-1$. A p-group G is said to be extraspecial if $G' = Z(G) = \Phi(G)$ is of order p. Also, a non-abelian group

2010 Mathematics Subject Classification: 20D45, 20D15
Keywords and phrases: Automorphism group; p-group; semicomplete group.

that has no non-trivial abelian direct factor is said to be purely non-abelian. Recall that a group G is called an (internal) central product of its subgroups $G_1, \dots, G_n$ if $G = G_1 \dots G_n$ and $[G_i, G_j] = 1$ for all $1 \le i < j \le n$. In this situation, we shall write $G = G_1 * \dots * G_n$. The terms of the lower central series and the upper central series of a group G are respectively denoted as $\Gamma_i(G)$ and $Z_i(G)$. If α is an automorphism of G and x is an element of G, we write x^α for the image of x under α. For a finite group G, $\Omega_i(G)$, $d(G)$, $\mathcal{M}(G)$, $\exp(G)$ and $\mathrm{cl}(G)$ respectively denote the subgroup of G generated by its elements of order dividing p^i, minimal number of generators, the set of all maximal subgroups, the exponent and the nilpotency class of G. Also the size of a finite group G is shown by $|G|$, $o(x)$ for the order of $x \in G$, C_n is the cyclic group of order n and X_{p^3} for non-abelian p-group of order p^3 and exponent p, where p is an odd prime. For $s \ge 1$, we use the notation G^{*s} for the iterated central product defined by $G^{*s} = G * G^{*(s-1)}$ with $G^{*1} = G$, where G is a finite p-group. We also make the convention $G^{*0} = 1$.

2. Some basic results

In this section, we give some known results which will be used in the rest of the paper.

An automorphism α of a group G is called central if $x^{-1}x^\alpha \in Z(G)$ for all $x \in G$. The set of all central automorphisms of G is denoted by $\mathrm{Aut}^Z(G)$, where $Z = Z(G)$. The following well-known results will be later used in the paper.

THEOREM 2.1. *([2, Theorem 1]) For a finite purely non-abelian group G, there is a 1-1 correspondence between* $\mathrm{Hom}(G, Z(G))$ *and* $\mathrm{Aut}^Z(G)$, *whence* $|\mathrm{Hom}(G/G', Z(G))| = |\mathrm{Aut}^Z(G)|$.

LEMMA 2.2. *([1, Lemma 2.1]) Let G be a finite group and N be a normal subgroup of G such that G/N is abelian. Let $G/N = \langle x_1N\rangle \times \dots \times \langle x_dN\rangle$, where $x_1, \dots, x_d \in G$ and $d = d(G/N)$. If $u_1, \dots, u_d \in Z(N)$ such that*

$$\begin{cases} (x_iu_i)^{n_i} = x_i^{n_i} & 1 \le i \le d \\ [x_i, u_j] = [x_j, u_i] & 1 \le i < j \le d \end{cases}$$

where $n_i = o(x_iN)$, then the mapping $x_i \mapsto x_iu_i, 1 \le i \le d$, can be extended to an automorphism of G leaving N elementwise fixed.

LEMMA 2.3. *([17, Lemma 2.2] Let G be a group and M, N be normal subgroups of G with $N \le M$ and $C_N(M) \le Z(G)$. Then* $\mathrm{Aut}_M^N(G) \cong \mathrm{Hom}(G/M, C_N(M))$.

3. Main results

In this section, we study the group $\mathrm{Aut}^{G'}(G)$ for a finite p-group G. For simplicity, we let $\Gamma_i = \Gamma_i(G)$, for all i.

Lemma 3.1. *Let G be a finite nilpotent group. If $\alpha \in \mathrm{Aut}^{G'}(G)$ and $a \in \Gamma_i$ $(i = 1, \ldots)$, then $a^\alpha \equiv a$ (mod Γ_{i+1}).*

Proof. The result is clearly true for $i = 1$. Proceeding by induction on i, and assume the validity of the lemma for some i. Let $a \in \Gamma_{i+1}$. Then a is a product of terms $b = [y, g]$, such that $y \in \Gamma_i$ and $g \in G$. Now

$$\begin{aligned} b^\alpha &= [y^\alpha, g^\alpha] = [yd, gx], && (d \in \Gamma_{i+1}, x \in G') \\ &= [y, gx]^d [d, gx] \equiv [y, gx]^d && (mod \quad \Gamma_{i+2}) \\ &= ([y, x][y, g]^x)^d = [y, x]^d [y, g]^{xd} \equiv [y, g]^{xd} && (mod \quad \Gamma_{i+2}) \\ &= [y, g][[y, g], xd] \equiv [y, g] = b && (mod \quad \Gamma_{i+2}), \end{aligned}$$

and the lemma follows. □

Theorem 3.2. *Let G be a finite nilpotent group of class c. Then*

(i) $\mathrm{Aut}^{G'}(G) = \mathrm{Aut}^{G'}_{\Gamma_c}(G)$*;*

(ii) $\mathrm{Aut}^{\Gamma_c}(G) \leq Z(\mathrm{Aut}^{G'}(G))$*;*

(iii) $\mathrm{Aut}^{G'}(G)/\mathrm{Aut}^{\Gamma_c}(G)$ *is isomorphic to the subgroup of automorphisms in* $\mathrm{Aut}^{G'/\Gamma_c}(G/\Gamma_c)$.

Proof. (i) Follows from Lemma 3.1.

To prove (ii), take $\alpha \in \mathrm{Aut}^{\Gamma_c}(G)$ and $\beta \in \mathrm{Aut}^{G'}(G)$. Then for $g \in G$, $g^\alpha = gd$ and $g^\beta = gx$, where $d \in \Gamma_c$ and $x \in G'$. Thus $g^{\alpha\beta} = (gd)^\beta = gxd = gdx$ and $g^{\beta\alpha} = (gx)^\alpha = gdx$, by (i) and since $\mathrm{Aut}(G, \Gamma_c) = \mathrm{Aut}_{G'}(G, \Gamma_c)$. Hence $\alpha\beta = \beta\alpha$ and $\alpha \in Z(\mathrm{Aut}^{G'}(G))$.

(iii) Clearly $\alpha \in \mathrm{Aut}^{G'}(G)$ induces an automorphism $\overline{\alpha}$ in $\frac{G}{\Gamma_c}$, defined by $(g\Gamma_c)^{\overline{\alpha}} = g^\alpha \Gamma_c$. It is easy to see that the mapping $\alpha \mapsto \overline{\alpha}$ defines a homomorphism of $\mathrm{Aut}^{G'}(G)$ into $\mathrm{Aut}(\frac{G}{\Gamma_c}, \frac{G'}{\Gamma_c})$. The kernel of this homomorphism is $\mathrm{Aut}^{\Gamma_c}(G)$, for $\overline{\alpha} = \overline{1}$ if and only if $g^{-1}g^\alpha \in \Gamma_c$, for all $g \in G$, which means that $\alpha \in \mathrm{Aut}(G, \Gamma_c)$. □

Theorem 3.3. *Let G be a finite nilpotent group. Then* $\mathrm{cl}(\mathrm{Aut}^{G'}(G)) = \mathrm{cl}(G) - 1$.

Proof. Suppose that $\mathrm{cl}(G) = c$. We use induction on c. For $c = 1$, it is clearly true. Assume that the result holds for any finite nilpotent group of nilpotency class less than c. Hence $\mathrm{cl}(\mathrm{Aut}(\frac{G}{\Gamma_c}, \frac{G'}{\Gamma_c})) = \mathrm{cl}(\frac{G}{\Gamma_c}) - 1 \leq \mathrm{cl}(G) - 2$. Since $\mathrm{Inn}(G) \leq \mathrm{Aut}^{G'}(G)$, $\mathrm{cl}(G) - 1 \leq \mathrm{cl}(\mathrm{Aut}^{G'}(G))$. Now by Theorem 3.2 (ii) and (iii) we have $\mathrm{cl}(G) - 2 \leq \mathrm{cl}(\mathrm{Aut}^{G'}(G)) - 1 = \mathrm{cl}(\frac{\mathrm{Aut}^{G'}(G)}{Z(\mathrm{Aut}^{G'}(G))}) \leq \mathrm{cl}(\frac{\mathrm{Aut}^{G'}(G)}{\mathrm{Aut}(G,\Gamma_c)}) \leq \mathrm{cl}(\mathrm{Aut}(\frac{G}{\Gamma_c}, \frac{G'}{\Gamma_c})) \leq \mathrm{cl}(G) - 2$.

Consequently, $\mathrm{cl}(\mathrm{Aut}^{G'}(G)) - 1 = \mathrm{cl}(G) - 2$ and $\mathrm{cl}(\mathrm{Aut}^{G'}(G)) = \mathrm{cl}(G) - 1$. □

Corollary 3.4. *Let G be a finite p-group of order p^n. Then G is of maximal class if and only if* $\mathrm{cl}(\mathrm{Aut}^{G'}(G)) = n - 2$.

THEOREM 3.5. *Let G be a finite p-group of class c. Then* $\mathrm{Aut}_Z^{\Gamma_c}(\Gamma_{c-1}) = \mathrm{Inn}(\Gamma_{c-1})$ *if and only if Γ_c is cyclic, where $Z = Z(\Gamma_{c-1})$.*

Proof. By Lemma 2.3, $\mathrm{Aut}_Z^{\Gamma_c}(\Gamma_{c-1}) \cong \mathrm{Hom}(\Gamma_{c-1}/Z(\Gamma_{c-1}), \Gamma_c)$. It is sufficient to prove that $\exp(\Gamma_{c-1}/Z(\Gamma_{c-1})) \leq \exp(\Gamma_c)$. Suppose that $\exp(\Gamma_c) = p^n$ and $g \in \Gamma_{c-1}$ such that $o(gZ(\Gamma_{c-1})) = \exp(\Gamma_{c-1}/Z(\Gamma_{c-1}))$. Now $[g^{p^n}, x] = [g, x]^{p^n} = 1$ for all $x \in G$. So $g^{p^n} \in Z(\Gamma_{c-1})$ and the proof is complete. □

As an application of Theorem 3.5, we get the following corollary which is the same as [17, Proposition 3.2].

COROLLARY 3.6. *Let G be a finite p-group of class 2. Then* $\mathrm{Aut}_Z^{G'}(G) = \mathrm{Inn}(G)$ *if and only if G' is cyclic, where $Z = Z(G)$.*

THEOREM 3.7. *Let G be a finite p-group of class c. Then* $\mathrm{Aut}_{\Gamma_c}^{\Gamma_c}(\Gamma_{c-1}) = \mathrm{Inn}(\Gamma_{c-1})$ *if and only if Γ_c is cyclic and $Z(\Gamma_{c-1}) = \Gamma_c{\Gamma_{c-1}}^{p^n}$, where $|\Gamma_c| = p^n$.*

Proof. Assume that Γ_c is cyclic and of order p^n. By Theorem 3.5, it is sufficient to prove that $\mathrm{Aut}_{\Gamma_c}^{\Gamma_c}(\Gamma_{c-1}) = \mathrm{Aut}_Z^{\Gamma_c}(\Gamma_{c-1})$, where $Z = Z(\Gamma_{c-1})$. Let $\alpha \in \mathrm{Aut}_{\Gamma_c}^{\Gamma_c}(\Gamma_{c-1})$ and $x \in \Gamma_{c-1}$. We may write $(x^{p^n})^\alpha = (xd)^{p^n} = x^{p^n}$ with $d \in \Gamma_c$, which shows that α fixes any element of $Z(\Gamma_{c-1})$, since $Z(\Gamma_{c-1}) = \Gamma_c{\Gamma_{c-1}}^{p^n}$. Consequently $\mathrm{Aut}_{\Gamma_c}^{\Gamma_c}(\Gamma_{c-1}) = \mathrm{Aut}_Z^{\Gamma_c}(\Gamma_{c-1}) = \mathrm{Inn}(\Gamma_{c-1})$.

Conversely, suppose that $\mathrm{Aut}_{\Gamma_c}^{\Gamma_c}(\Gamma_{c-1}) = \mathrm{Inn}(\Gamma_{c-1})$. By Theorem 3.5, Γ_c is cyclic. Since $\Gamma_c \leq \Gamma_c{\Gamma_{c-1}}^{p^n} \leq Z(\Gamma_{c-1}) \leq \Gamma_{c-1}$, it follows that

$$\begin{aligned}\mathrm{Inn}(\Gamma_{c-1}) \cong &\mathrm{Hom}(\Gamma_{c-1}/Z(\Gamma_{c-1}), \Gamma_c) \rightarrowtail \mathrm{Hom}(\Gamma_{c-1}/(\Gamma_c\Gamma_{c-1}^{p^n}), \Gamma_c)\\ \rightarrowtail &\mathrm{Hom}(\Gamma_{c-1}/\Gamma_c, \Gamma_c) \cong \mathrm{Aut}_{\Gamma_c}(\Gamma_{c-1}, \Gamma_c) = \mathrm{Inn}(\Gamma_{c-1}).\end{aligned}$$

Therefore $\mathrm{Hom}(\Gamma_{c-1}/(\Gamma_c\Gamma_{c-1}^{p^n}), \Gamma_c) \cong \mathrm{Inn}(\Gamma_{c-1})$, which gives $|\Gamma_{c-1}/(\Gamma_c\Gamma_{c-1}^{p^n})| = |\Gamma_{c-1}/Z(\Gamma_{c-1})|$. So $Z(\Gamma_{c-1}) = \Gamma_c{\Gamma_{c-1}}^{p^n}$, as required. □

S. Singh, D. Gumber, and H. Kalra [15] gave a necessary and sufficient condition on a finite p-group to be semicomplete. Our next corollary, which is a particular case of Theorem 3.7, gives an another interpretation of this result. This corollary is [17, Theorem 3.3].

COROLLARY 3.8. *Let G be a finite p-group of class 2. Then* $\mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G)$ *if and only if G' is cyclic and $Z(G) = G'G^{p^n}$, where $|G'| = p^n$.*

We now give an alternative proof for [15, Corollary 2.4].

COROLLARY 3.9. *Let G be a 2-generated finite nilpotent group of class 2. Then any IA-automorphism of G is an inner automorphism.*

Proof. Suppose that $G = \langle a, b\rangle$. Then $G' = \langle [a, b]^g | g \in G\rangle = \langle [a, b]\rangle$ and so G' is cyclic. Since G is a nilpotent group, $G = P_1 \times \ldots \times P_n$, where P_i is the Sylow p_i-subgroup of G, for $i = 1, \ldots, n$. Thus $G' = P_k'$, $\mathrm{Inn}(G) \cong \mathrm{Inn}(P_k)$ and by Lemma 2.3, $\mathrm{Aut}^{G'}(G) \cong \mathrm{Aut}^{P_1'}(P_1) \times \ldots \times \mathrm{Aut}^{P_n'}(P_n) = \mathrm{Aut}^{P_k'}(P_k) \cong \mathrm{Hom}(P_k/P_k', P_k')$ for some

$1 \le k \le n$. Next by [5, Theorem 3.2], $|\mathrm{Aut}^{G'}(G)| = |G'|^2$ and so $|\mathrm{Aut}^{P_k'}(P_k)| = |P_k'|^2$. Now since $P_k' \le Z(P_k)$, by [[14], Lemma 0.4], if $\exp(P_k/Z(P_k)) = p^m = \exp(P_k')$, then $P_k/Z(P_k)$ has the form $C_{p^m} \times C_{p^m} \times A$ for some (possibly trivial) abelian p-group A. So by Lemma 2.3, $|\mathrm{Aut}_Z^{P_k'}(P_k)| = |\mathrm{Hom}(P_k/Z(P_k), P_k')| \ge |P_k'|^2$, where $Z = Z(P_k)$. Thus $\mathrm{Aut}^{P_k'}(P_k) = \mathrm{Aut}_Z^{P_k'}(P_k)$, which together with Corollary 3.6 completes the proof. □

4. Groups G such that $(G, Z(G))$ is a Camina pair

Camina groups were introduced by A.R. Camina in [4] and were studied in past (see for example [11–13]). Let G be a finite group and N be non-trivial proper normal subgroup of G. Then (G, N) is called a *Camina pair* if $xN \subseteq x^G$ for all $x \in G - N$, where x^G denotes the conjugacy class of x in G. It follows that (G, N) is a Camina pair if and only if $N \subseteq [x, G]$ for all $x \in G - N$, where $[x, G] = \{[x, g] | g \in G\}$.

In this section, we give necessary and sufficient condition for a finite p-group G to be semicomplete when $(G, Z(G))$ is a Camina pair and G' is cyclic. We start with some results of I.D. Macdonald.

Lemma 4.1. *([12, Lemma 2.1]) Let (G, H) be a Camina pair and G have class c. Then $H = \Gamma_r(G)$ and $H = Z_{c-r+1}(G)$ for some r satisfying $1 < r \le c$.*

Theorem 4.2. *([12, Theorem 2.2]) Let (G, H) be a Camina pair, $H = Z(G)$, and G have class c. Then $Z_r(G)/Z_{r-1}(G)$ has exponent p whenever $1 \le r \le c$.*

Theorem 4.3. *Let G be a finite p-group such that G' is cyclic and $(G, Z(G))$ is a Camina pair. Then $\mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G)$ if and only if G is an extraspecial p-group or G is isomorphic to a central product $A * X_{p^3}^{*s}$, for some $s \ge 0$, p is an odd prime and A is a 2-generator subgroup which is either a metacyclic group or $A = \langle a\rangle\langle b\rangle\langle c\rangle, [a, c] = [b, c] = 1, [a, b] = cb^{p^k}$, where $k \ge 1$.*

Proof. Let $(G, Z(G))$ be a Camina pair and $\alpha \in \mathrm{Aut}^Z(G)$, where $Z = Z(G)$. Since $Z(G) \le G'$, $\frac{Z_2(G)}{Z(G)} \cong \mathrm{Aut}^Z(G) \bigcap \mathrm{Inn}(G) = \mathrm{Aut}^Z(G)$ and so by Theorem 4.2, $\mathrm{Aut}^Z(G)$ is elementary abelian. Now $Z(G) < Z(M)$ and $C_G(M) = Z(M)$, for all $M \in \mathcal{M}(G)$ [7, Remark 2]. Assume that $|G/\Phi(G)| = p^t$ and $|Z(G)| = p^r$. By [18, Theorem 3.1], $d(Z_2(G)/Z(G)) = d(G)$. Since G is purely non-abelian, we have $p^t = |\mathrm{Aut}^Z(G)| = |\mathrm{Hom}(G/G', Z(G))| = p^{rt}$, by Theorem 2.1. Whence $r = 1$ and $Z(G) \cong C_p$. If $G/Z(G)$ be an abelian then by Corollary 3.8, $G' = Z(G) = \Phi(G) \cong C_p$ and hence G is extraspecial. So we may assume that $G/Z(G)$ is not abelian.

We first assume that $p > 2$. Then by the main theorem of [8], we may write $G = A_1 * A_2 * \ldots * A_n * B$, where B is an abelian subgroup, $A_1, A_2, \ldots, A_n$ are 2-generator subgroups, and the classes of $A_2, \ldots, A_n$ are equal to 2. Now $G = A_1 * A_2 * \ldots * A_n$, since $B \le Z(G) \le \Phi(G)$. Next for $2 \le i \le n$, $(A_i, Z(A_i))$ is a Camina pair since $xZ(A_i) = xZ(G) \subseteq x^G = x^{A_1 \ldots A_n} = x^{A_i}$, for all $x \in A_i - Z(A_i)$. Thus $A_i' = Z(A_i) = Z(G) \cong C_p$ and A_i is an extraspecial p-group of order p^3 and exponent

p, where $2 \leq i \leq n$. So by the theorem mentioned earlier, it follows that $G \cong A * X_{p^3}^{*s}$, where $s \geq 0$ and A is a 2-generator subgroup which is either a metacyclic group or $A = \langle a \rangle \langle b \rangle \langle c \rangle, [a,c] = [b,c] = 1, [a,b] = cb^{p^k}, k \geq 1$.
Suppose next that $p = 2$. First we show that $Z(M) \leq Z_2(G)$, for all $M \in \mathcal{M}(G)$. Let $M \in \mathcal{M}(G), g \in G \setminus M$ and $x \in Z(M) \setminus Z(G)$. Since $g^2 \in M$, $[x, G] = [x, M\langle g \rangle] = \{[x, g^i] | 0 \leq i < 2\}$. By assumption $Z(G) \subseteq [x, G]$ and $|Z(G)| = 2$. Consequently $Z(G) = [x, G]$ and so $x \in Z_2(G)$. Next let $x \in Z_2(G) \setminus Z(G)$. It follows that $M = C_G(x)$ is a maximal subgroup of G, since $|C_G(x)| = |G|/|[x,G]| = |G|/2$. Let $(Z_2(G) \cap G')/Z(G) = \langle \bar{t} \rangle$ and $M = C_G(t)$, where $t \in Z_2(G) \cap G'$ and $\bar{t} = tZ(G)$. Then $M \in \mathcal{M}(G)$ and if $g \in G \setminus M$, it follows that $[t, g] \in Z(G)$. Hence $(gt)^2 = g^2 t^2 [t, g] = g^2$, since $o(t) = 4$ and $[t, g] = t^2$. Now since $t \in Z(M)$, the map α sending $g \mapsto gt$ and $m \mapsto m$, for all $m \in M$, can be extended to an automorphism of G by Lemma 2.2, which is an automorphism lying in $\mathrm{Aut}^{G'}(G)$. So that α is an inner automorphism of G induced by an element x_M in G. It follows that $x_M \in C_G(M) = Z(M) \leq Z_2(G)$. This means that $t = g^{-1} g^{\alpha} = [g, x_M] \in Z(G)$, which is impossible.

Conversely, if G is an extraspecial p-group then by Lemma 2.3, $\mathrm{Aut}^{G'}(G) \cong \mathrm{Hom}(G/G', G') \cong \mathrm{Inn}(G)$, and so G is semicomplete. Next let $G \cong A * X_{p^3}^{*s}$, for some $s \geq 0$ and $p > 2$. Then by Theorem 3.2.(i), Lemma 4.1 and [6, Theorem 3], $\mathrm{Aut}^{G'}(G) = \mathrm{Aut}_{\Gamma_c}^{G'}(G) = \mathrm{Aut}_Z^{G'}(G) = \mathrm{Inn}(G)$, which completes the proof. □

Acknowledgement. The author is grateful to the referee for his/her valuable suggestions. The paper was revised according to these comments. This research was in part supported by a grant from Payame Noor University.

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(received 13.06.2017; in revised form 15.01.2018; available online 02.02.2018)

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