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THE ZARISKI TOPOLOGY ON THE GRADED CLASSICAL PRIME SPECTRUM OF A GRADED MODULE OVER A GRADED COMMUTATIVE RING

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Abstract. Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. A proper graded submodule N of M is called a graded classical prime if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $sm \in N$. The graded classical prime spectrum $Cl.Spec^g(M)$ is defined to be the set of all graded classical prime submodules of M. In this paper, we introduce and study a topology on $Cl.Spec^g(M)$, which generalizes the Zariski topology of graded ring R to graded module M, called Zariski topology of M, and investigate several properties of the topology.

1. Introduction

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. There is a wide variety of applications of graded algebras in geometry and physics, (for example see [27, Introduction].)

The concept of graded prime ideal was introduced by M. Refai, M. Hailat and S. Obiedat in [26] and studied in [24, 25].

In the literature, there are several different generalizations of the notion of graded prime ideal to graded module. The concept of graded prime submodule was introduced by S.E. Atani in [9] and studied in [2, 4, 5, 10, 11, 23]. Also, the concept of graded classical prime submodule was introduced by A.Y Darani and S. Motmaen in [13] and studied in [1, 3, 4]. Every graded prime submodule is a graded classical prime submodule, but the converse is not true in general (see [4, Example 2.3].)

Zariski topology on the prime spectrum of a module over a commutative ring, have been already studied in [6–8, 16, 17, 19]. Also, some topologies on the spectrum of graded prime submodules of a graded module over a graded commutative ring has been studied in [12, 22]. Moreover, some topologies on the spectrum of graded

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classical prime submodules of a graded module over a graded commutative ring have been studied in [13]. These results will be used in order to obtain the main aims of this paper.

In this paper, we rely on the graded classical prime submodules, and then introduce and study a new topology on the graded classical prime spectrum, which generalizes the Zariski topology of a graded ring R to a graded R-module M, called Zariski topology of M and investigate several properties of the topology.

2. Preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [14, 20, 21] for these basic properties and more information on graded rings and modules.

Let G be a group with identity e and R be a commutative ring with identity 1_R. Then R is a *G-graded ring* if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are said to be *homogeneous* of degree g where R_g's are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \bigcup_{g\in G} R_g$. Let I be an ideal of R. Then I is called a *graded ideal* of (R, G) if $I = \bigoplus_{g\in G}(I \bigcap R_g)$. Thus, if $x \in I$, then $x = \sum_{g\in G} x_g$ with $x_g \in I$. An ideal of a G-graded ring need not be G-graded.

Let R be a G-graded ring and M an R-module. We say that M is a *G-graded R-module* (or *graded R-module*) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \bigoplus_{g\in G} M_g$ (as abelian groups) and $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. Here, R_gM_h denotes the additive subgroup of M consisting of all finite sums of elements r_gs_h with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g\in G} M_g$ and the elements of $h(M)$ are said to be *homogeneous*. Let $M = \bigoplus_{g\in G} M_g$ be a graded R-module and N a submodule of M. Then N is called a *graded submodule* of M if $N = \bigoplus_{g\in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the *g-component* of N.

Let R be a G-graded ring and M a graded R-module. A proper graded ideal I of R is said to be a graded prime ideal if whenever $rs \in I$, we have $r \in I$ or $s \in I$, where $r, s \in h(R)$ (see [26].) Let $Spec_g(R)$ denote the set of all graded prime ideals of R. The *graded radical* of I, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$ (see [26].) It is shown in [26, Proposition 2.5] that $Gr(I)$ is the intersection of all graded prime ideals of R containing I. A proper graded submodule N of M is said to be a graded prime submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $r \in (N :_R M) = \{r \in R : rM \subseteq N\}$ *or* $m \in N$ (see [9].) It is shown in [9, Proposition 2.7] that if N is a graded prime submodule of M, then $P := (N :_R M)$ is a graded prime ideal of R, and N is called *graded P-prime submodule*. Let $Spec_g(M)$

denote the set of all graded prime submodules of M.

A proper graded submodule N of M is called a graded classical prime submodule if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $sm \in N$ (see [4, 13].) Of course, every graded prime submodule is a graded classical prime submodule, but the converse is not true in general (see [4, Example 2.3].) Let $Cl.Spec_g(M)$ denote the set of all graded classical prime submodules of M. Obviously, some graded R-modules M have no graded classical prime submodules; such modules are called g-Cl.primeless. We know that $Spec_g(M) \subseteq Cl.Spec_g(M)$. As it is mentioned in [4, Example 2.3]), it happens sometimes that this containment is strict. We call M a *graded compatible* R-module if its graded classical prime submodules and graded prime submodules coincide, that is if $Spec_g(M) = Cl.Spec_g(M)$ (for example see [4, Theorem 3.2 and 3.3]). If R is a G-graded ring, then every graded classical prime ideal of R is a graded prime ideal. So, if we consider R as a graded R-module, it is graded compatible.

3. Topologies on the graded classical prime spectrum

Let R be a G-graded ring and M a graded R-module. For each graded ideal I of R, the graded variety of I is the set $V_R^g(I) = \{P \in Spec_g(R) \mid I \subseteq P\}$. Then the set $\xi^g(R) = \{V_R^g(I) \mid I$ is a graded ideal of $R\}$ satisfies the axioms for the closed sets of a topology on $Spec_g(R)$, called the Zariski topology on $Spec_g(R)$ (see [25, 26]).

In [12], $Spec_g(M)$ was endowed with quasi-Zariski topology. For each graded submodule N of M, let $V_*^g(N) = \{P \in Spec_g(M) \mid N \subseteq P\}$. In this case, the set $\zeta_*^g(M) = \{V_*^g(N) \mid N$ is a graded submodule of $M\}$ contains the empty set and $Spec_g(M)$, and it is closed under arbitrary intersections, but it is not necessarily closed under finite unions. The graded R-module M is said to be a g-Top module if $\zeta_*^g(M)$ is closed under finite unions. In this case $\zeta_*^g(M)$ satisfies the axioms for the closed sets of a unique topology τ_*^g on $Spec_g(M)$. The topology $\tau_*^g(M)$ on $Spec_g(M)$ is called the quasi-Zariski topology.

In [13], $Cl.Spec_g(M)$ was endowed with quasi-Zariski topology. For each graded submodule N of M, let $\mathbb{V}_*^g(N) = \{C \in Cl.Spec_g(M) \mid N \subseteq C\}$. In this case, the set $\eta_*^g(M) = \{\mathbb{V}_*^g(N) \mid N$ is a graded submodule of $M\}$ contains the empty set and $Cl.Spec_g(M)$, and it is closed under arbitrary intersections, but it is not necessarily closed under finite unions. The graded R-module M is said to be a g-$Cl.Top$ module module if $\eta_*^g(M)$ is closed under finite unions. In this case $\eta_*^g(M)$ satisfies the axioms for the closed sets of a unique topology ϱ_*^g on $Cl.Spec_g(M)$. In this case, the topology $\varrho_*^g(M)$ on $Cl.Spec_g(M)$ is called the quasi-Zariski topology.

Now we define another variety for a graded submodule N of a graded R-module M. We define the variety of N to be $\mathbb{V}^g(N) = \{C \in Cl.Spec_g(M) : (C :_R M) \supseteq (N :_R M)\}$.

The following proposition shows that this variety satisfies the topology axioms for closed sets.

THEOREM 3.1. *Let R be a G-graded ring and M a graded R-module. Then the following hold:*
(i) $\mathbb{V}^g(0) = Cl.Spec_g(M)$, *and* $\mathbb{V}^g(M) = \emptyset$.
(ii) If $\{N_\lambda\}_{\lambda\in\Delta}$ *is a family of graded submodules of* M*, then*

$$\bigcap_{\lambda\in\Delta} \mathbb{V}^g(N_\lambda) = \mathbb{V}^g\Big(\sum_{\lambda\in\Delta}(N_\lambda :_R M)M\Big).$$

(iii) For every pair N *and* K *of graded submodules of* M*, we have*

$$\mathbb{V}^g(N) \cup \mathbb{V}^g(K) = \mathbb{V}^g(N \cap K).$$

Proof. (i) It is clear.
(ii) Let $C \in \bigcap_{\lambda\in\Delta} \mathbb{V}^g(N_\lambda)$. Then $(N_\lambda :_R M) \subseteq (C :_R M)$, for all $\lambda \in \Delta$. It follows that $(N_\lambda :_R M)M \subseteq (C :_R M)M$, for all $\lambda \in \Delta$ and consequently $\sum_{\lambda\in\Delta}(N_\lambda :_R M)M \subseteq (C :_R M)M$. Since $(C :_R M)M \subseteq C$, we conclude that $(\sum_{\lambda\in\Delta}(N_\lambda :_R M)M :_R M) \subseteq (C :_R M)$. Therefore $C \in \mathbb{V}^g(\sum_{\lambda\in\Delta}(N_\lambda :_R M)M)$. Conversely, let $C \in \mathbb{V}^g(\sum_{\lambda\in\Delta}(N_\lambda :_R M)M)$. Then $(N_\lambda :_R M) \subseteq ((N_\lambda :_R M)M :_R M) \subseteq (\sum_{\lambda\in\Delta}(N_\lambda :_R M)M :_R M) \subseteq (C :_R M)$. It follows that $(N_\lambda :_R M) \subseteq (C :_R M)$ for all $\lambda \in \Delta$. Thus $C \in \mathbb{V}^g(N_\lambda)$ for all $\lambda \in \Delta$. Hence $P \in \bigcap_{\lambda\in\Delta} \mathbb{V}^g(N_\lambda)$.
(iii) Let $C \in \mathbb{V}^g(N \cap K)$. Then $(N \cap K :_R M) \subseteq (C :_R M)$, so that $(N :_R M)\ (K :_R M) \subseteq (N \cap K :_R M) \subseteq (C :_R M)$. Since C is a graded classical prime submodule by [4, Lemma 3.1], we have that $(C :_R M)$ is a graded prime ideal of R. It follows that $(N :_R M) \subseteq (C :_R M)$ or $(K :_R M) \subseteq (C :_R M)$ by [26, Proposition 1.2]. Thus $C \in \mathbb{V}^g(N)$ or $C \in \mathbb{V}^g(K)$. Hence $C \in \mathbb{V}^g(N) \cup \mathbb{V}^g(K)$. The opposite inclusion is easily seen. □

DEFINITION 3.2. Let R be a G-graded ring and M a graded R-module. Since $\eta^g(M) = \{\mathbb{V}^g(N) \mid N$ is a graded submodule of $M\}$ is closed under finite union, the family $\eta^g(M)$ satisfies the axioms of topological space for closed sets. So, there exists a topology on $Cl.Spec_g(M)$ called the Zariski topology and denoted by ϱ^g.

Let M be a G-graded R-Module and Y a subset of $Cl.Spec_g(M)$. We will denote $\bigcap_{C\in Y} C$ by $\Im(Y)$ (note that if $Y = \emptyset$, then $\Im(Y) = M$).

The assertions in the following lemma are straightforward to prove.

LEMMA 3.3. *Let R be a G-graded ring and M a graded R-module. Then:*
(i) $\mathbb{V}^g_*(N) \subseteq \mathbb{V}^g(N)$ *for every graded submodule* N *of* M.
(ii) If $Y_1 \subseteq Y_2$, *then* $\Im(Y_1) \supseteq \Im(Y_2)$.
(iii) $Y \subseteq \mathbb{V}^g_*(\Im(Y)) \subseteq \mathbb{V}^g(\Im(Y))$, *for every* $Y \subseteq Cl.Spec_g(M)$.
(iv) If $N \subseteq K \subseteq M$, *then* $\mathbb{V}^g(N) \supseteq \mathbb{V}^g(K)$ *and* $\mathbb{V}^g_*(N) \supseteq \mathbb{V}^g_*(K)$.
(v) For $Y \subseteq Cl.Spec_g(M)$, *we have* $\Im(\mathbb{V}^g(\Im(Y))) \subseteq \Im(\mathbb{V}^g_*(\Im(Y))) = \Im(Y)$; *in general the inclusion might be proper.*
(vi) Equalities $\mathbb{V}^g(N) = \mathbb{V}^g(\Im(\mathbb{V}^g(N)))$ *and* $\mathbb{V}^g_*(N) = \mathbb{V}^g_*(\Im(\mathbb{V}^g_*(N)))$ *hold for any graded submodule* N *of* M.

THEOREM 3.4. *Let R be a G-graded ring and M a graded R-module. Let N be a graded submodule of M and I a graded ideal of R. Then* $\mathbb{V}^g(IM) \cup \mathbb{V}^g(N) = \mathbb{V}^g(IN) = \mathbb{V}^g(IM \cap N)$.

Proof. Clearly $\mathbb{V}^g(IM) \cup \mathbb{V}^g(N) \subseteq \mathbb{V}^g(IM \cap N) \subseteq \mathbb{V}^g(IN)$. Let $C \in \mathbb{V}^g(IN)$. Then $(IN :_R M) \subseteq (C :_R M)$. It is easily seen that $I(N :_R M) \subseteq (IN :_R M)$. Since C is a graded classical prime submodule by [4, Lemma 3.1], we have $(C :_R M)$ is a graded prime ideal of R. Since $I(N :_R M) \subseteq (C :_R M)$ by [26, Proposition 1.2], we have $I \subseteq (C :_R M)$ or $(N :_R M) \subseteq (C :_R M)$ and consequently $(IM :_R M) \subseteq (C :_R M)$ or $(N :_R M) \subseteq (C :_R M)$. Hence $C \in \mathbb{V}^g(IM)$ or $C \in \mathbb{V}^g(N)$, i.e. $C \in \mathbb{V}^g(IM) \cup \mathbb{V}^g(N)$. Thus $\mathbb{V}^g(IN) \subseteq \mathbb{V}^g(IM) \cup \mathbb{V}^g(N)$. □

COROLLARY 3.5. *Let R be a G-graded ring and M a graded R-module. Let I and J be graded ideals of R. Then $\mathbb{V}^g(IM) \cup \mathbb{V}^g(JM) = \mathbb{V}^g(IJM) = \mathbb{V}^g(IM \cap JM)$.*

LEMMA 3.6. *Let R be a G-graded ring and M a graded R-module. Let N be a graded submodule of M and I a graded ideal of R. Then:*
(i) If $(N :_R M) = (K :_R M)$, then $\mathbb{V}^g(N) = \mathbb{V}^g(K)$ for all graded submodules N and K of M. The converse is true if N and K are graded classical prime submodules.
(ii) $\mathbb{V}^g(IM) = \mathbb{V}^g(Gr(I)M) = \mathbb{V}^g_*(IM) = \mathbb{V}^g_*(Gr(I)M)$;
(iii) $\mathbb{V}^g(N) = \mathbb{V}^g((N :_R M)M) = \mathbb{V}^g(Gr((N :_R M))M) = \mathbb{V}^g_*((N :_R M)M) = \mathbb{V}^g_*(Gr((N :_R M))M)$.

Proof. (i) It is clear.
(ii) Clearly $\mathbb{V}^g_*(Gr(I)M) \subseteq \mathbb{V}^g_*(IM) \subseteq \mathbb{V}^g(IM)$ by Lemma 3.3(i) and (iv). Next we claim that $\mathbb{V}^g(IM) \subseteq \mathbb{V}^g_*(Gr(I)M)$. To see this, let $C \in \mathbb{V}^g(IM)$. Then $I \subseteq (IM :_R M) \subseteq (C :_R M)$. Since C is a graded classical prime submodule by [4, Lemma 3.1], we have that $(C :_R M)$ is a graded prime ideal of R. It follows that $Gr(I) \subseteq Gr((C :_R M)) = (C : M)$ by [26, Proposition 2.4(5)]. Thus $Gr(I)M \subseteq (C :_R M)M \subseteq C$. Hence $C \in \mathbb{V}^g_*(Gr(I)M)$. Consequently, we have $\mathbb{V}^g_*(Gr(I)M) \subseteq \mathbb{V}^g_*(IM) \subseteq \mathbb{V}^g(IM) \subseteq \mathbb{V}^g_*(Gr(I)M)$, and so these terms are all equal. Replacing the middle I by $Gr(I)$, we get (i).
(iii) It suffices to show that $\mathbb{V}^g(N) = \mathbb{V}^g((N :_R M)M)$ by (i). Clearly $((N :_R M)M :_R M) = (N :_R M)$. Now $C \in \mathbb{V}^g(N) \Leftrightarrow ((N :_R M)M :_R M) = (N :_R M) \subseteq (C :_R M) \Leftrightarrow C \in \mathbb{V}^g((N :_R M)M)$. □

Throughout the rest of this paper, we assume that $Cl.Spec_g(M)$ is non-empty, unless stated otherwise, and is equipped with Zarisky topology for every graded R-module under consideration.

The map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$ where $\overline{R} = R/Ann(M)$ defined by $\psi(C) = \overline{(C :_R M)}$ for every $C \in Cl.Spec_g(M)$ will be called the natural map of $Cl.Spec_g(M)$.

THEOREM 3.7. *Let R be a G-graded ring and M a graded R-module. The natural map ψ of $Cl.Spec_g(M)$ is continuous for the Zariski topologies; more precisely, $\psi^{-1}(V^g_{\overline{R}}(\overline{I})) = \mathbb{V}^g(IM)$ for every graded ideal I of R containing $Ann(M)$.*

Proof. Let $\overline{I}$ be a graded ideal of $\overline{R}$, $V^g_{\overline{R}}(\overline{I}) \in \xi^g(\overline{R})$ and $C \in \psi^{-1}(V^g_{\overline{R}}(\overline{I}))$. Then $\psi(C) = \overline{(C :_R M)} \in V^g_{\overline{R}}(\overline{I})$, hence $\overline{I} \subseteq \overline{(C :_R M)}$. This implies $I \subseteq (C :_R M)$. So $IM \subseteq (C :_R M)M \subseteq C$ and hence $C \in \mathbb{V}^g_*(IM)$. By Lemma 3.6(ii), $C \in \mathbb{V}^g_*(IM) =$

$\mathbb{V}^g(IM)$. Therefore $\psi^{-1}(V^g_{\overline{R}}(\overline{I})) \subseteq \mathbb{V}^g(IM)$. For the converse inclusion, let $C \in \mathbb{V}^g(IM)$. Thus $IM \subseteq C \in Cl.Spec_g(M)$ and so $I \subseteq (C :_R M) \subseteq Spec_g(R)$ by [4, Lemma 3.1]. So we get $\psi(C) = \overline{(C :_R M)} \in V^g_{\overline{R}}(\overline{I})$. It follows that $C \in \psi^{-1}(V^g_{\overline{R}}(\overline{I}))$. Therefore $\psi^{-1}(V^g_{\overline{R}}(\overline{I})) = \mathbb{V}^g(IM)$. □

Let R be a G-graded ring and M be a graded R-module and $p \in Spec_g(R)$. Then the set $Cl.Spec^p_g(M)$ is defined to be $\{P \in Cl.Spec_g(M) \mid (P :_R M) = p\}$.

THEOREM 3.8. *Let R be a G-graded ring and M a graded R-module. The following statements are equivalent for any $C, D \in Cl.Spec_g(M)$:*
(i) The natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$ is injective.
(ii) If $\mathbb{V}^g(C) = \mathbb{V}^g(D)$, then $C = D$.
(iii) $|Cl.Spec^p_g(M)| \leq 1$ for every graded prime ideal p of R.

Proof. (i)$\Rightarrow$(ii): Suppose that $\mathbb{V}^g(C) = \mathbb{V}^g(D)$. Since $C, D \in Cl.Spec_g(M)$ by Lemma 3.6(i), we have $(C :_R M) = (D :_R M)$ and hence $\overline{(C :_R M)} = \overline{(D :_R M)}$. Thus $\psi(C) = \psi(D)$. Since ψ is injective, $C = D$.
(ii)$\Rightarrow$(iii): Let $|Cl.Spec^p_g(M)| > 1$ and let $C, D \in Cl.Spec^p_g(M)$ be such that $C \neq D$. Thus $(C :_R M) = (D :_R M) = p$. By Lemma 3.6(i), $\mathbb{V}^g(D) = \mathbb{V}^g(C)$ and by the hypothesis we get $D = C$, which is a contradiction.
(iii)$\Rightarrow$(i): Suppose that $\psi(C) = \psi(D)$. Hence $\overline{(C :_R M)} = \overline{(D :_R M)}$. By [4, Lemma 3.1], we conclude that $(C :_R M) = (D :_R M) = p$ for some graded prime ideal p of R and since $|Cl.Spec^p_g(M)| \leq 1$, we get $D = C$. □

THEOREM 3.9. *Let R be a G-graded ring and M a graded R-module. Let $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$ be the natural map of $Cl.Spec_g(M)$. If ψ is surjective, then ψ is both closed and open. More precisely, for every graded submodule N of M, $\psi(\mathbb{V}^g(N)) = V^g_{\overline{R}}(\overline{(N :_R M)})$ and $\psi(Cl.Spec_g(M) - \mathbb{V}^g(N)) = Spec_g(\overline{R}) - V^g_{\overline{R}}(\overline{(N :_R M)})$.*

Proof. By Theorem 3.7, ψ is a continuous map such that $\psi^{-1}(V^g_{\overline{R}}(\overline{I})) = \mathbb{V}^g(IM)$ for every graded ideal I of R containing $Ann(M)$. Hence for every graded submodule N of M, $\psi^{-1}(V^g_{\overline{R}}(\overline{(N :_R M)})) = \mathbb{V}^g((N :_R M)M) = \mathbb{V}^g(N)$ (Lemma 3.6(iii)). It follows that $\psi((\mathbb{V}^g(N)) = \psi \circ \psi^{-1}(V^g_{\overline{R}}(\overline{(N :_R M)})) = V^g_{\overline{R}}(\overline{(N :_R M)})$ as ψ is surjective. Similarly, $\psi(Cl.Spec_g(M) - \mathbb{V}^g(N)) = \psi(\psi^{-1}(Spec_g(\overline{R})) - \psi^{-1}(V^g_{\overline{R}}(\overline{N :_R M}))) = \psi(\psi^{-1}(Spec_g(\overline{R}) - V^g_{\overline{R}}(\overline{(N :_R M)}))) = \psi \circ \psi^{-1}(Spec_g(\overline{R}) - V^g_{\overline{R}}(\overline{(N :_R M)})) = Spec_g(\overline{R}) - V^g_{\overline{R}}(\overline{(N :_R M)})$. □

4. Zariski Topology on $Cl.Spec_g(M)$ and spectral spaces

For any $r \in h(R)$, the set $GX_r = Spec_g(R) - V^g_R(rR)$ is open in $Spec_g(R)$ and the family $\digamma = \{GX_t : t \in h(R)\}$ forms a base for the Zariski topology on $Spec_g(R)$

(see [26, Proposition 3.4]). Each GX_r is known to be quasi-compact (see [26, Proposition 3.8]). In this section, we introduce a base for the Zariski topology on $Cl.Spec_g(M)$ for any graded R-module M, which is similar to that on $Spec_g(R)$. Let $r \in h(R)$, we define $GX_r^{cl} = Cl.Spec_g(M) - \mathbb{V}^g((rM)$. Then every GX_r^{cl} is an open set of $Cl.Spec_g(M)$, $GX_0^{cl} = \emptyset$, and $GX_1^{cl} = Cl.Spec_g(M)$.

Theorem 4.1. *Let R be a G-graded ring and M a graded R-module with natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$ and $r \in h(R)$. Then:*
(i) $\psi^{-1}(GX_{\overline{r}}) = GX_r^{cl}$.
(ii) $\psi(GX_r^{cl}) \subseteq GX_{\overline{r}}$; the equality holds if ψ is surjective.
(iii) $GX_{rs}^{cl} = GX_r^{cl} \cap GX_s^{cl}$, for any $r, s \in h(R)$.

Proof. (i) $\psi^{-1}(GX_{\overline{r}}) = \psi^{-1}(Spec_g(\overline{R}) - V_{\overline{R}}^g(\overline{r}\overline{R})) = Cl.Spec_g(M) - \psi^{-1}(V_{\overline{R}}^g(\overline{r}\overline{R}))$. By Theorem 3.7, we have $\psi^{-1}(V_{\overline{R}}^g(\overline{r}\overline{R})) = \mathbb{V}^g(rM)$. It follows that $\psi^{-1}(GX_{\overline{r}}) = Cl.Spec_g(M) - \mathbb{V}^g(rM) = GX_r^{cl}$.
(ii) Follows from (i).
(iii) By (i), $GX_{rs}^{cl} = \psi^{-1}(GX_{\overline{rs}})$. By [26, Proposition 3.6(1)], $GX_{\overline{rs}} = GX_{\overline{r}} \cap GX_{\overline{s}}$. Thus $GX_{rs}^{cl} = \psi^{-1}(GX_{\overline{rs}}) = \psi^{-1}(GX_{\overline{r}} \cap GX_{\overline{s}}) = \psi^{-1}(GX_{\overline{r}}) \cap \psi^{-1}(GX_{\overline{s}}) = GX_r^{cl} \cap GX_s^{cl}$. □

Theorem 4.2. *Let R be a G-graded ring and M a graded R-module. The set $B = \{GX_r^{cl} : r \in h(R)\}$ is a base for the Zariski topology on $Cl.Spec_g(M)$.*

Proof. Let U be any open set in $Cl.Spec_g(M)$. By Lemma 3.6, $U = Cl.Spec_g(M) - \mathbb{V}^g(IM)$ for some graded ideal I of R. Note that $\mathbb{V}^g(IM) = \mathbb{V}^g(\sum_{r_i \in h(I)} r_i M) = \mathbb{V}^g(\sum_{r_i \in h(I)} (r_i M : M)M) = \bigcap_{r_i \in h(I)} \mathbb{V}^g(r_i M)$ by Theorem 3.1(ii). Hence $U = Cl.Spec_g(M) - \mathbb{V}^g(IM) = Cl.Spec_g(M) - \bigcap_{r_i \in h(I)} \mathbb{V}^g(r_i M) = \bigcup_{r_i \in h(I)} GX_{r_i}^{cl}$. Therefore the set B is a base for for the Zariski topology on $Cl.Spec_g(M)$. □

Theorem 4.3. *Let R be a G-graded ring and M a graded R-module. If the natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$ is surjective, then the open set GX_r^{cl} is quasi-compact for each $r \in h(R)$; In particular, the space $Cl.Spec_g(M)$ is quasi-compact.*

Proof. Since $B = \{GX_r^{cl} : r \in h(R)\}$ forms a base for the Zariski topology on $Cl.Spec_g(M)$ by Theorem 4.2, for every open cover of GX_r^{cl}, there is a set $\{r_\alpha \in h(R) \mid \alpha \in \Lambda\}$ such that $GX_r^{cl} \subseteq \bigcup_{\alpha \in \Lambda} GX_{r_\alpha}^{cl}$. By Theorem 4.1(ii) $GX_{\overline{r}} = \psi(GX_r^{cl}) \subseteq \bigcup_{\alpha \in \Lambda} \psi(GX_{r_\alpha}^{cl}) = \bigcup_{\alpha \in \Lambda} GX_{\overline{r}_\alpha}$. Since $GX_{\overline{r}}$ is quasi-compact ([26, Proposition 3.8]), there exists a finite subset $\Lambda' \subset \Lambda$ such that $GX_{\overline{r}} \subseteq \bigcup_{\alpha \in \Lambda'} GX_{\overline{r}_\alpha}$. By Theorem 4.1(i), we get $GX_r^{cl} = \psi^{-1}(GX_{\overline{r}}) \subseteq \bigcup_{\alpha \in \Lambda'} GX_{r_\alpha}^{cl}$. □

Let M be a G-graded R-module and Y be any subset of $Cl.Spec_g(M)$. We will denote the closure of Y in $Cl.Spec_g(M)$ for the Zariski topology by $Cl(Y)$.

Theorem 4.4. *Let R be a G-graded ring, M a graded R-module and $Y \subseteq Cl.Spec_g(M)$. Then $\mathbb{V}^g(\Im(Y)) = Cl(Y)$. In particular, Y is closed if and only if $\mathbb{V}_g(\Im(Y)) = Y$.*

Proof. By Lemma 3.3(iii), $Y \subseteq \mathbb{V}^g(\Im(Y))$. Let $\mathbb{V}^g(N)$ be any closed subset of $Cl.Spec_g(M)$ containing Y. Then $(N :_R M) \subseteq (C :_R M)$ for every $C \in Y$. This implies that $(N :_R M) \subseteq \bigcap_{C \in Y}(C :_R M) \subseteq (\Im(Y) :_R M)$. Hence, for every $D \in \mathbb{V}^g(\Im(Y))$, $(N :_R M) \subseteq (\Im(Y) :_R M) \subseteq (D :_R M)$, that is, $\mathbb{V}^g(\Im(Y)) \subseteq \mathbb{V}^g(N)$. Hence $\mathbb{V}^g(\Im(Y))$ is the smallest closed subset of $Cl.Spec_g(M)$ including Y, so $\mathbb{V}^g(\Im(Y)) = Cl(Y)$. □

THEOREM 4.5. *Let R be a G-graded ring, M a graded R-module and $C \in Cl.Spec_g(M)$. Then:*
(i) $Cl(\{C\}) = \mathbb{V}^g(C)$.
(ii) For any $D \in Cl.Spec_g(M)$, $D \in Cl(\{C\})$, if and only if $(C :_R M) \subseteq (D :_R M)$ if and only if $\mathbb{V}^g(D) \subseteq \mathbb{V}^g(C)$.

Proof. (i) Apply Theorem 4.4 by taking $Y = \{C\}$. (ii) This follows from (i). □

A topological space X is called *irreducible* if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A (non-empty) subset Y of a topology space X is called *an irreducible set* if the subspace Y of X is irreducible, equivalently if Y_1 and Y_2 are closed subsets of X and satisfy $Y \subseteq Y_1 \cup Y_2$, then $Y \subseteq Y_1$ or $Y \subseteq Y_2$ (see [18].)

The following corollary is a result of Theorem 4.5(i).

COROLLARY 4.6. *Let R be a G-graded ring and M a graded R-module. Then for every graded classical prime submodule C of M, $\mathbb{V}^g(C)$ is an irreducible closed subset of $Cl.Spec_g(M)$.*

THEOREM 4.7. *Let R be a G-graded ring, M a graded R-module and $Y \subseteq Cl.Spec_g(M)$. If $\Im(Y)$ is a graded classical prime submodule of M, then Y is irreducible.*

Proof. Assume that $C := \Im(Y)$ is a graded classical prime submodule of M. By Theorem 4.4, $cl(Y) = \mathbb{V}^g(C)$. Now let $Y \subseteq Y_1 \cup Y_2$, where Y_1, Y_2 are closed sets. Then we have $\mathbb{V}^g(C) = cl(Y) \subseteq Y_1 \cup Y_2$. Since $\mathbb{V}^g(C) \subseteq Y_1 \cup Y_2$ and by Corollary 4.6, $\mathbb{V}^g(C)$ is irreducible, $\mathbb{V}^g(C) \subseteq Y_1$ or $\mathbb{V}^g(C) \subseteq Y_2$. Hence, $Y \subseteq Y_1$ or $Y \subseteq Y_2$. Thus Y is irreducible. □

THEOREM 4.8. *Let R be a G-graded ring, M a graded R-module and $Y \subseteq Cl.Spec_g(M)$ such that $(\Im(Y) :_R M) = p$ is a graded prime ideal of R. If $Cl.Spec_g^p(M) \neq \emptyset$, then Y is irreducible.*

Proof. Let $C \in Cl.Spec_g^p(M)$. Since $(C :_R M) = p = (\Im(Y) :_R M)$ by Lemma 3.6(i), we have $\mathbb{V}^g(C) = \mathbb{V}^g(\Im(Y)$. Then $\mathbb{V}^g(C) = \mathbb{V}^g(\Im(Y)) = Cl(Y)$ by Theorem 4.4. Therefore, $Cl(Y)$ is irreducible and so is Y. □

Let Y be a closed subset of a topological space. An element $y \in Y$ is called *a generic point* of Y if $Y = cl(\{y\})$. Note that a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space (see [15]).

THEOREM 4.9. *Let R be a G-graded ring and M a graded R-module. Then:*

(i) Every $C \in Cl.Spec_g(M)$ *is a generic point of the irreducible closed subset* $\mathbb{V}^g(C)$.
(ii) Every finite irreducible closed subset of $Cl.Spec_g(M)$ *has a generic point.*

Proof. (i) is clear by Theorem 4.5(i).
(ii) Let Y be a finite irreducible closed subset of $Cl.Spec_g(M)$ and $Y = \{C_1, \dots, C_n\}$, where $C_i \in Cl.Spec_g(M)$, $n \in \mathbb{N}$. Then $Y = cl(Y) = \mathbb{V}^g(C_1 \cap \cdots \cap C_n) = \mathbb{V}^g(C_1) \cup \cdots \cup \mathbb{V}^g(C_n)$ by Theorem 4.4 and Theorem 3.1(iii). Since Y is irreducible, $Y = \mathbb{V}^g(C_i)$ for some $i(1 \le i \le n)$. Now by (i), C_i is a generic point of Y. □

THEOREM 4.10. *Let R be a G-graded ring and M a graded R-module. Then the following statements are equivalent for any* $C, D \in Cl.Spec_g(M)$:
(i) $Cl.Spec_g(M)$ *is a* T_0*-space.*
(ii) The natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$ *is injective.*
(iii) If $\mathbb{V}^g(C) = \mathbb{V}^g(D)$, *then* $C = D$.
(iv) $\mid Cl.Spec_g^p(M) \mid \le 1$ *for every graded prime ideal p of R.*

Proof. (i)$\Leftrightarrow$(iii) This follows from Theorem 4.5 and the fact that a topological space is a T_0-space if and only if the closures of distinct points are distinct.

The equivalences of (ii), (iii) and (iv) are proved in Theorem 3.8. □

LEMMA 4.11. *Let R be a G-graded ring and M a graded R-module. Let* $C \in Cl.Spec_g(M)$, *and* $\Omega = \{(D :_R M) \mid D \in Cl.Spec_g(M)\} \subseteq Spec_g(R)$. *Then the set* $\{C\}$ *is closed in* $Cl.Spec_g(M)$ *if and only if*
(i) $p = (C :_R M)$ *is a maximal element of the set* Ω, *and*
(ii) $Cl.Spec_g^p(M) = \{C\}$, *that is,* $|Cl.Spec_g^p(M)| = 1$.

Proof. Assume that $\{C\}$ is closed in $Cl.Spec_g(M)$. By Theorem 4.5(i), we have $\{C\} = Cl(\{C\}) = \mathbb{V}^g(C)$. Let $q \in \Omega$ such that $p \subseteq q$. Then there exists $D \in Cl.Spec_g(M)$ such that $q = (D :_R M)$. Hence $(C :_R M) = p \subseteq q = (D :_R M)$. Thus $D \in \mathbb{V}^g(C) = \{C\}$ so that $D = C$ and $p = q$, which proves (i). Let $C' \in Cl.Spec_g^p(M)$. Thus $(C' :_R M) = p = (C :_R M)$ and hence $C' \in \mathbb{V}^g(C) = \{C\}$. So $C' = C$ and (ii) follows. Conversely, we assume (i) and (ii), and show that $\mathbb{V}^g(C) \subseteq \{C\}$. If $D \in \mathbb{V}^g(C)$, then $q = (D :_R M) \supseteq (C :_R M) = p$. Therefore (i) implies $q = p$ and consequently (ii) implies $D = C$ by (ii). Thus $\mathbb{V}^g(C) \subseteq \{C\}$. Since C is a graded classical prime we have $\{C\} \subseteq \mathbb{V}^g(C)$, so that $\mathbb{V}^g(C) = \{C\}$. By Theorem 4.5(i), $Cl(\{C\}) = \{C\}$. Therefore the set $\{C\}$ is closed in $Cl.Spec_g(M)$. □

Let X be a topological space and let x_1 and x_2 be two points in X. We say that x_1 and x_2 can be separated if each lies in an open set which does not contain the other point. X is a T_1-space if any two distinct points in X can be separated. A topological space X is a T_1-space if and only if all points of X are closed in X (see [18]).

THEOREM 4.12. *Let R be a G-graded ring, M a graded R-module and* $\Omega = \{(D :_R M) \mid D \in Cl.Spec_g(M)\} \subseteq Spec_g(R)$. *Then* $Cl.Spec_g(M)$ *is a* T_1*-space if and only if*
(i) $(C :_R M) = p$ *is a maximal element of* Ω *for all* $C \in Cl.Spec_g(M)$, *and*
(ii) $|Cl.Spec_g^p(M)| = 1$ *for every graded prime ideal p of R.*

Proof. If $Cl.Spec_g(M)$ is a T_1-space then the singleton sets are closed in $Cl.Spec_g(M)$. So we get (i) and (ii) by Lemma 4.11.

Conversely, (i) and (ii) are equivalent so that the singleton set $\{C\}$ is closed in $Cl.Spec_g(M)$ for every $C \in Cl.Spec_g(M)$, that is, $Cl.Spec_g(M)$ is a T_1-space. □

A *spectral space* is a topological space homomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology. Spectral spaces have been characterized by Hochster [15] as the topological space W which satisfy the following conditions:
(i) W is a T_0-space.
(ii) W is quasi-compact.
(iii) the quasi-compact open subsets of W are closed under finite intersections and form an open basis.
(iv) each irreducible closed subset of W has a generic point.

THEOREM 4.13. *Let R be a G-graded ring and M a graded R-module such that $Cl.Spec_g(M)$ is finite and $|Cl.Spec^p_g(M)| \leq 1$ for every graded prime ideal p of R. Then $Cl.Spec_g(M)$ is a spectral space.*

Proof. Since $Cl.Spec_g(M)$ is finite, every subset of $Cl.Spec_g(M)$ is quasi-compact. Hence the quasi-compact open sets of $Cl.Spec_g(M)$ are closed under finite intersections and form an open basis. Also by Theorem 4.10, $Cl.Spec_g(M)$ a T_0-space. Moreover, every irreducible closed subset of $Cl.Spec_g(M)$ has a generic point by Theorem 4.9. Therefore $Cl.Spec_g(M)$ is a spectral space by Hochster's characterization. □

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