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Corrected proof
Available online 21.07.2018

research paper
оригинални научни рад

COINCIDENCE POINT AND COMMON FIXED POINT RESULTS FOR A HYBRID PAIR OF MAPPINGS VIA DIGRAPHS

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Abstract. In this paper, we introduce the concept of $(\alpha, \psi, \xi) - G$-contractive mappings in a metric space endowed with a directed graph G. We investigate the existence and uniqueness of points of coincidence and common fixed points for such mappings under some conditions. Our results extend and generalize several well-known comparable results in the literature. Some examples are provided to justify the validity of our results.

1. Introduction

Fixed point theory plays an important role in several branches of mathematics and applied sciences. In 1969, Nadler [19] extended the famous Banach contraction theorem to set-valued form. Afterwards, series of articles has been dedicated to the development of fixed point theory of multi-valued mappings in metric spaces (see [1,3]). Later on, hybrid fixed point theory for nonlinear single-valued and multi-valued mappings takes a prominent place in many aspects (see [16, 17]).

In 2012, Samet et al. [20] introduced the notion of $\alpha - \psi$-contractive mappings and obtained some fixed point theorems for such mappings in complete metric spaces. Some results in this direction are given in [1, 3, 14, 18]. In recent investigations, the study of fixed point theory endowed with a graph presents a new development in the domain of contractive type multi-valued theory. Many important results from [2–4, 7–9, 11, 12, 15, 20] have become the source of motivation for many researchers in fixed point theory. Motivated by the work in [1,17,21], we will introduce the notion of $(\alpha, \psi, \xi) - G$-contractive mappings of a hybrid pair of single-valued and multi-valued mappings and prove some coincidence point and common fixed point results for such mappings. As consequences of this study, we deduce several related results in metric fixed point theory. Finally, some examples are provided to illustrate the results.

2010 Mathematics Subject Classification: 54H25, 47H10
Keywords and phrases: Digraph; weakly compatible mappings; point of coincidence; common fixed point.

2. Some basic concepts

For a metric space (X, d), we let $CB(X)$ and $CL(X)$ be the set of all nonempty closed bounded subsets of X and the set of all nonempty closed subsets of X, respectively. A point $x \in X$ is called a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$. For every $A, B \in CL(X)$, let

$$H(A,B) = \begin{cases} \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}, & \text{if the maximum exists} \\ \infty, & \text{otherwise} \end{cases}$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. Such map H is called the generalized Hausdorff metric induced by the metric d.

Let Ψ be a class of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(ψ1) ψ is a nondecreasing function;

(ψ2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ.

Remark 2.1. ([17]) For each $\psi \in \Psi$, we see that the following assertions hold:
(i) $\lim_{n\to\infty} \psi^n(t) = 0$, for all $t > 0$; (ii) $\psi(t) < t$, for each $t > 0$; (iii) $\psi(0) = 0$.

Definition 2.2. ([3]) Let T be a self-mapping on a nonempty set X and $\alpha : X \times X \to [0, \infty)$ be another mapping. We say that T is α-admissible if the following condition holds: $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.

Definition 2.3. ([3]) Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a mapping. A mapping $T : X \to CL(X)$ is called α_*-admissible if $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha_*(Tx, Ty) \geq 1$, where $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$.

In 2014, Ali et al. [1] introduced a family Ξ of functions $\xi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(ξ1) ξ is continuous;

(ξ2) ξ is nondecreasing on $[0, \infty)$;

(ξ3) $\xi(0) = 0$ and $\xi(t) > 0$ for all $t \in (0, \infty)$;

(ξ4) ξ is subadditive.

Example 2.4. ([1]) Suppose that $\phi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, \infty)$, for each $\epsilon > 0$, $\int_0^{\epsilon} \phi(t)\, dt > 0$ and for each $a, b > 0$, we have

$$\int_0^{a+b} \phi(t)\, dt \leq \int_0^{a} \phi(t)\, dt + \int_0^{b} \phi(t)\, dt.$$

Define $\xi : [0, \infty) \to [0, \infty)$ by $\xi(t) = \int_0^t \phi(w)\, dw$ for each $t \in [0, \infty)$. Then, $\xi \in \Xi$.

Lemma 2.5. *([1]) Let (X, d) be a metric space and let $\xi \in \Xi$. Then $(X, \xi \circ d)$ is a metric space.*

Lemma 2.6. *([1]) Let (X,d) be a metric space, let $\xi \in \Xi$ and let $B \in CL(X)$. Assume that there exists $x \in X$ such that $\xi(d(x,B)) > 0$. Then there exists $y \in B$ such that $\xi(d(x,y)) < q\xi(d(x,B))$, where $q > 1$.*

Definition 2.7. ([17]) Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be a mapping. A mapping $T : X \to CL(X)$ is called α_*-admissible with respect to f (a self-mapping on X) if the following condition holds: $x, y \in X$, $\alpha(fx, fy) \geq 1 \Rightarrow \alpha_*(Tx,Ty) \geq 1$, where $\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx,\ b \in Ty\}$.

Definition 2.8. ([17]) Let (X,d) be a metric space. The mappings $T : X \to CL(X)$ and $f : X \to X$ are called (α,ψ,ξ)-contractive if there exist three functions $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0,\infty)$ such that $x, y \in X$, $\alpha(fx,fy) \geq 1 \Rightarrow \xi(H(Tx,Ty)) \leq \psi(\xi(M(x,y)))$, where $M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}$.

If $\psi \in \Psi$ is strictly increasing, then T and f are called strictly (α,ψ,ξ)-contractive mappings.

Definition 2.9. Let (X,d) be a metric space and $T : X \to CL(X)$ and $f : X \to X$ be two mappings. If $y = fx \in Tx$ for some x in X, then x is called a coincidence point of T and f and y is called a point of coincidence of T and f.

Definition 2.10. Let (X,d) be a metric space. The mappings $T : X \to CL(X)$ and $f : X \to X$ are called compatible if and only if $fTx \in CL(X)$ for all $x \in X$ and $H(Tfx_n, fTx_n) \to 0$ whenever (x_n) is a sequence in X such that $Tx_n \to M \in CL(X)$ and $fx_n \to t \in M$.

Definition 2.11. Let (X,d) be a metric space. The mappings $T : X \to CL(X)$ and $f : X \to X$ are called weakly compatible if they commute at their coincidence points, i.e., if $Tfx = fTx$ whenever $fx \in Tx$.

Proposition 2.12. *Let (X,d) be a metric space and $T : X \to CL(X)$ and $f : X \to X$ be weakly compatible. If T and f have a unique point of coincidence $y = fx \in Tx$, then y is the unique common fixed point of T and f in X.*

Proof. Let $y = fx \in Tx$ for some x in X. Since f and T are weakly compatible, $Tfx = fTx$. This implies that $fy \in Ty$ and hence fy is a point of coincidence of f and T. As y is the unique point of coincidence of f and T, it follows that $y = fy \in Ty$. This shows that y is a common fixed point of f and T.

Let z be another common fixed point of f and T in X i.e., $z = fz \in Tz$. Since f and T have a unique point of coincidence in X, it follows that $fy = fz$ and hence $y = z$. This proves that y is the unique common fixed point of f and T in X. □

We next review some basic notions in graph theory.

Let (X,d) be a metric space. We assume that G is a digraph with the set of vertices $V(G) = X$ and the set $E(G)$ of its edges contains all the loops, i.e., $\Delta \subseteq E(G)$ where $\Delta = \{(x,x) : x \in X\}$. We also assume that G has no parallel edges. So we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention, $E(\tilde{G}) = E(G) \cup E(G^{-1})$.

Our graph theory notations and terminology are standard and can be found in all graph theory books (for example [6,10,13]). If x, y are vertices of the digraph G, then a path in G from x to y of length n ($n \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

Definition 2.13. Let (X, d) be a metric space endowed with a graph G. The mappings $T : X \to CL(X)$ and $f : X \to X$ are called $(\alpha, \psi, \xi) - G$-contractive if there exist three functions $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0, \infty)$ such that $x, y \in X$ with $(fx, fy) \in E(\tilde{G})$, $\alpha(fx, fy) \geq 1 \Rightarrow \xi(H(Tx, Ty)) \leq \psi(\xi(M(fx, fy)))$, where $M(fx, fy) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}$.

If $\psi \in \Psi$ is strictly increasing, then T and f are called strictly $(\alpha, \psi, \xi) - G$-contractive mappings.

It is valuable to note that strictly (α, ψ, ξ)-contractive mappings are strictly $(\alpha, \psi, \xi)-G_0$-contractive. But strictly $(\alpha, \psi, \xi) - G$-contractive mappings need not be strictly (α, ψ, ξ)-contractive mappings (see Remark 3.18).

3. Main results

Assume that (X, d) is a metric space endowed with a reflexive digraph G such that $V(G) = X$ and G has no parallel edges. Let $f : X \to X$ and $T : X \to CL(X)$ be such that $T(X) \subseteq f(X)$. Let $x_0 \in X$ be arbitrary. Since $T(X) \subseteq f(X)$, there exists an element $x_1 \in X$ such that $fx_1 \in Tx_0$. Continuing in this way, we can construct a sequence (fx_n) such that $fx_n \in Tx_{n-1}$, $n = 1, 2, 3, \ldots$.

Definition 3.1. Let (X, d) be a metric space endowed with a graph G and $f : X \to X$ and $T : X \to CL(X)$ be such that $T(X) \subseteq f(X)$. Denote by C^{α}_{fT} the set of all elements x_0 of X such that $(fx_n, fx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \ldots$ and $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$.

Taking $f = I$, the identity map on X, C^{α}_{fT} becomes C^{α}_{T} which is the collection of all elements x_0 of X such that $(x_n, x_m) \in E(\tilde{G})$ for m, $n = 0, 1, \ldots$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, for every sequence (x_n) such that $x_n \in Tx_{n-1}$.

Theorem 3.2. *Let (X, d) be a metric space endowed with a graph G. Let $T : X \to CL(X)$ and $f : X \to X$ be strictly $(\alpha, \psi, \xi) - G$-contractive mappings. Suppose that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X with the following property:*

(*) If (fx_n) is a sequence in X such that $fx_n \to x$ and $(fx_n, fx_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$ and $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then there exists a subsequence (fx_{n_i}) of (fx_n) such that $(fx_{n_i}, x) \in E(\tilde{G})$ and $\alpha(fx_{n_i}, x) \geq 1$ for all $i \geq 1$.

Then f and T have a point of coincidence in X if $C^{\alpha}_{fT} \neq \emptyset$. Moreover, f and T have a unique point of coincidence in X if the graph G has the following property:

(**) If x, y are points of coincidence of f and T in X, then $(x, y) \in E(\tilde{G})$ and $\alpha(x, y) \geq 1$.

Furthermore, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof. Suppose that $C^{\alpha}_{fT} \neq \emptyset$. We choose an $x_0 \in C^{\alpha}_{fT}$ and keep it fixed. Since $Tx_0 \subseteq f(X)$, there exists $x_1 \in X$ such that $fx_1 \in Tx_0$ and $(fx_0, fx_1) \in E(\tilde{G})$ with $\alpha(fx_0, fx_1) \geq 1$. If $x_1 = x_0$, then f and T have a point of coincidence in X. So, we assume that $x_1 \neq x_0$. If $fx_1 \in Tx_1$, then we have nothing to prove. Therefore, let $fx_1 \notin Tx_1$.

Since T and f are strictly $(\alpha, \psi, \xi) - G$-contractive, we have

$$\begin{aligned}
\xi(H(Tx_0, Tx_1)) &\leq \psi\left(\xi\left(\max\left\{\begin{array}{l} d(fx_0, fx_1), d(fx_0, Tx_0), d(fx_1, Tx_1), \\ \frac{d(fx_0, Tx_1) + d(fx_1, Tx_0)}{2} \end{array}\right\}\right)\right) \\
&\leq \psi(\xi(\max\{d(fx_0, fx_1), d(fx_1, Tx_1), \frac{d(fx_0, Tx_1)}{2}\})) \\
&\leq \psi\left(\xi\left(\max\left\{\begin{array}{l} d(fx_0, fx_1), d(fx_1, Tx_1), \\ \frac{d(fx_0, fx_1) + d(fx_1, Tx_1)}{2} \end{array}\right\}\right)\right) \\
&= \psi(\xi(\max\{d(fx_0, fx_1), d(fx_1, Tx_1)\})). \qquad (1)
\end{aligned}$$

If $\max\{d(fx_0, fx_1), d(fx_1, Tx_1)\} = d(fx_1, Tx_1)$, then condition (1) implies

$$0 < \xi(d(fx_1, Tx_1)) \leq \xi(H(Tx_0, Tx_1)) \leq \psi(\xi(d(fx_1, Tx_1))), \qquad (2)$$

which is a contradiction, since $\psi(r) < r$ for each $r > 0$.

Therefore, $\max\{d(fx_0, fx_1), d(fx_1, Tx_1)\} = d(fx_0, fx_1)$.

From condition (1), we obtain

$$0 < \xi(d(fx_1, Tx_1)) \leq \xi(H(Tx_0, Tx_1)) \leq \psi(\xi(d(fx_0, fx_1))). \qquad (3)$$

By Lemma 2.6, for $q > 1$, there exists $fx_2 \in Tx_1$ such that

$$0 < \xi(d(fx_1, fx_2)) < q\xi(d(fx_1, Tx_1)). \qquad (4)$$

From conditions (3) and (4), we get $0 < \xi(d(fx_1, fx_2)) < q\psi(\xi(d(fx_0, fx_1)))$. Since ψ is strictly increasing, we have $0 < \psi(\xi(d(fx_1, fx_2))) < \psi(q\psi(\xi(d(fx_0, fx_1))))$. Put $q_1 = \frac{\psi(q\psi(\xi(d(fx_0, fx_1))))}{\psi(\xi(d(fx_1, fx_2)))}$. Then, $q_1 > 1$.

If $x_1 = x_2$ or $fx_2 \in Tx_2$, then we have nothing to prove. Therefore, we assume that $x_1 \neq x_2$ and $fx_2 \notin Tx_2$. Since $x_0 \in C^{\alpha}_{fT}$, $fx_1 \in Tx_0$, $fx_2 \in Tx_1$, it follows that $(fx_1, fx_2) \in E(\tilde{G})$ and $\alpha(fx_1, fx_2) \geq 1$.

Applying strictly $(\alpha, \psi, \xi) - G$-contractive condition, we get

$$\xi(H(Tx_1, Tx_2)) \leq \psi(\xi(\max\{d(fx_1, fx_2), d(fx_2, Tx_2)\})). \tag{5}$$

If $\max\{d(fx_1, fx_2), d(fx_2, Tx_2)\} = d(fx_2, Tx_2)$, then it follows from condition (5) that $0 < \xi(d(fx_2, Tx_2)) \leq \xi(H(Tx_1, Tx_2)) \leq \psi(\xi(d(fx_2, Tx_2)))$, which is a contradiction. Therefore, $\max\{d(fx_1, fx_2), d(fx_2, Tx_2)\} = d(fx_1, fx_2)$.

Now, by using condition (5), we obtain

$$0 < \xi(d(fx_2, Tx_2)) \leq \xi(H(Tx_1, Tx_2)) \leq \psi(\xi(d(fx_1, fx_2))). \tag{6}$$

By Lemma 2.6, for $q_1 > 1$, there exists $fx_3 \in Tx_2$ such that

$$0 < \xi(d(fx_2, fx_3)) < q_1\xi(d(fx_2, Tx_2)). \tag{7}$$

From conditions (6) and (7), we get

$$0 < \xi(d(fx_2, fx_3)) < q_1\psi(\xi(d(fx_1, fx_2))) = \psi(q\psi(\xi(d(fx_0, fx_1)))).$$

ψ being strictly increasing implies that $0 < \psi(\xi(d(fx_2, fx_3))) < \psi^2(q\psi(\xi(d(fx_0, fx_1))))$. Since $x_0 \in C^{\alpha}_{fT}$, $fx_1 \in Tx_0$, $fx_2 \in Tx_1$, $fx_3 \in Tx_2$, it follows that $(fx_n, fx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, 3$ and $\alpha(fx_n, fx_{n+1}) \geq 1$ for $n = 0, 1, 2$.

Continuing this process, we can construct a sequence (fx_n) in X such that $fx_n \in Tx_{n-1}$, $(fx_n, fx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \ldots$ and $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $0 < \xi(d(fx_{n+1}, fx_{n+2})) < \psi^n(q\psi(\xi(d(fx_0, fx_1))))$, $\forall n \in \mathbb{N} \cup \{0\}$.

We now show that (fx_n) is a Cauchy sequence in $f(X)$.

For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\xi(d(fx_m, fx_n)) \leq \sum_{i=n}^{m-1} \xi(d(fx_i, fx_{i+1})) \leq \sum_{i=n}^{m-1} \psi^{i-1}(q\psi(\xi(d(fx_0, fx_1)))).$$

By using $(\psi 2)$, it follows that $\lim\limits_{n,m\to\infty} \xi(d(fx_m, fx_n)) = 0$. By using $(\xi 1)$ and $(\xi 3)$, we get $\lim\limits_{n,m\to\infty} d(fx_m, fx_n) = 0$. This gives that (fx_n) is a Cauchy sequence in $f(X)$. As $f(X)$ is complete, there exists a $t \in f(X)$ such that $fx_n \to t = fu$ for some $u \in X$.

As $(fx_n, fx_{n+1}) \in E(\tilde{G})$ and $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \geq 1$, by property (*), there exists a subsequence (fx_{n_i}) of (fx_n) such that $(fx_{n_i}, fu) \in E(\tilde{G})$ and $\alpha(fx_{n_i}, fu) \geq 1$ for all $i \geq 1$.

Then by applying strictly $(\alpha, \psi, \xi) - G$-contractivity, we have

$$\xi(H(Tx_{n_i}, Tu)) \leq \psi\left(\xi\left(\max\left\{\begin{array}{c} d(fx_{n_i}, fu), d(fx_{n_i}, Tx_{n_i}), d(fu, Tu), \\ \frac{d(fx_{n_i}, Tu) + d(fu, Tx_{n_i})}{2} \end{array}\right\}\right)\right). \tag{8}$$

Suppose that $d(fu, Tu) \neq 0$. Let $\epsilon = \frac{d(fu,Tu)}{2} > 0$. Since $fx_{n_i} \to fu$, there exists $k_1 \in \mathbb{N}$ such that

$$d(fx_{n_i}, fu) < \frac{d(fu, Tu)}{2}, \quad \text{for each } i \geq k_1. \tag{9}$$

As $fx_n \to fu$, there exists $k_2 \in \mathbb{N}$ such that

$$d(fx_{n_i}, Tx_{n_i}) \leq d(fx_{n_i}, fx_{n_i+1}) < \frac{d(fu, Tu)}{2}, \quad \text{for each } i \geq k_2. \tag{10}$$

Moreover, there exists $k_3 \in \mathbb{N}$ such that

$$d(fu, Tx_{n_i}) \leq d(fu, fx_{n_i+1}) < \frac{d(fu, Tu)}{2}, \quad \text{for each } i \geq k_3. \tag{11}$$

As $d(fx_{n_i}, Tu) \leq d(fx_{n_i}, fu) + d(fu, Tu)$, it follows that

$$d(fx_{n_i}, Tu) < \frac{d(fu, Tu)}{2} + d(fu, Tu) = \frac{3}{2} d(fu, Tu), \quad \text{for each } i \geq k_1. \tag{12}$$

Put $k = \max\{k_1, k_2, k_3\}$. Then, for $i \geq k$, it follows from conditions (9), (10), (11) and (12) that

$$\max \left\{ \begin{array}{l} d(fx_{n_i}, fu), d(fx_{n_i}, Tx_{n_i}), d(fu, Tu), \\ \frac{d(fx_{n_i}, Tu) + d(fu, Tx_{n_i})}{2} \end{array} \right\} = d(fu, Tu).$$

Therefore, for $i \geq k$, we obtain from (8) that

$$\xi(H(Tx_{n_i}, Tu)) \leq \psi(\xi(d(fu, Tu))). \tag{13}$$

By triangle inequality and condition (13), for $i \geq k$, we have

$$\begin{aligned} &\xi(d(fu, Tu)) \leq \xi(d(fu, fx_{n_i+1})) + \xi(d(fx_{n_i+1}, Tu)) \\ \leq &\xi(d(fu, fx_{n_i+1})) + \xi(H(Tx_{n_i}, Tu)) \leq \xi(d(fu, fx_{n_i+1})) + \psi(\xi(d(fu, Tu))). \end{aligned}$$

Taking limit as $i \to \infty$, we get $\xi(d(fu, Tu)) \leq \psi(\xi(d(fu, Tu)))$, which is a contradiction, since $\xi(d(fu, Tu)) > 0$. Therefore, $d(fu, Tu) = 0$ and so, $t = fu \in Tu$, i.e., t is a point of coincidence of f and T.

For uniqueness, assume that there is another point of coincidence $s(\neq t)$ in X such that $s = fv \in Tv$ for some $v \in X$. By property $(**)$, we have $(fu, fv) \in E(\tilde{G})$ and $\alpha(fu, fv) \geq 1$. Then, $\xi(H(Tu, Tv)) \leq \psi(\xi(M(fu, fv)))$, where

$$\begin{aligned} M(fu, fv) &= \max\{d(fu, fv), d(fu, Tu), d(fv, Tv), \frac{d(fu, Tv) + d(fv, Tu)}{2}\} \\ &= \max\{d(fu, fv), \frac{d(fu, Tv) + d(fv, Tu)}{2}\} \\ &\leq \max\{d(fu, fv), \frac{d(fu, fv) + d(fv, fu)}{2}\} = d(fu, fv). \end{aligned}$$

Thus, $0 < \xi(d(fu, fv)) \leq \xi(H(Tu, Tv)) \leq \psi(\xi(d(fu, fv)))$, which is a contradiction, since $\psi(r) < r$ for each $r > 0$.

So, it must be the case that $d(fu, fv) = 0$ and hence, $fu = fv$. Therefore, f and T have a unique point of coincidence in X.

If f and T are weakly compatible, then by Proposition 2.12, f and T have a unique common fixed point in X. □

Corollary 3.3. *Let (X, d) be a complete metric space endowed with a graph G and let $T : X \to CL(X)$ be a strictly $(\alpha, \psi, \xi) - G$-contractive mapping. Suppose the triple (X, d, G) has the following property:*

(') If (x_n) is a sequence in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$, $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, x) \in E(\tilde{G})$ and $\alpha(x_{n_i}, x) \geq 1$ for all $i \geq 1$.*

Then T has a fixed point in X if $C_T^\alpha \neq \emptyset$. Moreover, T has a unique fixed point in X if the graph G has the following property:

*(**') If x, y are fixed points of T in X, then $(x,y) \in E(\tilde{G})$ and $\alpha(x,y) \geq 1$.*

Proof. The proof follows from Theorem 3.2 by taking the identity map on X for f. □

COROLLARY 3.4. *Let (X,d) be a metric space and let $f : X \to X$ and $T : X \to CL(X)$ be strictly (α,ψ,ξ)-contractive mappings. Suppose that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X with the following property:*

(†) If (fx_n) is a sequence in X such that $fx_n \to x$ and $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \geq 1$, then there exists a subsequence (fx_{n_i}) of (fx_n) such that $\alpha(fx_{n_i}, x) \geq 1$ for all $i \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$, then f and T have a point of coincidence in X. Moreover, f and T have a unique point of coincidence in X if the following property holds:

(‡) If x, y are points of coincidence of f and T in X, then $\alpha(x,y) \geq 1$.

Furthermore, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof. The proof follows from Theorem 3.2 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. □

COROLLARY 3.5. *Let (X,d) be a metric space endowed with a partial ordering $\preceq$. Let $f : X \to X$ and $T : X \to CL(X)$ be such that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Suppose that there exist $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0,\infty)$ such that $\alpha(fx, fy) \geq 1 \Rightarrow \xi(H(Tx,Ty)) \leq \psi(\xi(M(fx,fy)))$ for all $x, y \in X$ with $fx \preceq fy$ or, $fy \preceq fx$. Suppose the triple $(X,d,\preceq)$ has the following property:*

(†′) If (fx_n) is a sequence in X such that $fx_n \to x$ and fx_n, fx_{n+1} are comparable with $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \geq 1$, then there exists a subsequence (fx_{n_i}) of (fx_n) such that fx_{n_i}, x are comparable with $\alpha(fx_{n_i}, x) \geq 1$ for all $i \geq 1$.

If there exists $x_0 \in X$ such that fx_n, fx_m are comparable for $m, n = 0, 1, 2, \ldots$ and $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$, then f and T have a point of coincidence in X. Moreover, f and T have a unique point of coincidence in X if the following property holds:

(‡′) If x, y are points of coincidence of f and T in X, then x, y are comparable and $\alpha(x,y) \geq 1$.

Furthermore, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof. The proof can be obtained from Theorem 3.2 by taking $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. □

As an application of Theorem 3.2, we obtain the following theorem.

THEOREM 3.6. *Let (X, d) be a metric space and let $T : X \to CL(X)$ and $f : X \to X$ be a hybrid pair such that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Suppose that T and f are strictly (α, ψ, ξ)-contractive mappings satisfying the following conditions:*

(i) T is an α_*-admissible multi-valued mapping w.r.t. f;

(ii) there exists $x_0 \in X$ such that $\alpha(fx_0, fx_1) \geq 1$, $\forall fx_1 \in Tx_0$;

(iii) if (fx_n) is a sequence in X with $fx_n \to x$ and $\alpha(fx_n, fx_{n+1}) \geq 1$ for each $n \geq 1$, then there exists a subsequence (fx_{n_i}) of (fx_n) such that $\alpha(fx_{n_i}, x) \geq 1$ for all $i \geq 1$.

Then f and T have a point of coincidence in X. Moreover, f and T have a unique point of coincidence in X if the property (‡) holds. Furthermore, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof. We take $G = G_0 = (X, X \times X)$. Then, the mappings T and f are strictly $(\alpha, \psi, \xi) - G_0$-contractive mappings. By hypothesis (ii), there exists $x_0 \in X$ such that $\alpha(fx_0, fx_1) \geq 1$, $\forall fx_1 \in Tx_0$. By hypothesis (i), it follows that $\alpha_*(Tx_0, Tx_1) \geq 1$ and hence $\alpha(fx_1, fx_2) \geq 1$, $\forall fx_1 \in Tx_0, fx_2 \in Tx_1$. By repeated use of hypothesis (i), we get that $\alpha(fx_n, fx_{n+1}) \geq 1$, $\forall n \in \mathbb{N} \cup \{0\}$ and for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$. Moreover, $(fx_n, fx_m) \in E(\tilde{G_0})$ for $m,\ n = 0, 1, 2, \ldots$. This ensures that $x_0 \in C^{\alpha}_{fT}$ and hence $C^{\alpha}_{fT} \neq \emptyset$. Furthermore, hypothesis (iii) shows that property (*) holds. Thus, all the conditions of Theorem 3.2 are satisfied and the conclusion of Theorem 3.6 can be obtained from Theorem 3.2. □

THEOREM 3.7. *Let (X, d) be a complete metric space endowed with a graph G and let $f : X \to X$ and $T : X \to CL(X)$ be the continuous and compatible hybrid pair such that $T(X) \subseteq f(X)$. Suppose that T and f are strictly $(\alpha, \psi, \xi) - G$-contractive mappings. Then f and T have a point of coincidence in X if $C^{\alpha}_{fT} \neq \emptyset$. Moreover, f and T have a unique common fixed point in X if the graph G has the property (**).*

Proof. As in the proof of Theorem 3.2, we can construct a Cauchy sequence (fx_n) in X such that $fx_n \in Tx_{n-1}$, $(fx_n, fx_m) \in E(\tilde{G})$ for $m, n = 0, 1, \ldots$ and $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and

$$\xi(H(Tx_n, Tx_{n+1})) \leq \psi(\xi(d(fx_n, fx_{n+1}))). \tag{14}$$

(X, d) being complete, there exists $t \in X$ such that $fx_n \to t$ as $n \to \infty$.

Since (fx_n) is a Cauchy sequence in (X, d), it follows from the condition (14) that (Tx_n) is a Cauchy sequence in the complete metric space $(CL(X), H)$. So, there exists $M \in CL(X)$ such that $Tx_n \to M$. Now,

$$d(t, M) \leq d(t, fx_n) + d(fx_n, M) \leq d(t, fx_n) + H(Tx_{n-1}, M) \ \to 0 \text{ as } n \to \infty.$$

Since M is closed, $t \in M$. The compatibility of f and T gives that $H(Tfx_n, fTx_n) \to 0$ as $n \to \infty$. By continuity of f and T, we have

$$\begin{aligned} d(ft, Tt) &\leq d(ft, ffx_{n+1}) + d(ffx_{n+1}, Tt) \leq d(ft, ffx_{n+1}) + H(fTx_n, Tt) \\ &\leq d(ft, ffx_{n+1}) + H(fTx_n, Tfx_n) + H(Tfx_n, Tt) \to 0 \text{ as } n \to \infty, \end{aligned}$$

which implies that $ft \in Tt$, since Tt is closed. Taking $u = ft$, it follows that u is a point of coincidence of f and T in X.

By the argument similar to that used in Theorem 3.2, it follows that u is the unique point of coincidence of f and T in X.

Since compatibility implies weak compatibility, by Proposition 2.12, it follows that f and T have a unique common fixed point in X. □

Corollary 3.8. *Let (X, d) be a complete metric space endowed with a graph G and let $T : X \to CL(X)$ be a continuous strictly $(\alpha, \psi, \xi) - G$-contractive mapping. Then T has a fixed point in X if $C_T^\alpha \neq \emptyset$. Moreover, T has a unique fixed point in X if the graph G has the property (**').*

Proof. The proof follows from Theorem 3.7 by taking $f = I$. □

Corollary 3.9. *Let (X, d) be a complete metric space and let $f : X \to X$ and $T : X \to CL(X)$ be the continuous and compatible hybrid pair such that $T(X) \subseteq f(X)$. Suppose that T and f are strictly (α, ψ, ξ)-contractive mappings. If there exists $x_0 \in X$ such that $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$, then f and T have a point of coincidence in X. Moreover, f and T have a unique common fixed point in X if the property (‡) holds.*

Proof. The proof follows from Theorem 3.7 by taking $G = G_0$. □

Corollary 3.10. *Let (X, d) be a complete metric space endowed with a partial ordering $\preceq$. Let $f : X \to X$ and $T : X \to CL(X)$ be the continuous and compatible hybrid pair such that $T(X) \subseteq f(X)$. Suppose that there exist $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0, \infty)$ such that $\alpha(fx, fy) \geq 1 \Rightarrow \xi(H(Tx, Ty)) \leq \psi(\xi(M(fx, fy)))$ for all $x, y \in X$ with $fx \preceq fy$ or, $fy \preceq fx$. If there exists $x_0 \in X$ such that fx_n, fx_m are comparable for $m, n = 0, 1, 2, \dots$ and $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$, then f and T have a point of coincidence in X. Also, f and T have a unique common fixed point in X if the property (‡′) holds.*

Proof. The proof can be obtained from Theorem 3.7 by taking $G = G_2$. □

Corollary 3.11. *([16]) Let (X, d) be a complete metric space, $f : X \to X$ and $T : X \to CB(X)$ be compatible continuous mappings such that $T(X) \subseteq f(X)$ and*

$$H(Tx, Ty) \leq h \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\right\}$$

for all $x, y \in X$, where $0 < h < 1$. Then there exists a point $t \in X$ such that $ft \in Tt$.

Proof. Since $CB(X) \subseteq CL(X)$, the proof can be obtained from Theorem 3.7 by taking $G = G_0$, $\alpha(x, y) = 1$ for all $x, y \in X$, $\xi(t) = t$ for each $t \geq 0$ and $\psi(t) = ht$ for each $t \geq 0$, where $h \in (0, 1)$ is a fixed number. □

Remark 3.12. It is worth mentioning that in Corollary 3.11, f and T have a unique common fixed point in X.

Remark 3.13. Several special cases of our results can be obtained by restricting $T : X \to X$ and taking $\xi(t) = t$ for each $t \geq 0$, $\alpha(x, y) = 1$, $G = G_0$. Further special cases of our results can be obtained by considering $T : X \to CB(X)$ and $f = I$, $\xi(t) = t$ for each $t \geq 0$, $\psi(t) = ht$ for each $t \geq 0$, where $h \in (0, 1)$ is a fixed number, $\alpha(x, y) = 1$, $G = G_0$.

As an application of Theorem 3.7, we obtain the following theorem.

Theorem 3.14. *Let (X, d) be a complete metric space and let $T : X \to CL(X)$ and $f : X \to X$ be the continuous and compatible hybrid pair such that $T(X) \subseteq f(X)$. Suppose that T and f are strictly (α, ψ, ξ)-contractive mappings satisfying the following conditions:*

(i) T is an α_*-admissible multi-valued mapping w.r.t. f;

(ii) there exists $x_0 \in X$ such that $\alpha(fx_0, fx_1) \geq 1$, $\forall fx_1 \in Tx_0$.

Then f and T have a point of coincidence in X. Moreover, f and T have a unique common fixed point in X if the property (‡) holds.

Proof. The proof is similar to that of Theorem 3.6. □

We provide some examples in favour of our results.

Example 3.15. Let $X = [0, \infty)$ with the usual metric d. Then (X, d) is a complete metric space. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{n}) : n = 1, 2, 3, \dots\}$. Let $T : X \to CL(X)$ be defined by $Tx = \{0, \frac{x}{2}\}$, for all $x \in X$ and $fx = 4x$ for all $x \in X$. Obviously, $T(X) \subseteq f(X) = X$.

Let $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1] \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Take $\psi(t) = \frac{t}{2}$ and $\xi(t) = \sqrt{t}$ for each $t \geq 0$.

If $x = 0$, $y = \frac{1}{4n}$, then $fx = 0$, $fy = \frac{1}{n}$ and so $(fx, fy) \in E(\tilde{G})$ and $\alpha(fx, fy) = 1$. For $x = 0$, $y = \frac{1}{4n}$, we have $Tx = \{0\}$, $Ty = \{0, \frac{1}{8n}\}$ and $\xi(H(Tx, Ty)) = \xi(\frac{1}{8n}) = \frac{1}{2\sqrt{2n}}$. Moreover,

$$\begin{aligned} M(fx, fy) &= \max\left\{ d(fx, fy), d(fx, Tx), d(fy, Ty), \tfrac{d(fx,Ty)+d(fy,Tx)}{2} \right\} \\ &= \max\left\{ d(0, \tfrac{1}{n}), d(0, \{0\}), d(\tfrac{1}{n}, \{0, \tfrac{1}{8n}\}), \tfrac{d(0,\{0,\frac{1}{8n}\})+d(\frac{1}{n},\{0\})}{2} \right\} \\ &= \max\left\{ \frac{1}{n}, 0, \frac{7}{8n}, \frac{\frac{7}{8n} + \frac{1}{n}}{2} \right\} = \frac{1}{n}. \end{aligned}$$

So, $\psi(\xi(M(fx,fy))) = \psi(\xi(\frac{1}{n})) = \psi(\sqrt{\frac{1}{n}}) = \frac{1}{2\sqrt{n}}$. Thus, for all $x, y \in X$ with $(fx,fy) \in E(\tilde{G})$ and $\alpha(fx,fy) = 1$, $\xi(H(Tx,Ty)) \leq \psi(\xi(M(fx,fy)))$. Therefore, T and f are strictly $(\alpha,\psi,\xi) - G$-contractive mappings.

We can verify that $x_0 = 0 \in C^{\alpha}_{fT}$. In fact, $fx_n \in Tx_{n-1}$, $n = 1,2,\ldots$ gives that $fx_1 \in T0 = \{0\} \Rightarrow x_1 = 0$ and so $fx_2 \in Tx_1 = \{0\} \Rightarrow x_2 = 0$. Proceeding in this way, we get $fx_n = 0$ for $n = 0,1,\ldots$ and hence $(fx_n, fx_m) = (0,0) \in E(\tilde{G})$ for $m,n = 0,1,2\ldots$ and $\alpha(fx_n, fx_{n+1}) = 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Also, any sequence (fx_n) with the property $\alpha(fx_n, fx_{n+1}) = 1$ must be a sequence in $[0,1]$. Moreover, $(fx_n, fx_{n+1}) \in E(\tilde{G})$ must be either a constant sequence or a sequence of the following form

$$fx_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{1}{n}, & \text{if } n \text{ is even} \end{cases}$$

where the words *odd* and *even* are interchangeable. Consequently it follows that property (*) holds. Furthermore, the graph G has the property (**) and f and T are weakly compatible. Thus, we have all the conditions of Theorem 3.2 and 0 is the unique common fixed point of f and T in X.

We now show that property (**) in Theorem 3.2 is necessary for the unique point of coincidence.

Remark 3.16. In Example 3.15, if we take

$$Tx = \begin{cases} \{0, \frac{x}{2}\}, & \text{if } 0 \leq x < 1 \\ \{0\}, & \text{if } x = 1 \\ \left[x^2, \infty\right), & \text{if } x > 1 \end{cases} \qquad \text{instead of} \qquad Tx = \left\{0, \frac{x}{2}\right\}, \quad \text{for all } x \in X,$$

then all the conditions of Theorem 3.2 except property (**) are satisfied. We observe that f and T have infinitely many points of coincidence in X.

Example 3.17. Let $X = \{1,2,3\} \cup [4,\infty)$ with the usual metric d. Then (X,d) is a complete metric space. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(1,3)\}$. Let $T : X \to CL(X)$ be defined by

$$Tx = \begin{cases} \{2,\ 3\}, & \text{if } x = 1,3 \\ \{2\}, & \text{if } x = 2 \\ \left[x^2, \infty\right), & \text{if } x \geq 4 \end{cases} \qquad \text{and} \qquad fx = \begin{cases} x, & \text{if } x = 1,2,3 \\ x+1, & \text{if } x \geq 4. \end{cases}$$

Obviously, $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of (X,d).

Let $\alpha : X \times X \to [0,\infty)$ be defined by $\alpha(x,y) = 1$ for all $x,y \in X$. Take $\psi(t) = \frac{t}{2}$ and $\xi(t) = \sqrt{t}$ for each $t \geq 0$. Then it is easy to verify that $\xi(H(Tx,Ty)) \leq \psi(\xi(M(fx,fy)))$, for all $x,y \in X$ with $(fx,fy) \in E(\tilde{G})$ and $\alpha(fx,fy) = 1$.

Therefore, T and f are strictly $(\alpha,\psi,\xi) - G$-contractive mappings. Moreover, $2 \in C^{\alpha}_{fT}$ and property (*) holds. We find that 2 and 3 are points of coincidence of f

and T in X. In fact, 2 and 3 are common fixed points of f and T in X. However, f and T are weakly compatible, the uniqueness part of Theorem 3.2 does not hold due to lack of property (**) of the graph G.

REMARK 3.18. In Example 3.17, T and f are strictly $(\alpha,\psi,\xi)-G$-contractive but not strictly (α,ψ,ξ)-contractive. In fact, for $x=1$, $y=2$, we have $fx=1$, $fy=2$, $Tx=\{2,3\}$, $Ty=\{2\}$ and so $(fx,fy)\notin E(\tilde{G})$.

Then, $\xi(H(Tx,Ty))=\xi(1)=1$ and

$$M(fx,fy)=\max\left\{\begin{array}{l} d(fx,fy),d(fx,Tx),d(fy,Ty),\\ \frac{d(fx,Ty)+d(fy,Tx)}{2}\end{array}\right\}=\max\left\{1,1,0,\frac{1+0}{2}\right\}=1,$$

which implies that, $\xi(H(Tx,Ty))>\psi(\xi(M(fx,fy)))$. Consequently, T and f are not strictly (α,ψ,ξ)-contractive.

The following example supports our Theorem 3.7.

EXAMPLE 3.19. Let $X=[1,\infty)$ be endowed with the Euclidean metric d. Then (X,d) is a complete metric space. Let G be a digraph such that $V(G)=X$ and $E(G)=\Delta\cup\{(n,n+1):n\in\mathbb{N}\}$.

Let $fx=6x^2-5$ and $Tx=[1,x^2]$ for each $x\geq 1$. Then, T and f are continuous and $T(X)=f(X)=X$. It is to be noted that $fTx=[1,6x^4-5]\in CL(X)$ for all $x\in X$. Since $fx_n\to 1$ and $Tx_n\to\{1\}$ iff $x_n\to 1$, $H(fTx_n,Tfx_n)=30|x_n^4-2x_n^2+1|\to 0$ iff $x_n\to 1$, implying that f and T are compatible.

Let $\alpha: X\times X\to[0,\infty)$ be defined by $\alpha(x,y)=1$ for all $x,y\in X$. Take $\psi(t)=\frac{t}{3}$ and $\xi(t)=\frac{t}{2}$ for each $t\geq 0$. If $x=\sqrt{\frac{n+5}{6}}$, $y=\sqrt{\frac{n+6}{6}}$, $n\in\mathbb{N}$, then $fx=n$, $fy=n+1$ and so $(fx,fy)\in E(\tilde{G})$ and $\alpha(fx,fy)=1$. Then, $H(Tx,Ty)=|x^2-y^2|=|\frac{n+5}{6}-\frac{n+6}{6}|=\frac{1}{6}$ and $d(fx,fy)=1\leq M(fx,fy)$ which implies that, $\psi(\xi(d(fx,fy)))\leq\psi(\xi(M(fx,fy)))$. Now, $\xi(H(Tx,Ty))=\xi(\frac{1}{6})=\frac{1}{12}<\frac{1}{6}=\psi(\xi(d(fx,fy)))\leq\psi(\xi(M(fx,fy)))$.

Thus, T and f are strictly $(\alpha,\psi,\xi)-G$-contractive.

It is easy to verify that property (**) holds and $1\in C^{\alpha}_{fT}$ i.e., $C^{\alpha}_{fT}\neq\emptyset$. Thus, we have all the conditions of Theorem 3.7 and 1 is the unique common fixed point of f and T in X.

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(received 29.07.2017; in revised form 01.02.2018; available online 21.07.2018)

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