

## ON SOMOS'S IDENTITIES OF LEVEL TWENTY ONE AND THEIR PARTITION INTERPRETATIONS

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**Abstract.** In this paper, proofs of Somos's theta function identities of level 21 will be given. Further, we deduce certain interesting partition identities from them.

### 1. Introduction

Let  $\tau$  be a complex number satisfying  $\text{Im}(\tau) > 0$  and let  $q = e^{2\pi i\tau}$ . The Dedekind eta-function is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Let  $f(-q) = q^{-1/24}\eta(\tau)$  and  $f_n = f(-q^n)$ . The theta function identity which relates  $f_1, f_n, f_m$  and  $f_{mn}$ , is called the theta function identity of level  $mn$ . M. Somos [4] discovered around 6200 theta function identities of different levels using computer. Recently B. Yuttanan [7], K. R. Vasuki, R. G. Veerasha [6] and B. R. Srivatsa Kumar and D. AnuRadha [5] have obtained proofs for levels 8, 10, 12, 14 and 16. Somos discovered 13 identities of level 21, where three are equivalent to (1)–(3) below. In this paper, we provide a proof of identities of level 21.

Our proofs depend on the following three  $P$ - $Q$  identities of Ramanujan:

**THEOREM 1.1** ([1, p. 236], [3, p.323]). (i) Let  $P = \frac{f_1}{q^{1/4}f_7}$  and  $Q = \frac{f_3}{q^{3/4}f_{21}}$ . Then

$$PQ + \frac{7}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 - 3. \quad (1)$$

(ii) Let  $A = \frac{f_1}{q^{1/12}f_3}$  and  $B = \frac{f_7}{q^{7/12}f_{21}}$ . Then

$$(AB)^3 + \frac{27}{(AB)^3} = \left(\frac{B}{A}\right)^4 - 7\left(\frac{B}{A}\right)^2 + 7\left(\frac{A}{B}\right)^2 - \left(\frac{A}{B}\right)^4. \quad (2)$$

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(iii) Let  $L = \frac{f_3}{q^{1/6}f_7}$  and  $M = \frac{f_1}{q^{5/6}f_{21}}$ . Then

$$\left(\frac{M}{L}\right)^3 - 27\left(\frac{L}{M}\right)^3 = (LM)^2 - LM + \frac{7}{LM} - \frac{49}{(LM)^2}. \quad (3)$$

## 2. Somos's identities

First let us list Somos's identities of level 21.

THEOREM 2.1.

$$\begin{aligned} & q^2 f_1^4 f_{21}^4 + f_3^4 f_7^4 - f_1^3 f_3^3 f_7 f_{21} - 3q f_1^2 f_3^2 f_7^2 f_{21}^2 \\ & \quad - 7q^2 f_1 f_3 f_7^3 f_{21}^3 = 0 \quad (4) \\ & q^4 f_1^8 f_{21}^8 + f_1^7 f_3 f_7^7 f_{21} + 27q^4 f_1 f_3^7 f_7 f_{21}^7 + 7q f_1^2 f_3^6 f_7^6 f_{21}^2 \\ & \quad - f_3^8 f_7^8 - 7q^3 f_1^6 f_3^2 f_7^2 f_{21}^6 = 0 \\ & q f_1^4 f_3^4 f_7^2 f_{21}^2 + f_1^6 f_7^6 + 49q^4 f_1 f_3 f_7^5 f_{21}^5 - f_1^5 f_3^5 f_7 f_{21} \\ & \quad - 27q^4 f_3^6 f_{21}^6 - 7q^3 f_1^2 f_3^2 f_7^4 f_{21}^4 = 0 \\ & f_1^6 f_7 f_{21}^2 + f_3^2 f_7^7 + 36q^4 f_1 f_3 f_{21}^7 - 2f_1^3 f_3 f_7^4 f_{21} - 9q^2 f_3^4 f_7 f_{21}^4 = 0. \quad (5) \\ & 10f_1^7 f_{21} + 1323q^3 f_3^3 f_7 f_{21}^4 + 189q^2 f_1 f_3^4 f_{21}^3 + 49q f_1^3 f_7^4 f_{21} \\ & \quad + 81f_3^7 f_7 - 343q f_3 f_7^7 - 91f_1^4 f_3 f_7^3 = 0 \\ & q f_1^3 f_7^4 f_{21} + 1q^2 f_1 f_3^4 f_{21}^3 + f_1^4 f_3 f_7^3 + 10q^5 f_1 f_{21}^7 + 3q f_3 f_7^7 \\ & \quad - f_3^7 f_7 - 13q^3 f_3^3 f_7 f_{21}^4 = 0 \\ & 13f_1^7 f_{21} + 1323q^5 f_1 f_{21}^7 + 14f_1^4 f_3 f_7^3 + 196q f_1^3 f_7^4 f_{21} \\ & \quad + 378q^2 f_1 f_3^4 f_{21}^3 - 27f_3^7 f_7 - 49q f_3 f_7^7 = 0 \\ & f_1^7 f_{21} + 14q f_1^3 f_7^4 f_{21} + 14q^3 f_3^3 f_7 f_{21}^4 + 28q^2 f_1 f_3^4 f_{21}^3 \\ & \quad + 91q^5 f_1 f_{21}^7 - f_3^7 f_7 - 7q f_3 f_7^7 = 0 \\ & 147q^4 f_1^2 f_3 f_{21}^6 + 3f_3^7 f_7^2 + 4f_1^7 f_7 f_{21} - 42q^2 f_1 f_3^4 f_7 f_{21}^3 \\ & \quad - 7f_1^4 f_3 f_7^4 = 0 \\ & q f_1^5 f_7^2 f_{21}^2 + 2f_1^3 f_3^2 f_7^4 + 9q^2 f_3^5 f_7 f_{21}^3 - 2f_1^6 f_3 f_7 f_{21} \\ & \quad - 63q^4 f_1 f_3^2 f_{21}^6 - 7q f_1^2 f_3 f_7^5 f_{21} = 0 \\ & f_1^3 f_3 f_7^5 + 3q^3 f_1^2 f_3^2 f_{21}^5 + 6q^2 f_3^4 f_7^2 f_{21}^3 - f_1^6 f_7^2 f_{21} \\ & \quad - 3q f_1^5 f_3 f_7 f_{21}^2 - 42q^4 f_1 f_3 f_7 f_{21}^6 = 0 \\ & f_1^2 f_3^2 f_7^5 + 2q f_1^4 f_7^3 f_{21}^2 + 9q f_3^6 f_7 f_{21}^2 - f_1^5 f_3 f_7^2 f_{21} \\ & \quad - 14q f_1 f_3 f_7^6 f_{21} - 9q^3 f_1 f_3^3 f_{21}^5 = 0 \\ & f_1^5 f_7^3 f_{21} + 21q^3 f_1 f_3^2 f_7 f_{21}^5 + 6f_1 f_3^3 f_7 f_{21} - 3q f_3^5 f_7^2 f_{21}^2 \\ & \quad - 6q^2 f_1^2 f_3^3 f_{21}^4 - 7f_1^2 f_3 f_7^6 = 0. \end{aligned}$$

*Proof.* The first three identities are equivalent to (1)–(3) respectively. Let us prove the identity (5).

We shall set  $P = \frac{f_1}{q^{1/3}f_7}$ ,  $Q = \frac{f_3}{qf_{21}}$ ,  $A = \frac{f_1}{q^{1/12}f_3}$  and  $B = \frac{f_7}{q^{7/12}f_{21}}$ . Dividing (5) throughout by  $f_3^2 f_7^7$ , we find that  $\frac{P^6}{Q^2} + 1 - 2\frac{P^3}{Q} + \frac{9}{(AB)^3} \left(4\frac{P^4}{Q^4} - \frac{P^3}{Q}\right) = 0$  or  $(AB)^3 = \frac{9(Q^3 P^3 - 4P^4)}{Q^2(P^3 - Q)^2}$ . Using the equation above in (2) and factoring, we obtain

$$C(P, Q)D(P, Q) = 0, \quad (6)$$

where  $C(P, Q) = P^4 + Q^4 - P^3 Q^3 - 3P^2 Q^2 - 7PQ$  and  $D(P, Q) = P^8 Q^4 - 4P^7 Q^3 - P^5 Q^5 + P^4 Q^4 + 28P^5 Q + 9P^3 Q^3 - 52P^4 - 16P^2 Q^2 - 3Q^4 + 28PQ$ . From Theorem 1.1 (i) we have,  $C(P, Q) = 0$ . This implies that (6) holds, which verifies (5).

We shall omit the proof of the remaining identities as the proofs are similar to the previous one.  $\square$

### 3. Application to partitions

As usually, for complex numbers  $a$  and  $q$  set  $(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$ ,  $|q| < 1$ , and  $(q_{a_1}^{x_1 \pm 1}, q_{a_2}^{x_2 \pm 1}, \dots, q_{a_k}^{x_k \pm 1}; q^y)_\infty = (q^{x_1}; q^y)_\infty^{a_1} \times (q^{x_2}; q^y)_\infty^{a_2} \times \dots \times (q^{x_k}; q^y)_\infty^{a_k} \times (q^{y-x_1}; q^y)_\infty^{a_1} \times (q^{y-x_2}; q^y)_\infty^{a_2} \times \dots \times (q^{y-x_k}; q^y)_\infty^{a_k}$ .

Color partition was introduced by S.-S. Haung [2]. In the context of partitions, we say that a positive integer  $n$  has  $k$  colors if there are  $k$  copies of  $n$  available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called colored partitions. For example, if 1 is allowed to have 2 colors, then all the (colored) partitions of 2 are  $2$ ,  $1_r + 1_r$ ,  $1_g + 1_g$  and  $1_r + 1_g$ , where we use the indices's  $r$  (red) and  $g$  (green) to distinguish the two colors of 1. The generating function is given by  $\frac{1}{(q^a; q^b)_\infty^k}$ , where all the parts are congruent to  $a \pmod{b}$ . Through this section we shall set

$$q^m = q^{1\pm}, q^{2\pm}, q^{4\pm}, q^{5\pm}, q^{8\pm}, q^{10\pm}, \quad q^t = q^{3\pm}, q^{6\pm}, q^{9\pm} \quad \text{and} \quad q^s = q^{7\pm}, \quad (7)$$

where

$$\begin{aligned} (q^{1\pm}; q^{21})_\infty &= (q^1, q^{20}; q^{21})_\infty, & (q^{2\pm}; q^{21})_\infty &= (q^2, q^{19}; q^{21})_\infty, \\ (q^{3\pm}; q^{21})_\infty &= (q^3, q^{18}; q^{21})_\infty, & (q^{4\pm}; q^{21})_\infty &= (q^4, q^{17}; q^{21})_\infty, \\ (q^{5\pm}; q^{21})_\infty &= (q^5, q^{16}; q^{21})_\infty, & (q^{6\pm}; q^{21})_\infty &= (q^6, q^{15}; q^{21})_\infty, \\ (q^{7\pm}; q^{21})_\infty &= (q^7, q^{14}; q^{21})_\infty, & (q^{8\pm}; q^{21})_\infty &= (q^8, q^{13}; q^{21})_\infty, \\ (q^{9\pm}; q^{21})_\infty &= (q^9, q^{12}; q^{21})_\infty \quad \text{and} & (q^{10\pm}; q^{21})_\infty &= (q^{10}, q^{11}; q^{21})_\infty. \end{aligned}$$

**DEFINITION 3.1.** Let  $P(n, a, b, c)$  denote the number of partition of  $n$  into parts not congruent to  $0 \pmod{21}$ , with parts congruent to  $0 \pmod{3}$  having  $\mathbf{a}$  colors and parts congruent to  $0 \pmod{7}$  having  $\mathbf{b}$  colors and parts not congruent to  $0 \pmod{3}$  or  $0 \pmod{7}$  having  $\mathbf{c}$  colors.

**THEOREM 3.2.** *If  $q^2 f_1^4 f_{21}^4 + f_3^4 f_7^4 - f_1^3 f_3^3 f_7 f_{21} - 3q f_1^2 f_3^2 f_7^2 f_{21}^2 - 7q^2 f_1 f_3 f_7^3 f_{21}^3 = 0$ , then for  $n \geq 2$*

$$P(n-2, 2, 0, 0) + P(n, 2, 0, 4) - P(n, 0, 0, 1) \\ - 3P(n-1, 2, 0, 2) - 7P(n-2, 4, 0, 3) = 0,$$

where  $P(0) = 1$ .

*Proof.* Dividing (4) throughout by  $f_1^8$ , we find that

$$q^2 \frac{f_{21}^4}{f_1^4} + \frac{f_3^4 f_7^4}{f_1^8} - \frac{f_3^3 f_7 f_{21}}{f_1^5} - 3q \frac{f_3^2 f_7^2 f_{21}^2}{f_1^6} - 7q^2 \frac{f_3 f_7^3 f_{21}^3}{f_1^7} = 0.$$

Using (7) in the equation above gives

$$\frac{q^2}{(q_4^m, q_4^t, q_4^s; q^{21})_\infty} + \frac{1}{(q_8^m, q_4^t, q_4^s; q^{21})_\infty} \\ - \frac{1}{(q_5^m, q_2^t, q_4^s; q^{21})_\infty} - \frac{3q}{(q_6^m, q_4^t, q_4^s; q^{21})_\infty} - \frac{7q^2}{(q_7^m, q_6^t, q_4^s; q^{21})_\infty} = 0.$$

Multiplying the equation above with  $(q_4^m, q_2^t, q_4^s; q^{21})_\infty$  gives

$$\frac{q^2}{(q_2^t; q^{21})_\infty} + \frac{1}{(q_4^m, q_2^t; q^{21})_\infty} - \frac{1}{(q_1^m; q^{21})_\infty} - \frac{3q}{(q_2^m, q_2^t; q^{21})_\infty} - \frac{7q^2}{(q_3^m, q_4^t; q^{21})_\infty} = 0.$$

Using the definition of  $P(n, a, b, c)$  in the equation above, we obtain

$$\sum_{n=0}^{\infty} P(n-2, 2, 0, 0)q^n + \sum_{n=0}^{\infty} P(n, 2, 0, 4)q^n - \sum_{n=0}^{\infty} P(n, 0, 0, 1)q^n \\ - 3 \sum_{n=0}^{\infty} P(n-1, 2, 0, 2)q^n - 7 \sum_{n=0}^{\infty} P(n-2, 4, 0, 3)q^n = 0.$$

Comparing the coefficients of  $q^n$  in the above equation, we obtain the required result.  $\square$

As the proof of the remaining identities is similar to the one proved, we shall only state the results. The following identities are equivalent to the identities from Theorem 2.1.

**THEOREM 3.3.** *The following identities hold.*

$$\text{For } n \geq 4 \quad P(n-4, 0, 6, 0) + P(n, 0, 0, 1) + 27P(n-4, 0, 12, 7) + 7P(n-1, 0, 6, 6) \\ - P(n, 0, 6, 8) - 7P(n-3, 0, 6, 2) = 0.$$

$$\text{For } n \geq 4 \quad P(n-1, 2, 0, 2) + P(n, 4, 0, 0) + 49P(n-4, 8, 6, 5) - P(n, 0, 6, 1) \\ - 27P(n-4, 4, 12, 6) - 7P(n-3, 6, 6, 4) = 0.$$

$$\text{For } n \geq 4 \quad P(n, 0, 1, 0) + P(n, 4, 1, 6) + 36P(n-4, 4, 0, 5) \\ + 7P(n, 2, 1, 3) - 9P(n, 2, 7, 6) = 0.$$

$$\text{For } n \geq 3 \quad 10 + 1323P(n-3, 4, 6, 7) + 189P(n-2, 2, 6, 6) + 49P(n-1, 4, 0, 4) \\ + 81P(n, 0, 6, 7) - 343P(n, 6, 0, 7) - 91P(n, 2, 6, 3) = 0.$$

$$\text{For } n \geq 5 \quad P(n-1, 2, 0, 1) + P(n-2, 1, 6, 3) + 1 + 3P(n-1, 4, 0, 4)$$

$$\begin{aligned}
& +10P(n-5, 4, 6, 3) + P(n, 4, 0, 4) - 13P(n-3, 2, 6, 4) = 0. \\
\text{For } n \geq 5 & \quad 13 + P(n-5, 6, 6, 6) + 14P(n, 2, 0, 3) + 1 + 196P(n-1, 4, 0, 4) \\
& + 378P(n-2, 2, 3, 6) - 27P(n, 0, 7, 6) - 49P(n-1, 6, 0, 7) = 0. \\
\text{For } n \geq 5 & \quad 1 + 14P(n-1, 4, 0, 4) + 14P(n-3, 4, 6, 7) + 28P(n-2, 3, 7, 7) \\
& + 91P(n-5, 6, 6, 6) - P(n, 0, 6, 7) - 7P(n-1, 6, 0, 7) = 0. \\
\text{For } n \geq 4 & \quad 147P(n-4, 5, 7, 6) + 3P(n, 1, 7, 8) + 4 \\
& - 42P(n-2, 3, 7, 7) - 7P(n, 3, 1, 6) = 0. \\
\text{For } n \geq 4 & \quad P(n-1, 2, 0, 1) + 2P(n, 2, 0, 3) + 9P(n-2, 2, 6, 6) \\
& - 63P(n-4, 4, 6, 5) - 7P(n-1, 4, 0, 4) = 0. \\
\text{For } n \geq 4 & \quad P(n, 2, 0, 3) + 3P(n-3, 2, 6, 5) + 6P(n-2, 2, 6, 4) - 1 \\
& - 3P(n-1, 0, 1, 5) - 49P(n-4, 4, 6, 5) = 0. \\
\text{For } n \geq 5 & \quad P(n, 1, 0, 2) + 2P(n-1, 1, 0, 0) + 9P(n-1, 2, 6, 4) \\
& - P(n-1, 0, 0, 1) + 14P(n-1, 3, 0, 3) - 9P(n-5, 1, 5, 3) = 0. \\
\text{For } n \geq 2 & \quad P(n, 2, 0, 0) + 21P(n-3, 4, 6, 4) + 6P(n, 0, 6, 4) - 3P(n-1, 2, 6, 3) \\
& - 6P(n-2, 2, 2, 3) - 7P(n, 4, 0, 3) = 0.
\end{aligned}$$

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