MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 13.04.2019

research paper
оригинални научни рад

LOCAL PERSISTENCE OF GEOMETRIC STRUCTURES FOR BOUSSINESQ SYSTEM WITH ZERO VISCOSITY

Halima Meddour

ABSTRACT. The current paper deals with the local well-posedness problem for the two-dimensional partial viscous Boussinesq system when the initial vorticity belongs to the patch class. We prove in particular some results concerning the regularity persistence of the patch boundary and establish the convergence towards the inviscid limit when the molecular diffusivity goes to zero.

1. Introduction

We are mainly concerned with studying the local well-posedness theory for the partial viscous Boussinesq system given by the coupled equations,

$$\begin{cases} \partial_t v_\kappa + v_\kappa \cdot \nabla v_\kappa + \nabla \pi_\kappa = \rho_\kappa \vec{e}_2, \quad t \geq 0,\ x \in \mathbb{R}^2 \\ \partial_t \rho_\kappa + v_\kappa \cdot \nabla \rho_\kappa - \kappa \Delta \rho_\kappa = 0, \\ \text{div } v_\kappa = 0. \end{cases} \tag{1}$$

It describes the evolution of stratified incompressible fluids in $\mathbb{R}^2$ under the influence of the gravity force which is proportional to ρ_κ in the direction $\vec{e}_2 = (0,1)$; for the derivation of this model, see for instance [24]. Above, the velocity vector field $v_\kappa \in \mathbb{R}^2$ is solenoidal, $\pi_\kappa \in \mathbb{R}$ is the pressure and $\rho_\kappa \in \mathbb{R}_+$ is the density. The parameter $\kappa \geq 0$ denotes the molecular diffusivity of the fluid. We will consider the Cauchy problem to the Boussinesq system by prescribing the initial data $v_{\kappa|t=0} = v_\kappa^0, \quad \rho_{\kappa|t=0} = \rho_\kappa^0$. Note that if the initial density vanishes, $\rho_\kappa^0 \equiv 0$, then the system (1) reduces to the classical Euler equations given by

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla \pi = 0, \\ \text{div } v = 0, \\ v_{|t=0} = v^0. \end{cases} \tag{2}$$

2010 Mathematics Subject Classification: 35B65, 35Q35, 76D05
Keywords and phrases: Stratified system; vortex patches; local well-posedness; inviscid limit.

One of the basic features in the dynamics of Euler equations is related to the vorticity $\omega \triangleq \partial_1 v^2 - \partial_2 v^1$ which is transported by the flow,

$$\begin{cases} \partial_t \omega + v\cdot\nabla\omega = 0, \\ \omega_{|t=0} = \omega^0, \\ v = \mathcal{N}_2 \star \omega, \end{cases} \tag{3}$$

where $\mathcal{N}_2$ is the Biot-Savart kernel defined by $\mathcal{N}_2(x) = \nabla^\perp E_2(x)$, $E_2(x) = \frac{1}{2\pi}\log\|x\|$, $\nabla^\perp \triangleq (-\partial_2, \partial_1)$.

Let us denote by $\Psi = (\Psi_t)$ the flow (*particle-trajectory mapping*) associated to the time-dependent velocity vector field v, so that $\partial_t\Psi_t(x) = v(t,\Psi_t(x))$, $\Psi_0(x) = x$. Thus the solution of (3) is determined explicitly by $\omega(t,\Psi_t(x)) = \omega^0(x)$ and admits in turn infinite conservation laws. For example, all the L^p norms are time invariant, that is, $\|\omega(t)\|_{L^p} = \|\omega^0\|_{L^p}$ for $p \in [1,\infty]$. Under this pattern, Yudovich succeed in [26] to obtain global unique weak solutions for the system (2) whenever $\omega^0 \in L^1 \cap L^\infty$. Furthermore, the velocity vector field, which is not necessary Lipschitz, belongs to the class of log-Lipschitz functions and the corresponding flow map Ψ is a planar homeomorphism. Notice that Yudovich's class encompasses vortex patches, that is, ω^0 is represented by the characteristic function of a bounded domain $\Omega_0 \subset \mathbb{R}^2$. This structure is preserved during the time, meaning that $\omega(t) = \mathbf{1}_{\Omega_t}$, with $\Omega_t = \Psi_t(\Omega_0)$ is the patch that moves with the flow.

In-depth study of vortex patches, whose dynamics is governed by the motion of closed curves in the complex plane, has led to several questions especially about the boundary regularity. A remarkable result in this direction, due to Chemin [7] (see also P. Serfati [25]), ensures that when the boundary $\partial\Omega_0$ belongs to the Hölderian class $C^{1+\varepsilon}$, with $0<\varepsilon<1$, then the regularity of $\partial\Omega_t$ is shown to be retained over the time. Actually, Chemin's strategy requires essentially the control of the Liscphitz norm of the velocity with respect to the co-normal regularity $\partial_{X_t}\omega$ of the vorticity in Hölder spaces $C^{\varepsilon-1}$ by means of logarithmic estimate. The choice of the family $X_t = (X_{t,\lambda})_{\lambda\in\Lambda}$ can be done in such way that it is non-degenerate and being tangential to $\partial\Omega_t$. The vector field $X_{t,\lambda}$ is the push-forward of $X_{0,\lambda}$ by the flow $\Psi(t)$,

$$(\partial_t + v\cdot\nabla)X_{t,\lambda} = \partial_{X_{t,\lambda}} v. \tag{4}$$

Those vector fields commute with the transport operator $\partial_t + v\cdot\nabla$, and consequently

$$(\partial_t + v\cdot\nabla)\partial_{X_{t,\lambda}}\omega = 0. \tag{5}$$

This allows to follow the tangential regularity of the vorticity which is a central step in the study of the vortex patch issue.

As the Boussinesq system (1) is in some sense a perturbation of (2), it will be of interest to ask whether the known results for Euler equations can be extended to the Boussinesq system as well. The topic of local/global posedness for (1) for $\kappa > 0$ has drawn great attention and has been widely studied during the last years. Particularly, it is worth mentioning that Chae showed in [4] that (1) is globally well-posed whenever $(v^0,\rho^0) \in H^s \times H^s$, with $s>2$. This result was improved later by Hmidi and Keraani in [19], where they imposed that $(v^0,\rho^0) \in B_{p,1}^{1+\frac{2}{p}} \times B_{p,1}^{-1+\frac{2}{p}} \cap L^r$, with $r>2$. In the same fashion, Hmidi and Zerguine [20] established similar result in the setting of fractional laplacian $(-\Delta)^{\frac{\alpha}{2}}$, $\alpha \in]1,2]$. In [10], Danchin and Paicu extended weak solutions of Yudovich's type to the system (1). For further discussions about this subject, we refer to [2,5,6,9,14] and the references therein.

In this paper we intend to conduct a detailed study of the vortex patch problem for the system (1) and to investigate the convergence towards the inviscid system when the parameter κ tends to zero. Note that the limit system is simply obtained by taking $\kappa = 0$, that is

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla\pi = \rho\vec{e}_2, \quad t\geq 0, x\in\mathbb{R}^2 \\ \partial_t\rho + v\cdot\nabla\rho = 0, \\ \operatorname{div} v = 0. \end{cases} \tag{6}$$

We point out that for the latter system local well-posedness can be implemented in various function spaces similarly to Euler equations. For instance, Chae and Nam showed in [5] that (6) is locally well-posed in Sobolev spaces H^s with $s > 2$. This result was extended to critical Besov spaces $B^{1+\frac{2}{p}}_{p,1}$, $p\in]1,\infty[$ by Liu, Wang and Zhang in [22]. The global existence of classical solutions is an outstanding open problem.

A study of the vortex patch problem for the system (1) was done in [21] with $\kappa = 1$. It was shown that if the boundary of the initial vortex patch belongs to $C^{1+\varepsilon}$ for $0<\varepsilon<1$ then the velocity is a Lipschitz function globally in time and the transported patch, that is, Ω_t keeps its initial regularity. Furthermore, the vorticity is given by the decomposition $\omega(t) = \mathbf{1}_{\Omega_t} + \widetilde{\rho}(t)$, with $\widetilde{\rho}$ a smooth function. A similar result has been done recently in [27] for the system (1) with critical fractional dissipation where a sharper result has been obtained compared to the incompressible Euler equations [7]. In the same spirit, Hassainia and Hmidi [16] showed that the system (6) is locally well-posed whenever the initial patch has a regular/singular structure. The related subject about the aforementioned topics are selected in [11–13,15,17,23] and the references therein.

At this stage, the first main result of this paper is summarized in the following theorem where we deal with local theory for the vortex patch problem uniformly with respect to the parameter κ. More accurately we have the following.

THEOREM 1.1. *Let $\kappa\in[0,1]$ and consider a bounded domain Ω_0 in $\mathbb{R}^2$ whose boundary $\partial\Omega_0$ is a Jordan curve of $C^{1+\varepsilon}$-regularity, with $0<\varepsilon<1$. Let v^0_κ be a divergence-free vector field such that its vorticity $\omega^0_\kappa = \mathbf{1}_{\Omega_0}$ and the initial density $\rho^0_\kappa\in L^2\cap C^{1+\varepsilon}$ with $\nabla\rho^0_\kappa\in L^2$. Then there exists $T>0$ independent of κ such that the system* (1) *admits a unique local solution $(v_\kappa,\rho_\kappa)\in\big(L^\infty([0,T];\mathrm{Lip}(\mathbb{R}^2))\big)^2$. Furthermore, for all $t\in[0,T]$ the boundary $\partial\Omega_t$ is a Jordan curve of class $C^{1+\varepsilon}$, with $\Omega_t = \Psi_t(\Omega_0)$.*

REMARK 1.2. We note that the initial condition $\rho^0_\kappa\in C^{1+\varepsilon}$ does not persist in time, that is $\rho_\kappa(t)\in C^{1+\varepsilon}$ is false in general for any positive time. Hence, the velocity field requires more regularity than the Lipschitz one.

The main step in the proof of Theorem 1.1 is to get an estimate for the Lipschitz norm of the velocity locally in time uniformly on $\kappa\in[0,1]$. For this purpose, we will employ the original Chemin's approach [7]. Thus we shall control $\|\nabla v_\kappa(t)\|_{L^\infty}$ with respect to the co-normal regularity of the vorticity $\partial_{X_t}\omega_\kappa$ in $C^{\varepsilon-1}$, with $0<\varepsilon<1$ by means of logarithmic estimate. The family of vector fileds $X_t = (X_{t,\lambda})_{\lambda\in\Lambda}$ obeys the equation (4). The tangential derivative of the vorticity $\partial_{X_t}\omega_\kappa$ satisfies, similarly to (5), $(\partial_t + v_\kappa\cdot\nabla)\partial_{X_{t,\lambda}}\omega_\kappa = \partial_{X_{t,\lambda}}\partial_1\rho_\kappa$. This follows from the fact that the vorticity-density formulation of (1) is given by

$$\begin{cases}\partial_t\omega_\kappa + v_\kappa\cdot\nabla\omega_\kappa = \partial_1\rho_\kappa, \quad t\geq 0,\, x\in\mathbb{R}^2\\ \partial_t\rho_\kappa + v_\kappa\cdot\nabla\rho_\kappa - \kappa\Delta\rho_\kappa = 0,\\ \mathrm{div}\ v_\kappa = 0.\end{cases} \tag{7}$$

Writing $\partial_{X_t}\partial_1\rho_\kappa = \partial_1(\partial_{X_t}\rho_\kappa) + [\partial_{X_t},\partial_1]\rho_\kappa$ and keeping in mind that the commutator behaves well, the problem reduces to follow the regularity of $\partial_{X_t}\rho_\kappa$ in C^ε. It is straightforward that the quantity $\partial_{X_t}\rho_\kappa$ satisfies the following evolution equation $(\partial_t + v_\kappa\cdot\nabla - \kappa\Delta)X_{t,\lambda}\rho_\kappa = -\kappa[\Delta, X_{t,\lambda}]\rho_\kappa$. Observe that for the inviscid case, we can check easily that the co-normal derivative of the density is transported by the flow which simplifies a lot the analysis, see [16]. In our context the commutator term contributes with additional drawbacks. The remedy is to treat carefully the commutator using the maximal smoothing effect of the transport diffusion equation in the spirit of the approach developed in [8, 17].

Our second main result deals with the inviscid limit problem. To be precise we have the following.

THEOREM 1.3. *Let* (v_κ,ρ_κ) *and* (v,ρ) *be the solutions of* (1) *and* (6) *respectively with the same initial data given by Theorem 1.1. Then the following assertions hold true.*
(i) For every $p\in[2,\infty]$ $\quad \sup_{t\in[0,T]}\big(\|v_\kappa(t)-v(t)\|_{L^p} + \|\rho_\kappa(t)-\rho(t)\|_{L^p}\big) \leq C_0\kappa^{1/4+1/2p}$.
(ii) If Ψ_κ *and* Ψ *denote the flow associated to* v_κ *and* v *respectively. Then we have* $\sup_{t\in[0,T]}\|\Psi_\kappa(t)-\Psi(t)\|_{L^\infty} \leq C_0\kappa^{1/4}$*, where* $C_0 = C(\|\nabla\rho^0\|_{L^2\cap L^\infty}, T)$.

The proof of the above theorem relies on some classical L^p-estimates, the classical complex interpolation between Lebesgue spaces and the so-called Gagliardo Nerenberg inequality.

The next section starts with a brief overview of the Littlewood-Paley theory, particularly the cut-off operators, paradifferential calculus. Thereafter, we undertake the concept of Besov, Hölder spaces and their connections with worthwhile lemmas concerning the persistence of Besov spaces and maximal regularity for a transport-diffusion equation. In Section 3, we state the general version of the Thoerem 1.1. For the sake of clarity, we divide its proof in several steps. Section 4 encloses the proof of Theorem 1.3.

2. Basic tools

This preparatory section comprises some basic tools that we shall freely use during this work. It starts with a short introduction to the Littlewood-Paley theory through the dyadic decomposition of unity, cut-off operators and Besov spaces. Afterwards, we state Bernstein's inequalities and Bony's decomposition which are required in particular, when it comes to the analysis of the commutator estimates. At the end, we state some technical lemmas freely used troughout this work.

2.1 Notation

Throughout this article, we will adopt the following notation.

- We denote by C a positive constant which may be different in each occurrence but it does not depend on the initial data. We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of the type $X \leq CY$, with C independent of X and Y. The notation C_0 means a constant depending on the involved norms of the initial data.
- $\mathscr{S}'(\mathbb{R}^2)$ stands for the space of tempered distributions defined on $\mathbb{R}^2$.
- For any $u \in \mathscr{S}'(\mathbb{R}^2)$ both $\widehat{u}$ and $\mathscr{F}u$ (resp. $\mathscr{F}^{-1}u$) denote the Fourier transform (resp. inverse Fourier transform) of u.
- For every $p \in [1,\infty]$, $\|\cdot\|_{L^p}$ denotes the norm in the Lebesgue space L^p.
- The norm in the mixed space time Lebesgue space $L^p([0,T],L^r(\mathbb{R}^2))$ is denoted by $\|\cdot\|_{L^p_T L^r}$.
- For any pair of operators P and Q, the commutator $[P,Q]$ is defined by $PQ-QP$.
- $B(0,R)$ denotes the ball centered at origin with radius $R>0$.
- $\mathscr{A}(0,R_1,R_2)$ represents the annulus centered at origin with radii $R_1<R_2$.

2.2 Brief review of the Littlewood-Paley theory

We start by the so-called Littlewood-Paley decomposition, based on a nonhomogeneous dyadic partition of unity with respect to the Fourier variables. For this purpose, let $\chi \in \mathscr{D}(\mathbb{R}^2)$ be a radial cut-off function, monotonically decaying along rays and so that

$$\chi(\xi) = \begin{cases} 1 & \text{if } \|\xi\| \leq \frac{1}{2} \\ 0 & \text{if } \|\xi\| \geq 1, \end{cases}$$

with $\operatorname{supp}\chi \subset B(0,1)$. Furthermore, define $\varphi(\xi) \triangleq \chi(\frac{\xi}{2}) - \chi(\xi)$, $\varphi \geq 0$; thus we have $\operatorname{supp}\varphi \subset \mathscr{A}(0,\frac{1}{2},2)$ and for all $\xi \in \mathbb{R}^2$, $\quad \chi(\xi) + \sum_{q\geq 0}\varphi(2^{-q}\xi) = 1$.

The Littlewood-Paley or frequency cut-off operators $(\Delta_q)_{q\geq -1}$ are defined for $u \in \mathscr{S}'(\mathbb{R}^2)$ by

$$\Delta_q u = \begin{cases} \chi(\mathrm{D})u & \text{if } q=-1 \\ \varphi(2^{-q}\mathrm{D})u & \text{if } q \geq 0, \end{cases}$$

where in general case $f(\mathrm{D})$ stands for the pseudo-differential operator $u \mapsto \mathscr{F}^{-1}(f\mathscr{F}u)$ with constant symbol. Also, the sequence $(S_q)_{q\geq 0}$ of lower frequencies for $q \geq 0$ is defined by $S_q u \triangleq \sum_{j\leq q-1}\Delta_j u$.

A few basic properties of the cut-off operators $(\Delta_q)_{q\geq -1}$ and $(S_q)_{q\geq 0}$ are listed in the following proposition.

PROPOSITION 2.1. *Let $u,v \in \mathscr{S}'(\mathbb{R}^2)$. Then we have*
(i) $|p-q| \geq 2 \Longrightarrow \Delta_p\Delta_q u \equiv 0$, *(ii)* $|p-q| \geq 4 \Longrightarrow \Delta_q(S_{p-1}u\Delta_p v) \equiv 0$,

(iii) $\Delta_q, S_q : L^p \to L^p$ *uniformly with respect to q and p,* *(iv)* $u = \sum_{q\geq -1}\Delta_q u$.

With these notations at our disposal, we now provide the definition of the inhomogeneous Besov space.

DEFINITION 2.2. For $(p,r,s)\in[1,+\infty]^2\times\mathbb{R}$, the inhomogeneous Besov space $B^s_{p,r}$ is defined by $B^s_{p,r}=\{u\in\mathscr{S}'(\mathbb{R}^2):\|u\|_{B^s_{p,r}}<+\infty\}$, where

$$\|u\|_{B^s_{p,r}}\triangleq\begin{cases}\Big(\sum_{q\ge-1}2^{rqs}\|\Delta_q u\|^r_{L^p}\Big)^{1/r} & \text{if } r\in[1,+\infty[,\\ \sup_{q\ge-1}2^{qs}\|\Delta_q u\|_{L^p} & \text{if } r=+\infty.\end{cases}$$

REMARK 2.3. We notice that:
(i) If $s\in\mathbb{R}_+\backslash\mathbb{N}$, the Hölder space denoted by C^s coincides with $B^s_{\infty,\infty}$.

(ii) $\big(C^s,\|\cdot\|_{C^s}\big)$ as a Banach space coincides with the usual Hölder space C^s with equivalent norms, $\|u\|_{C^s}\lesssim\|u\|_{L^\infty}+\sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^s}\lesssim\|u\|_{C^s}$.

(iii) If $s\in\mathbb{N}$, the obtained space is so-called Hölder-Zygmund space, still denoted by $B^s_{\infty,\infty}$.

2.3 Paradifferential calculus

The well-known *Bony's* decomposition [3] enables us to split formally the product of two tempered distributions u and v into three pieces.

DEFINITION 2.4. For a given $u,v\in\mathscr{S}'$ we have $uv=T_uv+T_vu+\mathscr{R}(u,v)$, where $T_uv=\sum_q S_{q-1}u\Delta_q v$ and $\mathscr{R}(u,v)=\sum_q\Delta_q u\widetilde{\Delta}_q v$, with the notation $\widetilde{\Delta}_q=\Delta_{q-1}+\Delta_q+\Delta_{q+1}$. T_uv is called the paraproduct of v by u and $\mathscr{R}(u,v)$ the remainder term.

The mixed space-time spaces are stated as follows.

DEFINITION 2.5. Let $T>0$ and $(\beta,p,r,s)\in[1,\infty]^3\times\mathbb{R}$. The spaces $L^\beta_TB^s_{p,r}$ and $\widetilde{L}^\beta_TB^s_{p,r}$ are defined respectively by:

$$L^\beta_TB^s_{p,r}\triangleq\Big\{u:[0,T]\to\mathscr{S}';\|u\|_{L^\beta_TB^s_{p,r}}=\big\|\big(2^{qs}\|\Delta_q u\|_{L^p}\big)_{\ell^r}\big\|_{L^\beta_T}<\infty\Big\},$$
$$\widetilde{L}^\beta_TB^s_{p,r}\triangleq\Big\{u:[0,T]\to\mathscr{S}';\|u\|_{\widetilde{L}^\beta_TB^s_{p,r}}=\big(2^{qs}\|\Delta_q u\|_{L^\beta_TL^p}\big)_{\ell^r}<\infty\Big\}.$$

The relationship between these spaces is given by the following embeddings. Let $\varepsilon>0$, then

$$\begin{cases}L^\beta_TB^s_{p,r}\hookrightarrow\widetilde{L}^\beta_TB^s_{p,r}\hookrightarrow L^\beta_TB^{s-\varepsilon}_{p,r} & \text{if } r\ge\beta,\\ L^\beta_TB^{s+\varepsilon}_{p,r}\hookrightarrow\widetilde{L}^\beta_TB^s_{p,r}\hookrightarrow L^\beta_TB^s_{p,r} & \text{if } \beta\ge r.\end{cases}$$

Accordingly, we have the following interpolation result.

COROLLARY 2.6. *Let $T>0$, $s_1<s<s_2$ and $\eta\in]0,1[$ such that $s=\eta s_1+(1-\eta)s_2$. Then we have $\|u\|_{\widetilde{L}^\beta_TB^s_{p,r}}\le C\|u\|^\eta_{\widetilde{L}^\beta_TB^{s_1}_{p,\infty}}\|u\|^{1-\eta}_{\widetilde{L}^\beta_TB^{s_2}_{p,\infty}}$.*

Now we shall state Bernstein's inequalities, see for instance [1,7].

LEMMA 2.7. *There exists a constant $C>0$ such that for $1\le a\le b\le\infty$, for every function u and every $q\in\mathbb{N}\cup\{-1\}$, we have*
(i) $\sup_{|\alpha|=k}\|\partial^\alpha S_q u\|_{L^b}\le C^k2^{q\left(k+2\left(\frac{1}{a}-\frac{1}{b}\right)\right)}\|S_q u\|_{L^a}$,

(ii) $C^{-k}2^{qk}\|\Delta_q u\|_{L^a} \le \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^a} \le C^k 2^{qk}\|\Delta_q u\|_{L^a}$.

A noteworthy consequence of Bernstein's inequality (i) is the following embedding $B^s_{p,r} \hookrightarrow B^{\widetilde{s}}_{\widetilde{p},\widetilde{r}}$ whenever $\widetilde{p} \ge p$, with $\widetilde{s} < s - 2\Big(\frac{1}{p} - \frac{1}{\widetilde{p}}\Big)$ or $\widetilde{s} = s - 2\Big(\frac{1}{p} - \frac{1}{\widetilde{p}}\Big)$ and $\widetilde{r} \le r$.

2.4 Useful results

Most of the results concerning the system (7) rely strongly on a priori estimates in Besov spaces for the transport-diffusion equation:

$$\begin{cases} \partial_t a + v\cdot\nabla a - \kappa\Delta a = f \\ a_{|t=0} = a^0. \end{cases} \tag{8}$$

We start by the persistence of Besov regularity for (8), whose proof may be found for example in [1, 18].

PROPOSITION 2.8. *Let* $(s,r,p) \in]-1,1[\times[1,\infty]^2$ *and* v *be a smooth divergence-free vector field. We assume that* $a^0 \in B^s_{p,r}$ *and* $f \in L^1_{loc}(\mathbb{R}_+;B^s_{p,r})$*. Then for every smooth solution* a *of* (8) *and* $t \ge 0$ *we have*

$$\|a(t)\|_{B^s_{p,r}} \le Ce^{CV(t)}\Big(\|a^0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|f(\tau)\|_{B^s_{p,r}}d\tau\Big),$$

with
$$V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau$$

and C *being a constant which depends only on* s *and not on* κ*.*

Next, we state the maximal smoothing effect result for (8) in mixed time-space spaces, whose proof can be found in [1, 18].

PROPOSITION 2.9. *Let* $(s,p_1,p_2,r) \in]-1,1[\times[1,+\infty]^3$ *and* v *be a divergence- free vector field belonging to* $L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathrm{Lip})$*. Then for every smooth solution* a *of* (8) *we have*

$$\kappa^{\frac{1}{r}}\|a\|_{\widetilde{L}^r_t B^{s+\frac{2}{r}}_{p_1,p_2}} \le Ce^{CV(t)}(1+\kappa t)^{\frac{1}{r}}\Big(\|a^0\|_{B^s_{p_1,p_2}} + \|f\|_{L^1_t B^s_{p_1,p_2}}\Big), \quad \forall t\in\mathbb{R}_+.$$

We end this paragraph by the Calderón-Zygmund estimate which is a deep result of harmonic analysis.

PROPOSITION 2.10. *Let* $p \in]1,\infty[$ *and* v *be a divergence-free vector field whose vorticity* $\omega \in L^p$*. Then* $\nabla v \in L^p$ *and* $\|\nabla v\|_{L^p} \le C\frac{p^2}{p-1}\|\omega\|_{L^p}$*, with* C *being a universal constant.*

2.5 Vortex patch tool box

In this section we state some aspects and properties about admissible family of vector fields often used in the definition of anisotropic Hölder spaces.

DEFINITION 2.11. Let $\varepsilon \in]0,1[$. A family of vector fields $X = (X_\lambda)_{\lambda\in\Lambda}$ is said to be admissible if and only if the following assertions hold.
(i) *Regularity*: $X_\lambda, \mathrm{div} X_\lambda \in C^\varepsilon \quad \forall\lambda\in\Lambda$.

(ii) *Non-degeneracy*: $I(X) \triangleq \inf_{x\in\mathbb{R}^2} \sup_{\lambda\in\Lambda} \left|X_\lambda(x)\right| > 0$.
The class X is equipped with the norm $\widetilde{\|}X_\lambda\|_{C^\varepsilon} \triangleq \|X_\lambda\|_{C^\varepsilon} + \|\mathrm{div}X_\lambda\|_{C^\varepsilon}$.

DEFINITION 2.12. Let $X = (X_\lambda)_{\lambda\in\Lambda}$ be an admissible family. The action of each member X_λ on $u \in L^\infty$ is defined as the directional derivative of u along X_λ by the formula $\partial_{X_\lambda} u = \mathrm{div}(uX_\lambda) - u\,\mathrm{div}X_\lambda$.

Now, we are in position to define the anisotropic Hölder spaces.

DEFINITION 2.13. Let $\varepsilon \in]0,1[$ and $X = (X_\lambda)_{\lambda\in\Lambda}$ be an admissible family of vector fields. We say that $u \in C^\varepsilon(X)$ if and only if $u \in L^\infty$ and satisfies $\partial_{X_\lambda} u \in C^{\varepsilon-1}$ and $\sup_{\lambda\in\Lambda} \|\partial_{X_\lambda} u\|_{C^{\varepsilon-1}} < +\infty$, for all $\lambda \in \Lambda$. The set $C^\varepsilon(X)$ is equipped with the canonical norm

$$\|u\|_{C^\varepsilon(X)} \triangleq \frac{1}{I(X)}\Big(\|u\|_{L^\infty} \sup_{\lambda\in\Lambda} \widetilde{\|}X_\lambda\|_{C^\varepsilon} + \sup_{\lambda\in\Lambda} \|\partial_{X_\lambda} u\|_{C^{\varepsilon-1}}\Big).$$

Let v be a time-dependent Lipschitz vector field and let $\Psi(t)$ be its flow. The time evolution of a given initial family $X_0 = (X_{0,\lambda})_{\lambda\in\Lambda}$ is defined by

$$X_{t,\lambda}(x) \triangleq X_{0,\lambda}\Psi(t,\Psi^{-1}(t,x)). \tag{9}$$

Notice that X_t is nothing but the push-forward of X_0 by the flow $\Psi(t)$, and from straightforward algebraic computations one finds that

$$\begin{cases} (\partial_t + v\cdot\nabla)X_{t,\lambda} = \partial_{X_{t,\lambda}} v & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ X_{t,\lambda|t=0} = X_{0,\lambda}. \end{cases} \tag{10}$$

One of the main features of the family $(X_{t,\lambda})_{\lambda\in\Lambda}$ is its commutativity with the transport operator $\partial_t + v\cdot\nabla$. This implies an important consequence about the dynamics of the tangential regularity of the vorticity subject to the system (7). Actually, one obtains easily the following result.

PROPOSITION 2.14. *Let* $(\omega_\kappa, \rho_\kappa)$ *be a solution of the system* (7) *and* $X_t \triangleq (X_{t,\lambda})_{\lambda\in\Lambda}$ *be a family of vector fields satisfying* (10). *Then we have* $\big(\partial_t + v_\kappa\cdot\nabla\big)\partial_{X_{t,\lambda}}\omega_\kappa = \partial_{X_{t,\lambda}}\partial_1\rho_\kappa$.

The following result deals with a special logarithmic result involving striated regularity for the vorticity, see for instance [7].

THEOREM 2.15. *Let* $\varepsilon \in]0,1[$ *and* $X = (X_\lambda)_{\lambda\in\Lambda}$ *be a family of vector fields as in Definition 2.11. Let* v *be a divergence-free vector field such that its vorticity* ω *belongs to* $L^2 \cap C^\varepsilon(X)$. *Then there exists a constant* C *depending only on* ε, *such that*

$$\|\nabla v(t)\|_{L^\infty} \le C\bigg(\|\omega(t)\|_{L^2} + \|\omega(t)\|_{L^\infty}\log\bigg(e + \frac{\|\omega(t)\|_{C^\varepsilon(X)}}{\|\omega(t)\|_{L^\infty}}\bigg)\bigg). \tag{11}$$

We end this section with the following geometric definition.

DEFINITION 2.16. Let $\varepsilon > 0$. A closed curve Σ is said to be of class $C^{1+\varepsilon}$, if there exists $f \in C^{1+\varepsilon}(\mathbb{R}^2)$ such that Σ is locally a zero set of f, i.e., there exists a neighborhood V of Σ such that $\Sigma = f^{-1}(\{0\}) \cap V, \quad \nabla f(x) \neq 0 \quad \forall x \in V$.

3. Smooth vortex patch

This section cares with more general class of initial data than the vortex patches stated in Theorem 1.1. This theorem is a consequence of the following one.

THEOREM 3.1. *Let $\kappa\in[0,1]$, $\varepsilon\in]0,1[$ and take an admissible family of vector fields $X_0=(X_{0,\lambda})_{\lambda\in\Lambda}$ according to the Definition 2.11. Let v_κ^0 be the initial velocity with* $\operatorname{div} v_\kappa^0=0$, *and such that its vorticity $\omega_\kappa^0\in L^2\cap C^\varepsilon(X_0)$. Assume that the initial density $\rho_\kappa^0\in L^2\cap C^{1+\varepsilon}(X_0)$ with $\nabla\rho_\kappa^0\in L^2$. Then there exist a time $T>0$ independent of κ and a unique solution (v_κ,ρ_κ) for the system* (1)*, such that*
(i) $v_\kappa\in L^\infty\big([0,T];\mathrm{Lip}(\mathbb{R}^2)\big)$ and $\omega_\kappa\in L^\infty\big([0,T];L^2\cap L^\infty\big)$;

(ii) $\rho_\kappa\in L^\infty\big([0,T];L^2\cap\mathrm{Lip}(\mathbb{R}^2)\big)$.
Moreover, the family of vector fields transported by the flow defined in (9) *still remains, at every time, admissible of the class C^ε and $\rho_\kappa(t)\in C^{1+\varepsilon}(X_t)$, $\omega_\kappa(t)\in C^\varepsilon(X_t)$. We emphasize that the estimates of the solution in the above spaces are uniform with respect to $\kappa\in[0,1]$.*

The proof of Theorem 3.1 follows several steps that will be stated in details in the following subsections. To simplify the notation, we will omit the index κ.

3.1 A priori estimates for the vorticity and density

We intend to establish the following elementary persistence results on weak regularities.

PROPOSITION 3.2. *Let (v,ρ) be a smooth solution of the system* (1) *defined on $[0,T]$. Then, for every $p\in[1,\infty]$ and $t\in[0,T]$ the following assertions hold.*
(i) $\|\nabla\rho(t)\|_{L^p}\le\|\nabla\rho^0\|_{L^p}e^{CV(t)}$,

(ii) $\kappa\|\nabla\rho\|_{\widetilde{L}^1_tB^2_{\infty,\infty}}\le C(1+\kappa t)\|\nabla\rho^0\|_{L^\infty}e^{CV(t)}$,

(iii) $\|\omega(t)\|_{L^p}\le\|\omega^0\|_{L^p}+\|\nabla\rho^0\|_{L^p}e^{CV(t)}t$,
with the notation $V(t)=\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau$.

Proof. (i) Applying the partial derivative ∂_j to the density equation of (1), one obtains

$$\partial_t\partial_j\rho+v\cdot\nabla(\partial_j\rho)-\kappa\Delta(\partial_j\rho)=-\partial_jv\cdot\nabla\rho \tag{12}$$

from which we infer the following classical estimate

$$\|\nabla\rho(t)\|_{L^p}\le\|\nabla\rho^0\|_{L^p}+\int_0^t\|\nabla\rho(\tau)\|_{L^p}\|\nabla v(\tau)\|_{L^\infty}d\tau.$$

Gronwall's inequality ensures that $\|\nabla\rho(t)\|_{L^p}\le\|\nabla\rho^0\|_{L^p}e^{V(t)}$.

(ii) Applying Proposition 2.9 to (12), one obtains

$$\kappa\|\nabla\rho\|_{\widetilde{L}^1_tB^2_{\infty,\infty}}\le Ce^{CV(t)}(1+\kappa t)\Big(\|\nabla\rho^0\|_{B^0_{\infty,\infty}}+\int_0^t\|\nabla v(\tau)\cdot\nabla\rho(\tau)\|_{B^0_{\infty,\infty}}d\tau\Big).$$

Then using the embedding $L^\infty\hookrightarrow B^0_{\infty,\infty}$, one gets

$$\begin{aligned}\kappa\|\nabla\rho\|_{\widetilde{L}^1_tB^2_{\infty,\infty}}&\le Ce^{CV(t)}(1+\kappa t)\Big(\|\nabla\rho^0\|_{L^\infty}+\int_0^t\|\nabla v(\tau)\|_{L^\infty}\|\nabla\rho(\tau)\|_{L^\infty}d\tau\Big)\\&\le Ce^{CV(t)}(1+\kappa t)\big(\|\nabla\rho^0\|_{L^\infty}+\|\nabla\rho\|_{L^\infty_tL^\infty}\big).\end{aligned}$$

Inserting the estimate (i) for $p=\infty$ into the last quantity of the above inequality, we finally get

$$\kappa\|\nabla\rho\|_{\widetilde{L}_t^1 B^2_{\infty,\infty}} \leq C(1+\kappa t)\|\nabla\rho^0\|_{L^\infty} e^{CV(t)}. \tag{13}$$

(iii) The L^p-estimate for the vorticity can be derived without any difficulty from the first equation of (7), $\|\omega(t)\|_{L^p} \leq \|\omega^0\|_{L^p} + \int_0^t \|\nabla\rho(\tau)\|_{L^p} d\tau$ that we combine with (i) in order to get the desired estimate. □

3.2 A priori estimates for the co-normal regularity of the density

The main result of this paragraph is to prove the persistence of the tangential regularity for the density. This latter unknown obeys to the following transport-diffusion equation

$$\begin{cases} \partial_t\rho + v\cdot\nabla\rho - \kappa\Delta\rho = 0 \\ \rho_{|t=0} = \rho^0. \end{cases} \tag{14}$$

PROPOSITION 3.3. *Let v be a smooth free-divergence vector field and $X_t = (X_{t,\lambda})_{\lambda\in\Lambda}$ be the family defined in* (9)*. Assume that ρ is a smooth solution of* (14)*; then for every $t \geq 0$ we have*

$$\|\partial_{X_{t,\lambda}}\rho(t)\|_{C^\varepsilon} \lesssim e^{CV(t)}(1+\kappa t)\big(\|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon} + \|\nabla\rho^0\|_{L^\infty}\|X\|_{\widetilde{L}_t^\infty C^\varepsilon}\big).$$

Proof. Applying the directional derivative $\partial_{X_{t,\lambda}}$ to (14), one gets

$$\begin{cases} \partial_t\partial_{X_{t,\lambda}}\rho + v\cdot\nabla\partial_{X_{t,\lambda}}\rho - \kappa\Delta\partial_{X_{t,\lambda}}\rho = -\kappa[\Delta, X_{t,\lambda}]\rho \\ \partial_{X_{t,\lambda}}\rho_{|t=0} = \partial_{X_{0,\lambda}}\rho^0, \end{cases} \tag{15}$$

where $[\Delta, X_{t,\lambda}]$ stands for the commutator between Δ and $X_{t,\lambda}$. According to [8, 17], the commutator $\kappa[\Delta, X_{t,\lambda}]\rho$ can be decomposed as follows $\kappa[\Delta, X_{t,\lambda}]\rho = F + \kappa G$, where

$$F = 2\kappa\mathscr{R}(\nabla X^i_{t,\lambda}, \partial_i\nabla\rho) + \kappa\mathscr{R}(\Delta X^i_{t,\lambda}, \partial_i\rho) := \kappa F_1 + \kappa F_2$$

and

$$G = 2T_{\nabla X^i_{t,\lambda}}\partial_i\nabla\rho + 2T_{\partial_i\nabla\rho}\nabla X^i_{t,\lambda} + T_{\Delta X^i_{t,\lambda}}\partial_i\rho + T_{\partial_i\rho}\Delta X^i_{t,\lambda}.$$

To bound $\partial_{X_{t,\lambda}}\rho$ in C^ε we apply [17, Theorem 2, pp. 1461] to (15) which implies that

$$\|\partial_{X_{t,\lambda}}\rho(t)\|_{C^\varepsilon} \leq Ce^{CV(t)}\Big(\|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon} + \|F\|_{\widetilde{L}_t^1 C^\varepsilon} + (1+\kappa t)\|G\|_{\widetilde{L}_t^\infty C^{\varepsilon-2}}\Big) \tag{16}$$

- Estimate of $\|F\|_{\widetilde{L}_t^1 C^\varepsilon}$. Using Bernstein's inequality, one gets

$$\begin{aligned}\|\Delta_q F_1\|_{L_t^1 L^\infty} &\leq C\sum_{j\geq q-4}\|\Delta_j\nabla X\|_{L_t^\infty L^\infty}\|\Delta_j\partial_i\nabla\rho\|_{L_t^1 L^\infty} \\ &\leq C\sum_{j\geq q-4}\|\Delta_j X\|_{L_t^\infty L^\infty}2^{2j}\|\Delta_j\nabla\rho\|_{L_t^1 L^\infty} \leq C\|X\|_{\widetilde{L}_t^\infty C^\varepsilon}\|\nabla\rho\|_{\widetilde{L}_t^1 B^2_{\infty,\infty}} 2^{-q\varepsilon}.\end{aligned}$$

Multiplying both sides by $2^{q\varepsilon}$ and taking the supremum over q, it holds

$$\|F_1\|_{\widetilde{L}_t^1 C^\varepsilon} \leq C\|X\|_{\widetilde{L}_t^\infty C^\varepsilon}\|\nabla\rho\|_{\widetilde{L}_t^1 B^2_{\infty,\infty}}. \tag{17}$$

The estimate of F_2 can be done in a similar way and one finds that

$$\|F_2\|_{\widetilde{L}_t^1 C^\varepsilon} \leq C\|X\|_{\widetilde{L}_t^\infty C^\varepsilon}\|\nabla\rho\|_{\widetilde{L}_t^1 B^2_{\infty,\infty}}. \tag{18}$$

Finally, combining (17) and (18), we end up with

$$\|F\|_{\widetilde{L}^1_t C^\varepsilon} \le C\kappa \|X\|_{\widetilde{L}^\infty_t C^\varepsilon} \|\nabla\rho\|_{\widetilde{L}^1_t B^2_{\infty,\infty}}. \tag{19}$$

- Estimate of $\|G\|_{\widetilde{L}^\infty_t C^{\varepsilon-2}}$. From the definition we have the splitting

$$G = 2T_{\nabla X^i_{t,\lambda}} \partial_i \nabla\rho + 2T_{\partial_i\nabla\rho} \nabla X^i_{t,\lambda} + T_{\Delta X^i_{t,\lambda}} \partial_i\rho + T_{\partial_i\rho} \Delta X^i_{t,\lambda} = 2G_1 + 2G_2 + G_3 + G_4.$$

We start by estimating G_1 in $\widetilde{L}^\infty_t C^{\varepsilon-2}$. For every $q \ge -1$ we have from Bernstein's inequality

$$\|\Delta_q G_1\|_{L^\infty} \le \sum_{|j-q|\le 4} \|\Delta_q\big(S_{j-1}\nabla X \Delta_j \partial_i \nabla\rho\big)\|_{L^\infty} \le C \sum_{|j-q|\le 4} 2^j \|S_{j-1}\nabla X\|_{L^\infty} \|\Delta_j \nabla\rho\|_{L^\infty}.$$

Multiplying both sides by $2^{q(\varepsilon-2)}$ and using once again Bernstein's inequality we deduce that

$$\begin{aligned} 2^{q(\varepsilon-2)} \|\Delta_q G_1\|_{L^\infty} &\le C \sum_{|j-q|\le 4} 2^j 2^{q(\varepsilon-2)} \sum_{l\le j-2} \|\Delta_l \nabla X\|_{L^\infty} \|\Delta_j \nabla\rho\|_{L^\infty} \\ &\le C \sum_{|j-q|\le 4} 2^{q(\varepsilon-2)} 2^j \|\Delta_j \nabla\rho\|_{L^\infty} \sum_{l\le j-2} 2^{l(1-\varepsilon)} 2^{l\varepsilon} \|\Delta_l X\|_{L^\infty} \\ &\le C\|X\|_{C^\varepsilon} \sum_{|j-q|\le 4} 2^{(j-q)(2-\varepsilon)} \|\Delta_j \nabla\rho\|_{L^\infty}. \end{aligned}$$

Since $L^\infty \hookrightarrow B^0_{\infty,\infty}$, then

$$\|G_1\|_{\widetilde{L}^\infty_t B^{\varepsilon-2}_{\infty,\infty}} \le C\|X\|_{L^\infty_t C^\varepsilon} \|\nabla\rho\|_{L^\infty_t L^\infty}. \tag{20}$$

The estimate of G_2 is quite similar to G_1 and one obtains

$$\|G_2\|_{\widetilde{L}^\infty_t B^{\varepsilon-2}_{\infty,\infty}} \le C\|X\|_{L^\infty_t C^\varepsilon} \|\nabla\rho\|_{L^\infty_t L^\infty}. \tag{21}$$

As to G_3 we write for every $q \ge -1$

$$\begin{aligned} \|\Delta_q G_3\|_{L^\infty} &\le C \sum_{|j-q|\le 4} \|\Delta_q\big(S_{j-1}\Delta X \Delta_j \partial_i\rho\big)\|_{L^\infty} \le C \sum_{|j-q|\le 4} \|S_{j-1}\Delta X\|_{L^\infty} \|\Delta_j \nabla\rho\|_{L^\infty} \\ &\le C\|\nabla\rho\|_{L^\infty} \sum_{|j-q|\le 4} \sum_{l\le j-2} 2^{2l} \|\Delta_l X\|_{L^\infty}. \end{aligned}$$

Multiplying both sides by $2^{q(\varepsilon-2)}$, we obtain

$$2^{q(\varepsilon-2)} \|\Delta_q G_3\|_{L^\infty} \le C\|X\|_{C^\varepsilon} \|\nabla\rho\|_{L^\infty} \sum_{\substack{|j-q|\le 4 \\ l\le j-2}} 2^{(q-l)(\varepsilon-2)} \le C\|X\|_{C^\varepsilon} \|\nabla\rho\|_{L^\infty}.$$

Consequently,

$$\|G_3\|_{\widetilde{L}^\infty_t B^{\varepsilon-2}_{\infty,\infty}} \le C\|X\|_{L^\infty_t C^\varepsilon} \|\nabla\rho\|_{L^\infty_t L^\infty}. \tag{22}$$

The estimate of G_4 is quite similar to the preceding ones

$$\|G_4\|_{\widetilde{L}^\infty_t B^{\varepsilon-2}_{\infty,\infty}} \le C\|X\|_{L^\infty_t C^\varepsilon} \|\nabla\rho\|_{L^\infty_t L^\infty}. \tag{23}$$

Putting together (20), (21), (22) and (23), we get

$$\|G\|_{\widetilde{L}^\infty_t B^{\varepsilon-2}_{\infty,\infty}} \le C\|X\|_{L^\infty_t C^\varepsilon} \|\nabla\rho\|_{L^\infty_t L^\infty}. \tag{24}$$

Now, substituting (19) and (24) in (16), we end up with

$$\begin{aligned} \|\partial_{X_{t,\lambda}} \rho(t)\|_{C^\varepsilon} \le Ce^{CV(t)} \big(&\|\partial_{X_{0,\lambda}} \rho^0\|_{C^\varepsilon} + \kappa \|\nabla\rho\|_{\widetilde{L}^1_t B^2_{\infty,\infty}} \|X\|_{\widetilde{L}^\infty_t C^\varepsilon} \\ &+ (1+\kappa t)\|\nabla\rho\|_{L^\infty_t L^\infty} \|X\|_{L^\infty_t C^\varepsilon}\big). \end{aligned} \tag{25}$$

By invoking Proposition 3.2–(i)–(ii) we obtain

$$\|\partial_{X_{t,\lambda}}\rho(t)\|_{C^\varepsilon} \le Ce^{CV(t)}\big(\|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon}+(1+\kappa t)\|\nabla\rho^0\|_{L^\infty}\|X\|_{\widetilde{L}^\infty_t C^\varepsilon}\big).$$

Hence
$$\|\partial_{X_{t,\lambda}}\rho(t)\|_{C^\varepsilon} \le C_0 e^{CV(t)}(1+\kappa t)\big(1+\|X\|_{\widetilde{L}^\infty_t C^\varepsilon}\big).$$
□

3.3 A priori estimates for the co-normal regularity $\partial_{X_{t,\lambda}}\omega$

In this paragraph we shall focus on the estimate of the conormal regularity $\partial_{X_{t,\lambda}}\omega$ in the Hölder space $C^{\varepsilon-1}$. For this aim, we prove

PROPOSITION 3.4. *Let (v,ρ) be any smooth solution of the system* (1) *on* $[0,T]$, *and take any time dependent family of vector field $X_t=(X_{t,\lambda})_{\lambda\in\Lambda}$ transported by the flow of v. Then we have for all $t\in[0,T]$, $\lambda\in\Lambda$*
(i) $I(X_{t,\lambda})\ge I(X_{0,\lambda})e^{-V(t)}$, *(ii)* $\|\operatorname{div}X_{t,\lambda}\|_{C^\varepsilon}\le\|\operatorname{div}X_{0,\lambda}\|_{C^\varepsilon}e^{CV(t)}$ *for every* $\lambda\in\Lambda$,

(iii) $\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}}+\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}\le C\Big(\|\partial_{X_{0,\lambda}}\omega^0\|_{C^{\varepsilon-1}}+\widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon}+\|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon}\Big)e^{C\Phi(t)}$,
with $\Phi(t):=(t+\kappa t^2)\|\nabla\rho^0\|_{L^\infty}e^{CV(t)}+V(t)+t$.

Proof. (i) Let us bound $I(X_{t,\lambda})$ from below by applying the time derivative to $\partial_{X_{0,\lambda}}\Psi(t,x)$ and invoking the fact $X_{t,\lambda}(\Psi(t,x))=\partial_{X_{0,\lambda}}\Psi(t,x)$ and $\partial_t\Psi(t,x)=v(t,\psi(t,x))$ with $\Psi(0,x)=x$. We deduce that $\partial_t\partial_{X_{0,\lambda}}\Psi(t,x)=\nabla v(t,\Psi(t,x))\cdot\partial_{X_{0,\lambda}}\Psi(t,x)$, $\partial_{X_{0,\lambda}}\Psi(0,x)=X_{0,\lambda}$. The time reversibility of this equation combined with Gronwall's inequality tells us $|X_{0,\lambda}(x)|\le e^{V(t)}\big|\partial_{X_{0,\lambda}}\Psi(t,x)\big|$, for all $(\lambda,x)\in\Lambda\times\mathbb{R}^2$. In view of Definition 2.11 (ii) we get the desired estimate.

(ii) Applying "div" operator to (10), an easy computation combined with $\operatorname{div}v=0$ shows us that $\operatorname{div}X_{t,\lambda}$ satisfies $(\partial_t+v\cdot\nabla)\operatorname{div}X_{t,\lambda}=0$. Proposition 2.8 yields $\|\operatorname{div}X_{t,\lambda}\|_{C^\varepsilon}\le Ce^{CV(t)}\|\operatorname{div}X_{0,\lambda}\|_{C^\varepsilon}$.

(iii) To bound $\partial_{X_{t,\lambda}}\omega$ in $C^{\varepsilon-1}$, we first recall from Proposition 2.14 that $(\partial_t+v\cdot\nabla)\partial_{X_{t,\lambda}}\omega=\partial_{X_{t,\lambda}}\partial_1\rho$. In accordance with Proposition 2.8, we readily get

$$\big\|\partial_{X_{t,\lambda}}\omega(t)\big\|_{C^{\varepsilon-1}}\le Ce^{CV(t)}\Big(\big\|\partial_{X_{0,\lambda}}\omega^0\big\|_{C^{\varepsilon-1}}+\int_0^t e^{-CV(\tau)}\|\partial_{X_{\tau,\lambda}}\partial_1\rho(\tau)\|_{C^{\varepsilon-1}}d\tau\Big). \tag{26}$$

Let us estimate $\|\partial_{X_{\tau,\lambda}}\partial_1\rho(\tau)\|_{C^{\varepsilon-1}}$. The identity $\partial_{X_{\tau,\lambda}}\partial_1\rho=\partial_1(\partial_{X_{\tau,\lambda}}\rho)-\partial_{\partial_1X_{\tau,\lambda}}\rho$ combined with the following estimate proved in [16, Corollary 1] $\|\partial_jX\cdot\nabla f\|_{C^{\varepsilon-1}}\le C\|\nabla f\|_{L^\infty}\widetilde{\|}X\|_{C^\varepsilon}$ yields to $\|\partial_{X_{t,\lambda}}\partial_1\rho(t)\|_{C^{\varepsilon-1}}\le\|\partial_{X_{t,\lambda}}\rho(t)\|_{C^\varepsilon}+\|\nabla\rho(t)\|_{L^\infty}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}$. Plugging the last estimate into (26), we get

$$\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}}\le Ce^{CV(t)}\bigg(\|\partial_{X_{0,\lambda}}\omega^0\|_{C^{\varepsilon-1}}+\int_0^t e^{-CV(\tau)}\|\partial_{X_{\tau,\lambda}}\rho(\tau)\|_{C^\varepsilon}d\tau + \int_0^t e^{-CV(\tau)}\|\nabla\rho(\tau)\|_{L^\infty}\widetilde{\|}X_{\tau,\lambda}\|_{C^\varepsilon}d\tau\bigg). \tag{27}$$

For the term $\|\partial_{X_{t,\lambda}}\rho(t)\|_{C^\varepsilon}$, we may apply (25) and (13) and therefore (27) becomes

$$e^{-CV(t)}\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}}\lesssim\|\partial_{X_{0,\lambda}}\omega^0\|_{C^{\varepsilon-1}}+\|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon}t + \int_0^t e^{-CV(\tau)}(1+\kappa\tau)\big(\|\nabla\rho^0\|_{L^\infty}+\|\nabla\rho\|_{L^\infty_\tau L^\infty}\big)\widetilde{\|}X_\lambda\|_{L^\infty_\tau C^\varepsilon}d\tau. \tag{28}$$

To estimate $\widetilde{\|}X_\lambda\|_{L_t^\infty C^\varepsilon}$, we apply again Proposition 2.8 to (10),

$$\|X_{t,\lambda}\|_{C^\varepsilon} \le Ce^{CV(t)}\Big(\|X_{0,\lambda}\|_{C^\varepsilon} + \int_0^t e^{-CV(\tau)}\|\partial_{X_{\tau,\lambda}} v(\tau)\|_{C^\varepsilon} d\tau\Big).$$

As to $\|\partial_{X_{t,\lambda}} v(t)\|_{C^\varepsilon}$ we use the following estimate proved in [1,7],

$$\|\partial_{X_{t,\lambda}} v(t)\|_{C^\varepsilon} \lesssim \|\nabla v(t)\|_{L^\infty}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} + \|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}}.$$

That we get

$$\|X_{t,\lambda}\|_{C^\varepsilon} \le Ce^{CV(t)}\bigg(\|X_{0,\lambda}\|_{C^\varepsilon} + \int_0^t e^{-CV(\tau)}\big(\|\nabla v(\tau)\|_{L^\infty}\widetilde{\|}X_{\tau,\lambda}\|_{C^\varepsilon} + \|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}}\big)d\tau\bigg).$$

Since $\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} = \|X_{t,\lambda}\|_{C^\varepsilon} + \|\operatorname{div} X_{t,\lambda}\|_{C^\varepsilon}$, then the last estimate combined with (ii) provides

$$e^{-CV(t)}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \lesssim \widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon} + \int_0^t e^{-CV(\tau)}\big(\|\nabla v(\tau)\|_{L^\infty}\widetilde{\|}X_{\tau,\lambda}\|_{C^\varepsilon} + \|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}}\big)d\tau. \quad (29)$$

Adding (28) and (29) and setting $\Pi(t) := e^{-CV(t)}\big(\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}} + \widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}\big)$, we find

$$\Pi(t) \lesssim \Pi(0) + \|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon} t + \int_0^t \big((1+\kappa\tau)(\|\nabla\rho^0\|_{L^\infty} + \|\nabla\rho\|_{L_\tau^\infty L^\infty}) + \|\nabla v(\tau)\|_{L^\infty} + 1\big)\Pi(\tau)d\tau.$$

Using Gronwall's inequality we obtain

$$\Pi(t) \lesssim \big(\Pi(0) + \|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon}\big)e^{(1+\kappa t)t\|\nabla\rho^0\|_{L^\infty} + (1+\kappa t)t\|\nabla\rho\|_{L_t^\infty L^\infty} + CV(t) + Ct}.$$

Finally, from Proposition 3.2 (i) we deduce that

$$\Pi(t) \lesssim \big(\Pi(0) + \|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon}\big)e^{(1+\kappa t)t\|\nabla\rho^0\|_{L^\infty}e^{CV(t)} + CV(t) + Ct}. \qquad \square$$

3.4 Regularity persistence

This part is concerned with the regularity persistence of the prescribed initial regularity. The basic ingredient is to get an estimate for the Lipschitz norm of the velocity for short time. The main result is the following.

PROPOSITION 3.5. *Under the assumptions of Theorem 3.1, the solution* (v,ρ) *of* (1) *can be defined on an interval* $[0,T]$ *such that* T *is related to the size of the initial data with the persistence result: for all* $t \in [0,T]$, $\quad \|\omega(t)\|_{L^2\cap L^\infty} + \|\omega(t)\|_{C^\varepsilon(X_t)} + \|\nabla v(t)\|_{L^\infty} + \widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \le C_0$*, with* C_0 *a constant depending on the initial data.*

Proof. The basic ingredient of the proof is to get an a priori estimate for the Lipschitz norm of the velocity over a time interval $[0,T]$ that can be quantified with respect to the initial data. By virtue of Proposition 3.2 (iii) and Proposition 3.4 (iii) we deduce that

$$\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}} + \|\omega(t)\|_{L^\infty}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \le C_0 e^{C\Phi(t)},$$

with the estimate

$$0 \le \Phi(t) \le C_0(1+t^2)e^{CV(t)}. \quad (30)$$

Therefore combining this estimate with the definition 2.13 and Proposition 3.4 (i) yields

$$\|\omega(t)\|_{C^\varepsilon(X_t)} \le C_0 e^{C\Phi(t)}. \quad (31)$$

Thus plugging this estimate into the logarithmic estimate (11) and using Proposition 3.2 we find

$$\|\nabla v(t)\|_{L^\infty} \le C\Big(\|\omega^0\|_{L^2} + t\|\nabla\rho^0\|_{L^2} e^{CV(t)}\Big) + C\|\omega(t)\|_{L^\infty} \log\left(e + \frac{\|\omega(t)\|_{C^\varepsilon(X_t)}}{\|\omega(t)\|_{L^\infty}}\right). \quad (32)$$

As the function $]0,+\infty[\ni x \mapsto x\log(e+a/x)$ is strictly increasing and $]0,+\infty[\ni x \mapsto \log(e+a/x)$ is strictly decreasing, we obtain

$$\|\omega(t)\|_{L^\infty} \log\left(e + \frac{\|\omega(t)\|_{C^\varepsilon(X_t)}}{\|\omega(t)\|_{L^\infty}}\right) \le C_0(1+t)e^{CV(t)} \log\left(e + \frac{\|\omega(t)\|_{C^\varepsilon(X_t)}}{C_0}\right).$$

Notice that we have used the following estimate which follows from Proposition 3.2 (iii) $\|\omega(t)\|_{L^\infty} \le C_0(1+t)e^{CV(t)}$. Consequently (32) becomes

$$\|\nabla v(t)\|_{L^\infty} \le C_0(1+t)e^{CV(t)} + C_0(1+t)e^{CV(t)} \log\left(e + \frac{\|\omega(t)\|_{C^\varepsilon(X_t)}}{C_0}\right).$$

Applying (30) and (31) we get

$$\|\nabla v(t)\|_{L^\infty} \le C_0(1+t)e^{CV(t)} + C_0(1+t)e^{CV(t)}\big(1+\Phi(t)\big) \le C_0(1+t^2)e^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}.$$

From this we deduce the existence of $T>0$ depending on the initial data through C_0 such that

$$\forall t \in [0,T], \quad \|\nabla v(t)\|_{L^\infty} \le 2C_0, \quad (33)$$

which implies in turn that all the involved norms are bounded over the time interval $[0,T]$. The proof of Theorem 3.1 follows easily from Proposition 3.5. Indeed, up to now we have established the suitable a priori estimates required for the regularity persistence which are enough to construct a unique solution for short time. This latter part concerning the construction of the solutions is classical and is well-detailed in various references such as [7, 16]. □

3.5 Proof of Theorem 1.1

The proof of Theorem 1.1 follows from Theorem 3.1. To see this, it suffices to build an initial admissible family $X_0 = (X_{0,\lambda})_{0\le\lambda\le1}$ such that $\mathbf{1}_{\Omega_0} \in C^\varepsilon(X_0)$ and to check the regularity persistence of the boundary. This is very classical and was done first in [7], and for the convenience of the reader we shall reproduce here the basic ingredients. Since the initial boundary $\partial\Omega_0$ is a Jordan curve in the class $C^{1+\varepsilon}$, then according to the definition 2.16, there exists a local chart (f_0, V_0), with V_0 being a neighborhood of $\partial\Omega_0$ such that

$$\begin{cases} f_0 \in C^{1+\varepsilon}(\mathbb{R}^2), \quad \nabla f_0(x) \neq 0 \quad \text{on } V_0 \\ \partial\Omega_0 = f_0^{-1}(\{0\}) \cap V_0. \end{cases}$$

On the other hand, take $\chi \in \mathscr{D}(\mathbb{R}^2)$, with $0 \le \chi \le 1$, $\operatorname{supp}\chi \subset V_0$ and $\chi(x) = 1$ for all $x \in W_0$, where W_0 is a small neighborhood of $\partial\Omega_0$ strictly contained in V_0. Next, define the family $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ by $X_{0,0}(x) = \nabla^\perp f_0(x)$ and $X_{0,1}(x) = (1-\chi(x))\begin{pmatrix}1\\0\end{pmatrix}$. We observe that $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ is non-degenerate, and each member $X_{0,\lambda}$ with its divergence belong to $C^\varepsilon(\mathbb{R}^2)$, then according to the Definition 2.11, we conclude that X_0 is an admissible family. Moreover, from the identity $\nabla\omega_0(x) = -\vec{n}(x)d\sigma_{\partial\Omega_0}$ with $\vec{n}$ being the outward unit normal vector to the boundary and $d\sigma_{\partial\Omega_0}$ is the arc-length measure on $\partial\Omega_0$, we check easily that $\forall\lambda \in \{0,1\}$, $X_{0,\lambda}(x)\cdot\nabla\omega_0(x) = 0$. In addition, since $\rho^0 \in C^{1+\varepsilon}(\mathbb{R}^2)$ then $\rho^0 \in C^{1+\varepsilon}(X_0)$.

Consequently, in view of the Theorem 3.1, the system (1) is locally well-posed, with the persistence regularity detailed in Proposition 3.5. i.e., there exists a unique local solution $(v_\kappa, \rho_\kappa) \in \big(L^\infty([0,T];\mathrm{Lip}(\mathbb{R}^2))\big)^2$ for (1).

Now, it remains to check the regularity of the transported boundary $\partial\Omega_t$. We parametrize the boundary $\partial\Omega_0$ by defining the periodic curve $\gamma^0 \in C^{1+\varepsilon}([0,2\pi];\mathbb{R}^2)$ as the solution of the following ordinary differential equation

$$\begin{cases} \partial_\sigma \gamma^0(\sigma) = X_{0,0}(\gamma^0(\sigma)) \\ \gamma^0(0) = x_0, \quad x_0 \in \partial\Omega_0. \end{cases}$$

To define the evolution parametrization of $\partial\Omega_t$, we simply set for $t \geq 0$, $\gamma(t,\sigma) \triangleq \Psi(t,\gamma_0(\sigma))$. Clearly, the curve $\gamma(t,\cdot)$ is the transport of γ^0 by the flow Ψ_t and by the criterion of differentiation with respect to σ, we readily get $\partial_\sigma\gamma(t,\sigma) = \big(\partial_{X_{0,0}}\Psi\big)(t,\gamma^0(\sigma))$. Since $\partial_{X_{0,0}}\Psi(t) \equiv X_{t,0} \circ \Psi(t)$, with $X_{t,0}$ is the push-forward of $X_{0,0}$ by the flow $\Psi(t)$, then in view of the classical estimate $\|X_{t,0} \circ \Psi(t)\|_{C^\varepsilon} \leq \|X_{t,0}\|_{C^\varepsilon} \|\nabla\Psi(t)\|_{L^\infty}^\varepsilon \leq \|X_{t,0}\|_{C^\varepsilon} e^{CV(t)} \leq C_0$, where we have used the fact $\|\nabla\Psi(t)\|_{L^\infty} \leq e^{CV(t)}$ and the estimates of Proposition 3.5. Therefore $\partial_{X_{0,0}}\Psi(t) \in L^\infty([0,T];C^\varepsilon)$ and $[0,T] \ni t \mapsto \gamma_t$ belongs to $L^\infty([0,T];C^{1+\varepsilon}([0,2\pi];\mathbb{R}^2))$. This concludes the regularity persistence of the boundary $\partial\Omega_t$ and so the proof of Theorem 1.1 is finished.

4. Inviscid limit for velocities and densities

This section cares essentially with the proof of Theorem 1.3.

4.1 Proof of Theorem 1.3

(i) Set $U = v_\kappa - v$, $\Theta = \rho_\kappa - \rho$ and $P = \pi_\kappa - \pi$. Then, a straightforward computation provides that (U,Θ,P) satisfies

$$\begin{cases} \partial_t U + v_\kappa \cdot \nabla U + \nabla P = \Theta e_2 - U \cdot \nabla v & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t \Theta + v_\kappa \cdot \nabla\Theta - \kappa\Delta\Theta = -U \cdot \nabla\rho + \kappa\Delta\rho & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \mathrm{div}\, U = 0, \\ U_{|t=0} = U_0 = 0, \quad \Theta_{|t=0} = \Theta_0 = 0. \end{cases} \tag{34}$$

• *First case:* $p = 2$. Dotting U-equation (resp. Θ-equation) by U (resp. Θ). After some integration by parts since convective terms integrate to zero, and due to the fact the div $v_\kappa =$ div $v = 0$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|U(t)\|_{L^2}^2 &\leq \|\Theta(t)\|_{L^2}\|U(t)\|_{L^2} + \|U(t)\|_{L^2}^2\|\nabla v(t)\|_{L^\infty} \\ &\leq \frac{1}{2}\big(\|\Theta(t)\|_{L^2}^2 + \|U(t)\|_{L^2}^2\big) + \|U(t)\|_{L^2}^2\|\nabla v(t)\|_{L^\infty} \\ &\leq C\big(\|\nabla v(t)\|_{L^\infty} + 1\big)\big(\|\Theta(t)\|_{L^2}^2 + \|U(t)\|_{L^2}^2\big) \end{aligned} \tag{35}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\Theta(t)\|_{L^2}^2 + \kappa\|\nabla\Theta(t)\|_{L^2}^2 \leq \kappa\|\nabla\rho(t)\|_{L^2}\|\nabla\Theta(t)\|_{L^2} + \|U(t)\|_{L^2}\|\nabla\rho(t)\|_{L^\infty}\|\Theta(t)\|_{L^2}$$

$$\leq \frac{\kappa}{2}\|\nabla\rho(t)\|_{L^2}^2+\frac{\kappa}{2}\|\nabla\Theta(t)\|_{L^2}^2+\frac{1}{2}\|\nabla\rho(t)\|_{L^\infty}\big(\|U(t)\|_{L^2}^2+\|\Theta(t)\|_{L^2}^2\big).$$

Here, we have used two times Young's inequality. Carrying over the term $\frac{\kappa}{2}\|\nabla\Theta(t)\|_{L^2}^2$, to the left-hand side, we have

$$\frac{1}{2}\frac{d}{dt}\|\Theta(t)\|_{L^2}^2\leq\frac{\kappa}{2}\|\nabla\rho(t)\|_{L^2}^2+\frac{1}{2}\|\nabla\rho(t)\|_{L^\infty}\big(\|U(t)\|_{L^2}^2+\|\Theta(t)\|_{L^2}^2\big). \tag{36}$$

Collecting (35) and (36), and defining $\Sigma(t)=\|U(t)\|_{L^2}^2+\|\Theta(t)\|_{L^2}^2$, it follows that

$$\frac{d}{dt}\Sigma(t)\leq\kappa\|\nabla\rho(t)\|_{L^2}^2+\Sigma(t)\big(\|\nabla v(t)\|_{L^\infty}+\|\nabla\rho(t)\|_{L^\infty}+1\big).$$

Gronwall's inequality then implies $\Sigma(t)\leq e^{V(t)+\|\nabla\rho\|_{L^1_tL^\infty}+t}\big(\Sigma(0)+\kappa\|\nabla\rho\|_{L^1_tL^2}^2\big)$.

For the two terms $\|\nabla\rho\|_{L^1_tL^\infty}$ and $\|\nabla\rho\|_{L^1_tL^2}^2$, we employ the Proposition 3.2 (i) (which remains true for $\kappa=0$) for $p=\infty$ and $p=2$. Then, in view of $\Sigma(0)=0$ we have $\Sigma(t)\leq \kappa e^{C(V(t)+\|\nabla\rho^0\|_{L^\infty}(e^{CV(t)}+1)t)}\|\nabla\rho^0\|_{L^2}^2 t$. Even though, all the involved norms are bounded over $[0,T]$, we infer that

$$\sup_{t\in[0,T]}\big(\|v_\kappa(t)-v(t)\|_{L^2}+\|\rho_\kappa(t)-\rho(t)\|_{L^2}\big)\leq C_0\kappa^{1/2}. \tag{37}$$

• *Second case:* $2<p\leq\infty$. Using in general case the following classical complex interpolation $\|f\|_{L^p}\leq C\|f\|_{L^2}^{2/p}\|f\|_{L^\infty}^{1-2/p}$, then, in view of (37) we deduce that

$$\|v_\kappa(t)-v(t)\|_{L^p}+\|\rho_\kappa(t)-\rho(t)\|_{L^p}\leq C_0\kappa^{\frac{1}{p}}\big(\|v_\kappa(t)-v(t)\|_{L^\infty}^{1-2/p}+\|\rho_\kappa(t)-\rho(t)\|_{L^\infty}^{1-2/p}\big). \tag{38}$$

To get the bound for the two last quantities, we employ in general case the so-called Gagliardo-Nirenberg inequality $\|f\|_{L^\infty}\lesssim\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^\infty}^{1/2}$. Thus we get in view of (33), (37) and Proposition 3.2 (i) that

$$\begin{aligned}\|v_\kappa(t)-v(t)\|_{L^\infty}&\leq\|v_\kappa(t)-v(t)\|_{L^2}^{1/2}\|\nabla v_\kappa(t)-\nabla v(t)\|_{L^\infty}^{1/2}\\&\leq\|v_\kappa(t)-v(t)\|_{L^2}^{1/2}\big(\|\nabla v_\kappa(t)\|_{L^\infty}+\|\nabla v(t)\|_{L^\infty}\big)^{1/2}\leq C_0\kappa^{1/4}.\end{aligned}$$

Similarly
$$\begin{aligned}\|\rho_\kappa(t)-\rho(t)\|_{L^\infty}&\leq\|\rho_\kappa(t)-\rho(t)\|_{L^2}^{1/2}\|\nabla\rho_\kappa(t)-\nabla\rho(t)\|_{L^\infty}^{1/2}\\&\leq\|\rho_\kappa(t)-\rho(t)\|_{L^2}^{1/2}\big(\|\nabla\rho_\kappa(t)\|_{L^\infty}+\|\nabla\rho(t)\|_{L^\infty}\big)^{1/2}\\&\leq C_0\kappa^{1/4}\|\nabla\rho^0\|_{L^\infty}^{1/2}e^{C(V_\kappa(t)+V(t))}\leq C_0\kappa^{1/4}.\end{aligned}$$

Plugging the last two estimates in (38), then with the notation $C_0=C(\|\nabla\rho^0\|_{L^2\cap L^\infty},T)$, we could obtain for $p\in[2,\infty]$

$$\sup_{t\in[0,T]}\big(\|v_\kappa(t)-v(t)\|_{L^p}+\|\rho_\kappa(t)-\rho(t)\|_{L^p}\big)\leq C_0\kappa^{1/4+1/2p} \tag{39}$$

which finishes the proof of (i)

(ii) Recall that $\Psi_\kappa(t,x)=x+\int_0^t v_\kappa(\tau,\Psi_\kappa(\tau,x))d\tau,\ \Psi(t,x)=x+\int_0^t v(\tau,\Psi(\tau,x))d\tau$. We intend to prove that $(\Psi_\kappa)_\kappa$ converges uniformly towards Ψ locally in time when κ goes to 0.

To do this, we have for every $\kappa > 0$

$$\begin{aligned}|\Psi_\kappa(t,x)-\Psi(t,x)| \leq & \underbrace{\int_0^t \big|v_\kappa\big(\tau,\Psi_\kappa(\tau,x)\big)-v\big(\tau,\Psi_\kappa(\tau,x)\big)\big|d\tau}_{\text{(I)}} \\ & + \underbrace{\int_0^t \big|v\big(\tau,\Psi_\kappa(\tau,x)\big)-v\big(\tau,\Psi(\tau,x)\big)\big|d\tau}_{\text{(II)}}\end{aligned} \tag{40}$$

The term (I) comes immediately from (39), that is for $t \in [0,T]$ (I) $\leq C_0\kappa^{1/4}$.

Concerning (II), using the following general estimate

$$\big|f\circ\Psi_\kappa - f\circ\Psi\big| = \frac{\big|f\circ\Psi_\kappa - f\circ\Psi\big|}{|\Psi_\kappa-\Psi|}|\Psi_\kappa-\Psi| \leq \|\nabla f\|_{L^\infty}\|\Psi_\kappa-\Psi\|_{L^\infty}.$$

Thus we have (II) $\leq \int_0^t \|\nabla v(\tau)\|_{L^\infty}\|\Psi_\kappa(\tau)-\Psi(\tau)\|_{L^\infty}d\tau$.

Adding (I) and (II) and inserting them in (40), we shall get for $x \in \mathbb{R}^2$

$$|\Psi_\kappa(t,x)-\Psi(t,x)| \lesssim C_0\kappa^{1/4} + \int_0^t \|\nabla v(\tau)\|_{L^\infty}\|\Psi_\kappa(\tau)-\Psi(\tau)\|_{L^\infty}d\tau.$$

Gronwall's inequality implies for every $t \in [0,T]$ $\|\Psi_\kappa(t)-\Psi(t)\|_{L^\infty} \lesssim C_0\kappa^{1/4}$, which achieves the proof of (ii), so of Theorem 1.3.

REFERENCES

[1] H. Bahouri, J-Y. Chemin, R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften 343. Springer-Verlag (2011).

[2] H. Berestycki, P. Constantin, L. Ryzhik, *Non-planar fronts in Boussinesq reactive flows*, Ann. Inst. H. Poincar. Anal. Non Linaire **23(4)** (2006), 407–437.

[3] M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. de l'École Norm. Sup. **14** (1981), 209–246.

[4] D. Chae, *Global regularity for the 2D−Boussinesq equations with partial viscous terms*, Advances in Math. **203(2)** (2006), 497–513.

[5] D. Chae, H.-S. Nam, *Local existence and blow-up criterion for the Boussinesq equations*, Proc .Roy. Soc. Edinburgh, Sect. A **127(5)** (1997), 935–946.

[6] D. Chae, H.-S. Nam, *Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations*, Nagoya Math. J. **155** (1999), 55–80.

[7] J.-Y. Chemin, *Perfect incompressible Fluids*, Oxford University Press 1998.

[8] R. Danchin, *Poches de tourbillon visqueuses*, J. Math. Pures Appl. (9) **76(7)** (1997), 609–647.

[9] R. Danchin, M. Paicu, *Le théorème de Leary et le théorème de Fujita-Kato pour le système de Boussinesq partiellement visqueux*, Bulletin de la SMF **136** (2008), 261–309.

[10] R. Danchin, M. Paicu. *Global well-posedness issues for the inviscid Boussinesq system with Yudovich's type data*, Comm. Math. Phys. **290(1)** (2009), 1–14 .

[11] R. Danchin, X. Zhang, *Global persistence of geometrical structures for the Boussinesq equation with no diffusion*, Commun. Part. Differ. Eq., **42(1)** (2017), 68–99.

[12] N. Depauw, *Poche de tourbillon pour Euler 2D dans un ouvert à bord*, J. Math. Pures Appl. (9) **78(3)** (1999), 313–351.

[13] F. Fanelli, *Conservation of geometric structures for non-homogeneous inviscid incompressible fluids*, Comm. Partial Differential Equations, **37(9)** (2012), 1553–1595.

[14] E. Feireisl, A. Novotný, *The Oberbeck-Boussinesq Approximation as a singular limit of the full Navier-Stokes-Fourier system*, J. Math. Fluid. Mech. **11** (2009), 274–302.

[15] P. Gamblin, X. Saint-Raymond, *On three-dimensional vortex patches*, Bull. Soc. Math. France **123(3)** (1995), 375–424.

[16] Z. Hassainia, T. Hmidi, *On the inviscid Boussinesq system with rough initial data*, J. Math. Anal. App. **430(2)** (2015), 777–809.

[17] T. Hmidi, *Régularité höldérienne des poches de tourbillon visqueuses*, J. Math. Pures Appl. (9) **84(11)** (2005), 1455–1495.

[18] T. Hmidi, S. Keraani, *Inviscid limit for the two-dimensional Navier-Stokes equation in a critical Besov space*, Asymptotic Anal., **53(3)** (2007), 125–138.

[19] T. Hmidi, S. Keraani, *On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity*, Adv. Differ. Eq., **12(4)** (2007), 461–480.

[20] T. Hmidi, M. Zerguine, *On the global well-posedness of the Euler-Boussinesq system with fractional dissipation*, Physica D. **239** (2010), 1387–1401.

[21] T. Hmidi, M. Zerguine, *Vortex patch problem for stratified Euler equations*, Comm. Math. Sci. **12(8)** (2014), 1541–1563.

[22] X. Liu, M. Wang, Z. Zhang, *Local well-posedness and blow-up criterion of the Boussinesq equations in critical Besov spaces*, J. Math. Fluid Mech., **12** (2010), 280–292.

[23] H. Meddour, M. Zerguine, *Optimal rate of convergence in stratified Boussinesq system*, Dynam. Part. Differ. Eq., **15(4)** (2018), 235–263.

[24] J. Pedlosky, *Geophysical fluid dynamics*, Springer-Verlag, New-York 1987.

[25] P. Serfati, *Une preuve directe d'existence globale des vortex patches 2D*, C. R. Acad. Sci. Paris Sér. I Math, **318(6)** (1994), 515–518.

[26] V. I. Yudovich, *Non-stationnary flows of an ideal incompressible fluid*, Zhurnal Vych Matematika, **3** (1963), 1032-106.

[27] M. Zerguine, *The regular vortex patch for stratified Euler equations with critical fractional dissipation*, J. Evol. Equ. **15** (2015), 667–698.

(received 03.04.2018; in revised form 08.10.2018; available online 13.04.2019)

LEDPA, Université de Batna –2–, Faculté des Mathématiques et d'Informatique, Département de Mathématiques, 05000 Batna Algérie

E-mail: h.meddour@univ-batna2.dz