MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 26.07.2019

research paper
оригинални научни рад

# LEXICOGRAPHIC PRODUCT OF VAGUE GRAPHS WITH APPLICATION

**R. A. Borzooei, B. Sheikh Hoseini and Young Bae Jun**

**Abstract**. A vague graph is a generalized structure of a fuzzy graph which gives more precision, flexibility, and compatibility to a system when it is compared with the systems which are designed by using fuzzy graphs. The present study aims to introduce the notion of lexicographic min-product and max-product of two vague graphs. Then, the degree of a vertex in the lexicographic products of two vague graphs is obtained. Finally, a relationship is obtained between the lexicographic min-product and lexicographic max-product.

## 1. Preliminaries

In this section, we introduce some preliminary notions and definitions which are used in this paper.

Let $G^* = (V, E)$ be a simple graph and suppose that $\sigma$ and $\mu$ are two fuzzy sets on $V$ and $E$, respectively. Then $G = (\sigma, \mu)$ is called a *fuzzy graph* on $G^*$ if $\mu(uv) \leqslant \sigma(u) \wedge \sigma(v)$, for every $uv \in E$. The degree of a vertex $u$ of fuzzy graph $G = (\sigma, \mu)$ is denoted by $d_G(u)$ and defined as $d_G(u) = \sum_{u \neq v, uv \in E} \mu(uv)$. Note that, if the degree of a vertex of fuzzy graph $G$ is zero, then the degree of the edge that is connected to this vertex should be zero. Let $G{=}(\sigma, \mu)$ be a fuzzy graph on $G^*$. Then $G'{=}(\sigma', \mu')$ is called a spanning fuzzy subgraph of $G$ if $\sigma{=}\sigma'$ and $\mu' \subseteq \mu$ (see [10]).

A *vague set* $A$ on a non-empty set $X$ is a pair $(t_A, f_A)$, where $t_A : X \to [0, 1]$ and $f_A : X \to [0, 1]$ are true and false membership functions, respectively such that $0 \leq t_A(x) + f_A(x) \leq 1$ for all $x \in X$. Let $G^* = (V, E)$ be a simple graph. Then a *vague graph* on $G^*$ is a pair $G = (A, B)$, where $A = (t_A, f_A)$ is a vague set on $V$ and $B = (t_B, f_B)$ is a vague set on $E$ such that for each $u, v \in V$, $t_B(uv) \leq t_A(u) \wedge t_A(v)$, $f_B(uv) \geq f_A(u) \vee f_A(v)$. The vague graph $G = (A, B)$ on $G^*$ is called *strong*, if for every edge $uv \in E$, $t_B(uv) = t_A(u) \wedge t_A(v)$ and $f_B(uv) = f_A(u) \vee f_A(v)$; it is called *complete*, if $G^*$ is complete and for every $uv \in E$, $t_B(uv) = t_A(u) \wedge t_A(v)$ and $f_B(uv) = f_A(u) \vee f_A(v)$. A *complete vague graph* with $n$ nodes is denoted by $K_n$.

*2010 Mathematics Subject Classification*:
*Keywords and phrases*: Vague graph; lexicographic min-product; lexicographic max-product.

Let $G = (V, E)$ be a vague graph on $G^*$. If each vertex in $G$ has the same degree $(l_1, l_2)$, then $G$ is said to be a *regular* vague graph. Moreover, if each vertex in $G$ has the same total degree $(t_1, t_2)$, then $G$ is said to be a *totally regular* vague graph (see [3, 7]).

## 2. Lexicographic product of vague graphs

In this section we introduce the concept of lexicographic (min)max-product of two vague graphs. Then we get some results on connected, strong and complete lexicographic (min)max-product of two vague graphs. Also degree of vertexes and edges are studied and concept of regularity is obtained. Then we prove a relationship between the lexicographic min-product and lexicographic max-product.

### 2.1 Lexicographic min-product

In this subsection, we define the notion of lexicographic min-product of two vague graphs and provide the main results such as strongness, connectedness and regularity on it.

DEFINITION 2.1. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two vague graphs on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Let $V = V_1 \times V_2$, $E = \{((u_1, v_1)(u_2, v_2)) \mid (u_1u_2 \in E_1, v_1v_2 \in E_2) \text{ or } (u_1 = u_2 \in V_1, v_1v_2 \in E_2) \text{ or } (v_1 = v_2 \in V_2, u_1u_2 \in E_1)\}$ and $G^* = (V, E)$. Now we define vague sets $A = (t_A, f_A)$ on $V$ and $B = (t_B, f_B)$ on $E$ by $t_A(u, v) = t_{A_1}(u) \wedge t_{A_2}(v)$, $f_A(u, v) = f_{A_1}(u) \vee f_{A_2}(v)$ for all $(u, v) \in V$ and

$$t_B((u_1, v_1)(u_2, v_2)) = \begin{cases} t_{B_1}(u_1u_2) \wedge t_{B_2}(v_1v_2), & \text{if} \quad u_1u_2 \in E_1, v_1v_2 \in E_2 \\ t_{A_1}(u_1) \wedge t_{B_2}(v_1v_2), & \text{if} \quad u_1 = u_2 \in V_1, v_1v_2 \in E_2 \\ t_{A_2}(v_1) \wedge t_{B_1}(u_1u_2), & \text{if} \quad v_1 = v_2 \in V_2, u_1u_2 \in E_1 \end{cases}$$

and
$$f_B((u_1, v_1)(u_2, v_2)) = \begin{cases} f_{B_1}(u_1u_2) \vee f_{B_2}(v_1v_2), & \text{if} \quad u_1u_2 \in E_1, v_1, v_2 \in E_2 \\ f_{A_1}(u_1) \vee f_{B_2}(v_1v_2), & \text{if} \quad u_1 = u_2 \in V_1, v_1v_2 \in E_2 \\ f_{A_2}(v_1) \vee f_{B_1}(u_1u_2), & \text{if} \quad v_1 = v_2 \in V_2, u_1u_2 \in E_1 \end{cases}$$

for any $(u_1, v_1), (u_2, v_2) \in V$. Then $G = (A, B)$ is called the *lexicographic min-product of* $G_1$ *and* $G_2$ and denoted by $G = G_1[G_2]^L_{min}$.

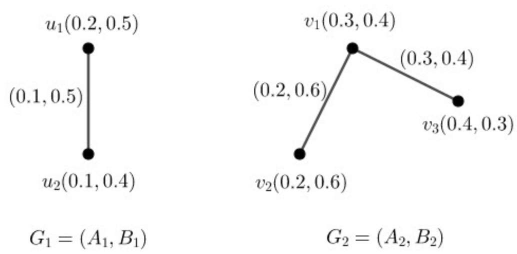


Figure 1: Vague graphs $G_1$ and $G_2$

EXAMPLE 2.2. Consider the vague graphs $G_1$ and $G_2$ in Figure 1. Then we can see that the lexicographic min-product of two vague graphs $G_1$ and $G_2$ in Figure 2, where the values of vertices and edges of the vague graph $G_1[G_2]^L_{min}$ and $G_2[G_1]^L_{min}$ are given by Tables 1 and 2.

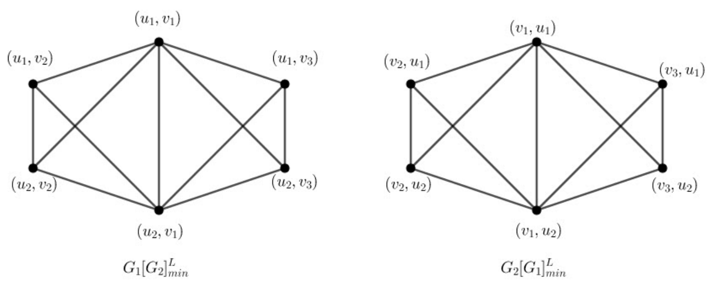


Figure 2: Vague graphs $G_1[G_2]^L_{min}$ and $G_2[G_1]^L_{min}$

| Vertices of $G_1[G_2]^L_{min}$ | $(t_A, f_A)$ |
|---|---|
| $(u_1, v_2)$ | $(0.2, 0.6)$ |
| $(u_1, v_1)$ | $(0.2, 0.5)$ |
| $(u_1, v_3)$ | $(0.2, 0.5)$ |
| $(u_2, v_2)$ | $(0.1, 0.6)$ |
| $(u_2, v_1)$ | $(0.1, 0.4)$ |
| $(u_2, v_3)$ | $(0.1, 0.4)$ |

| Vertices of $G_2[G_1]^L_{min}$ | $(t_A, f_A)$ |
|---|---|
| $(v_2, u_1)$ | $(0.2, 0.6)$ |
| $(v_1, u_1)$ | $(0.2, 0.5)$ |
| $(v_3, u_1)$ | $(0.2, 0.5)$ |
| $(v_2, u_2)$ | $(0.1, 0.6)$ |
| $(v_1, u_2)$ | $(0.1, 0.4)$ |
| $(v_3, u_2)$ | $(0.1, 0.4)$ |

Table 1: Value of vertices of vague graph $G_1[G_2]^L_{min}$ and $G_2[G_1]^L_{min}$

| Edges of $G_1[G_2]^L_{min}$ | $(t_B, f_B)$ |
|---|---|
| $(u_1, v_1)(u_1, v_2)$ | $(0.2, 0.6)$ |
| $(u_1, v_1)(u_2, v_2)$ | $(0.1, 0.6)$ |
| $(u_1, v_1)(u_2, v_1)$ | $(0.1, 0.5)$ |
| $(u_1, v_1)(u_2, v_3)$ | $(0.1, 0.5)$ |
| $(u_1, v_1)(u_1, v_3)$ | $(0.2, 0.5)$ |
| $(u_1, v_2)(u_2, v_2)$ | $(0.1, 0.6)$ |
| $(u_1, v_2)(u_2, v_1)$ | $(0.1, 0.6)$ |
| $(u_1, v_3)(u_2, v_1)$ | $(0.1, 0.5)$ |
| $(u_1, v_3)(u_2, v_3)$ | $(0.1, 0.5)$ |
| $(u_2, v_1)(u_2, v_2)$ | $(0.1, 0.6)$ |
| $(u_2, v_1)(u_2, v_3)$ | $(0.1, 0.4)$ |

| Edges of $G_2[G_1]^L_{min}$ | $(t_B, f_B)$ |
|---|---|
| $(v_1, u_1)(v_2, u_1)$ | $(0.2, 0.6)$ |
| $(v_1, u_1)(v_2, u_2)$ | $(0.1, 0.6)$ |
| $(v_1, u_1)(v_1, u_2)$ | $(0.1, 0.5)$ |
| $(v_1, u_1)(v_3, u_2)$ | $(0.1, 0.5)$ |
| $(v_1, u_1)(v_3, u_1)$ | $(0.2, 0.5)$ |
| $(v_2, u_1)(v_2, u_2)$ | $(0.1, 0.6)$ |
| $(v_2, u_1)(v_1, u_2)$ | $(0.1, 0.6)$ |
| $(v_3, u_1)(v_1, u_2)$ | $(0.1, 0.5)$ |
| $(v_3, u_1)(v_3, u_2)$ | $(0.1, 0.5)$ |
| $(v_1, u_2)(v_2, u_2)$ | $(0.1, 0.6)$ |
| $(v_1, u_2)(v_3, u_2)$ | $(0.1, 0.4)$ |

Table 2: Value of edges of vague graphs $G_1[G_2]^L_{min}$ and $G_2[G_1]^L_{min}$

Remark 2.3. The lexicographic min-product of two vague graphs is not commutative, in general. That is $G_1[G_2]^L_{min}$ is different from $G_2[G_1]^L_{min}$ (see Figure 2).

Proposition 2.4. *Lexicographic min-product of two vague graphs is a vague graph.*

*Proof.* Assume that $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are two vague graphs on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively and $G^* = (V, E)$ is as in Definition 2.1. Then for any $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$, we have;

$$\begin{aligned} t_B((u_1, v_1)(u_2, v_2)) &= t_{B_1}(u_1u_2) \wedge t_{B_2}(v_1v_2) \\ &\leq (t_{A_1}(u_1) \wedge t_{A_1}(u_2)) \wedge (t_{A_2}(v_1) \wedge t_{A_2}(v_2)) \\ &= (t_{A_1}(u_1) \wedge t_{A_2}(v_1)) \wedge (t_{A_1}(u_2) \wedge t_{A_2}(v_2)) \\ &= t_A(u_1, v_1) \wedge t_A(u_2, v_2) \end{aligned}$$

and
$$\begin{aligned} f_B((u_1, v_1)(u_2, v_2)) &= f_{B_1}(u_1u_2) \vee f_{B_2}(v_1v_2) \\ &\geq (f_{A_1}(u_1) \vee f_{A_1}(u_2)) \vee (f_{A_2}(v_1) \vee f_{A_2}(v_2)) \\ &= (f_{A_1}(u_1) \vee f_{A_2}(v_1)) \vee (f_{A_1}(u_2) \vee f_{A_2}(v_2)) \\ &= f_A(u_1, v_1) \vee f_A(u_2, v_2) \end{aligned}$$

In a similar way, for any $u_1 = u_2 \in V_1$ and $v_1v_2 \in E_2$ or $v_1 = v_2$ and $u_1u_2 \in E_1$, we have:

$$\begin{aligned} t_B((u_1, v_1)(u_2, v_2)) &\leq t_A(u_1, v_1) \wedge t_A(u_2, v_2) \\ f_B((u_1, v_1)(u_2, v_2)) &\geq f_A(u_1, v_1) \vee f_A(u_2, v_2). \end{aligned}$$

Therefore, lexicographic min-product $G_1[G_2]^L_{min}$ is a vague graph on $G^*$. In a similar way $G_2[G_1]^L_{min}$ is a vague graph on $G^*$, too. □

Theorem 2.5. *The lexicographic min-product of two strong vague graphs is a strong vague graph,too.*

*Proof.* Let $G = G_1[G_2]^L_{min}$ be a lexicographic min-product of two strong vague graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ . Then by Proposition 2.4, $G_1[G_2]^L_{min}$ is a vague graph on $G^*$. Now for any, $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$, we have

$$\begin{aligned} t_B((u_1, v_1)(u_2, v_2)) &= t_{B_1}(u_1u_2) \wedge t_{B_2}(v_1v_2) \\ &= (t_{A_1}(u_1) \wedge t_{A_1}(u_2)) \wedge (t_{A_2}(v_1) \wedge t_{A_2}(v_2)) \\ &= (t_{A_1}(u_1) \wedge t_{A_2}(v_1)) \wedge (t_{A_1}(u_2) \wedge t_{A_2}(v_2)) \\ &= t_A(u_1, v_1) \wedge t_A(u_2, v_2) \end{aligned}$$

In a similar way $f_B((u_1, v_1)(u_2, v_2)) = f_A(u_1, v_1) \vee f_A(u_2, v_2)$. Moreover, for any $u_1 = u_2 \in V_1$ and $v_1v_2 \in E_2$ or $v_1 = v_2$ and $u_1u_2 \in E_1$, the proof is similar to the proof of last case.

Hence for all edges in the lexicographic min-product $G_1[G_2]^L_{min}$, we have

$$\begin{aligned} t_B((u_1, v_1)(u_2, v_2)) &= t_A(u_1, v_1) \wedge t_A(u_2, v_2) \\ f_B((u_1, v_1)(u_2, v_2)) &= f_A(u_1, v_1) \vee f_A(u_2, v_2) \end{aligned}$$

Therefore, $G_1[G_2]^L_{min}$ is a strong vague graph on $G^*$. In a similar way $G_2[G_1]^L_{min}$ is a strong vague graph on $G^*$, too. □

It is easy to see that the lexicographic min-product of two complete vague graphs is a complete vague graph.

REMARK 2.6. If the lexicographic min-product $G = G_1[G_2]^L_{min}$ is strong, then $G_1$ or $G_2$ need not be strong vague graphs, in general.

EXAMPLE 2.7. Consider the vague graphs $G_1$, $G_2$ and $G_1[G_2]^L_{min}$ as in Figure 3.

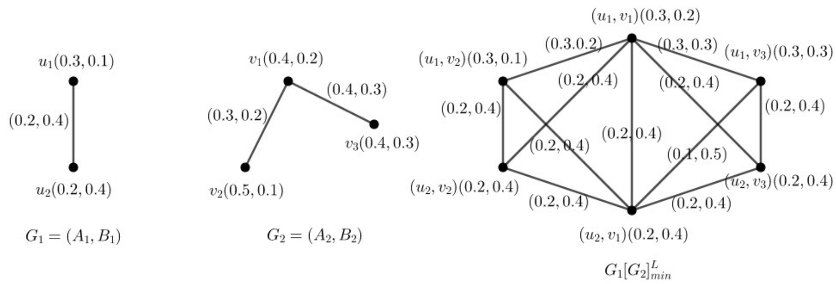


Figure 3: Vague graph $G_1$, $G_2$ and $G_1[G_2]^L_{min}$

Then $G_1$ and $G_1[G_2]^L_{min}$ are strong vague graphs, but $G_2$ is not a strong vague graph. Since $t_{B_2}(v_1, v_2) = 0.3$, but $t_{A_2}(v_1) \wedge t_{A_2}(v_2) = 0.4 \wedge 0.5 = 0.4$. Hence $t_{B_2}(v_1, v_2) \neq t_{A_2}(v_1) \wedge t_{A_2}(v_2)$.

THEOREM 2.8. *The lexicographic min-product $G_1[G_2]^L_{min}$ of two vague graphs $G_1$ and $G_2$ is a connected vague graph if and only if $G_1$ is connected.*

*Proof.* Let $G = G_1[G_2]^L_{min}$ be lexicographic min-product of two vague graphs $G_1$ and $G_2$. We know that $G_1[G_2]^L_{min}$ has $|V_2|$ copies of $G_1$, that is for each vertex in $G_2$ there is a copy of $G_1$ in $G_1[G_2]^L_{min}$. Now if $G_1$ is connected, then $G_1[G_2]^L_{min}$ is connected, too.

Conversely, assume that $G_1$ and $G_2$ be two vague graphs such that $G_1[G_2]^L_{min}$ is connected. Now suppose that $G_1$ is not connected, by contrary. Then there exist at least two different vertices $u_1, u_2 \in V_1$ such that there is no path between them. But since $G_1[G_2]^L_{min}$ is connected, for any two vertices of the form $(u_1, v_i)$ and $(u_2, v_j) \in V_1 \times V_2$ there is at least one path between them. This implies that there must be at least one path between the vertices $u_1$, $u_2$, which is impossible. Hence $G_1$ is connected. □

REMARK 2.9. Let $G_1$ and $G_2$ be two vague graphs. If $G_1$ is not connected and $G_2$ is connected, then $G_1[G_2]^L_{min}$ is not connected, in general.

THEOREM 2.10. *Let $G_1$ and $G_2$ be two strong vague graphs. Then the number of connected components in $G_1[G_2]^L_{min}$ is equal to the number of connected components in $G_1$.*

*Proof.* Let $G = G_1[G_2]^L_{min}$ be the lexicographic min-product of two strong vague graphs $G_1$ and $G_2$ and $G^* = (V, E)$ be as in Definition 2.1. We consider two cases:

**Case 1.** Let $G_1$ be connected. Then by Theorem 2.8, the lexicographic min-product $G_1[G_2]^L_{min}$ is connected. This implies that the number of connect component of both $G_1$ and $G_1[G_2]^L_{min}$ are equal.

**Case 2.** Suppose that $G_1$ is not connected. Then it has '$m$' disjoint connected components. Then we can rename the vertices of $G_1$ in such a way that $\{u_1, u_2, \ldots, u_{k1}\}$, $\{u_{k1+1}, u_{k1+2}, \ldots, u_{k2}\}$, ..., $\{u_{km+1}, u_{km+2}, \ldots, u_{km+n}\}$ are the vertex sets of the '$m$' disjoint connected components of $G_1$. If $\{v_1, v_2, \ldots, v_n\}$ is the vertex set of $G_2$, then for each vertex $v_i$ in $G_2$, there is a copy of each connected component of $G_1$ in $G_1[G_2]^L_{min}$. Hence there are no edges between these components. Because, if there is an edge between $u_1v_i$, $u_{k1+1}v_i$, then there must be an edge between $u_1$, $u_{k1+1}$ in $G_1$, which is a contradiction.

Thus, each connected component in $G_1[G_2]^L_{min}$ is disjoint with every other component and hence the theorem holds. $\square$

DEFINITION 2.11. Let $G = G_1[G_2]^L_{min}$ be the lexicographic min-product of two vague graphs $G_1$ and $G_2$ and $G^* = (V, E)$ be as in Definition 2.1. Then the degree of any vertex $(u, v) \in V = V_1 \times V_2$ in $G_1[G_2]^L_{min}$ is defined by $d_{G_1[G_2]^L_{min}}(u, v) = (s_1, s_2)$, where

$$s_1 = \sum_{\substack{uu_k \in E_1, \\ vv_l \in E_2}} t_{B_1}(uu_k) \wedge t_{B_2}(vv_l) + \sum_{\substack{u=u_k \in V_1, \\ vv_l \in E_2}} t_{A_1}(u) \wedge t_{B_2}(vv_l) + \sum_{\substack{v=v_l \in V_2, \\ uu_k \in E_1}} t_{A_2}(v) \wedge t_{B_1}(uu_k),$$

$$s_2 = \sum_{\substack{uu_k \in E_1, \\ vv_l \in E_2}} f_{B_1}(uu_k) \vee f_{B_2}(vv_l) + \sum_{\substack{u=u_k \in V_1, \\ vv_l \in E_2}} f_{A_1}(u) \vee f_{B_2}(vv_l) + \sum_{\substack{v=v_l \in V_2, \\ uu_k \in E_1}} f_{A_2}(v) \vee f_{B_1}(uu_k).$$

EXAMPLE 2.12. In Example 2.2, for $G_1[G_2]^L_{min}$ we have

$d_{G_1[G_2]^v_{min}}(u_1, v_1) = (0.2 + 0.1 + 0.1 + 0.1 + 0.2, 0.6 + 0.6 + 0.5 + 0.5 + 0.5) = (0.7, 2.7)$

$d_{G_1[G_2]^v_{min}}(u_1, v_2) = (0.2 + 0.1 + 0.1, 0.6 + 0.6 + 0.6) = (0.4, 1.8)$

$d_{G_1[G_2]^v_{min}}(u_2, v_1) = (0.1 + 0.1 + 0.1 + 0.1 + 0.1, 0.5 + 0.6 + 0.5 + 0.6 + 0.4) = (0.5, 2.5)$

$d_{G_1[G_2]^v_{min}}(u_2, v_2) = (0.1 + 0.1 + 0.1, 0.6 + 0.6 + 0.6) = (0.3, 1.8)$

$d_{G_1[G_2]^v_{min}}(u_1, v_3) = (0.2 + 0.1 + 0.1, 0.5 + 0.5 + 0.5) = (0.4, 1.5)$

$d_{G_1[G_2]^v_{min}}(u_2, v_3) = (0.1 + 0.1 + 0.1, 0.5 + 0.5 + 0.4) = (0.3, 1.4)$

DEFINITION 2.13. The lexicographic min-product $G = G_1[G_2]^L_{min}$ of two vague graphs $G_1$ and $G_2$, is called $(k_1, k_2)$*-regular* if $d_{G_1[G_2]^L_{min}}(u, v) = (k_1, k_2)$, for all $(u, v) \in V_1 \times V_2$. Moreover, $G_1[G_2]^L_{min}$ is called *regular vague graph of degree* $(k_1, k_2)$.

EXAMPLE 2.14. Consider two vague graphs $G_1$ and $G_2$ and lexicographic product vague graph $G_1[G_2]^L_{min}$ as in Figure 4.

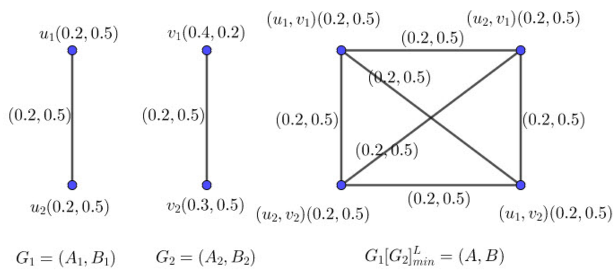


Figure 4: Vague graph $G_1$, $G_2$ and $G_1[G_2]^L_{min}$

Hence $G_1[G_2]^L_{min}$ is a regular vague graph of degree $(0.6, 1.5)$.

If $G_1$ and $G_2$ are two regular vague graphs, then the lexicographic min-product of $G_1$ and $G_2$ is not a regular vague graph, in general.

### 2.2 Lexicographic max-product

In this section, we define the notion of lexicographic max-product. Even though concepts like strongness, connectedness and regularity are similarly defined as lexicographic min-product of two vague graphs, there is not any relation between lexicographic min-product and lexicographic max-product, in general.

Definition 2.15. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two vague graphs on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Let $V = V_1 \times V_2$, $E = \{((u_1, v_1)(u_2, v_2)) \mid (u_1u_2 \in E_1, v_1v_2 \in E_2) \text{ or } (u_1 = u_2 \in V_1, v_1v_2 \in E_2) \text{ or } (v_1 = v_2 \in V_2, u_1u_2 \in E_1)\}$ and $G^* = (V, E)$. Now we define vague sets $A = (t_A, f_A)$ on $V$ and $B = (t_B, f_B)$ on $E$ by $t_A(u, v) = t_{A_1}(u) \vee t_{A_2}(v)$, $f_A(u, v) = f_{A_1}(u) \wedge f_{A_2}(v)$ for all $(u, v) \in V$ and

$$t_B((u_1, v_1)(u_2, v_2)) = \begin{cases} t_{B_1}(u_1u_2) \vee t_{B_2}(v_1v_2), & \text{if } u_1u_2 \in E_1, v_1v_2 \in E_2 \\ t_{A_1}(u_1) \vee t_{B_2}(v_1v_2), & \text{if } u_1 = u_2 \in V_1, v_1v_2 \in E_2 \\ t_{A_2}(v_1) \vee t_{B_1}(u_1u_2), & \text{if } v_1 = v_2 \in V_2, u_1u_2 \in E_1 \end{cases}$$

and $$f_B((u_1, v_1)(u_2, v_2)) = \begin{cases} f_{B_1}(u_1u_2) \wedge f_{B_2}(v_1v_2), & \text{if } u_1u_2 \in E_1, v_1, v_2 \in E_2 \\ f_{A_1}(u_1) \wedge f_{B_2}(v_1v_2), & \text{if } u_1 = u_2 \in V_1, v_1v_2 \in E_2 \\ f_{A_2}(v_1) \wedge f_{B_1}(u_1u_2), & \text{if } v_1 = v_2 \in V_2, u_1u_2 \in E_1 \end{cases}$$

for any $(u_1, v_1), (u_2, v_2) \in V$. Then $G = (A, B)$ is called the *lexicographic max-product of* $G_1$ *and* $G_2$ and denoted by $G = G_1[G_2]^L_{max}$.

Example 2.16. Consider the vague graphs $G_1$ and $G_2$ as in Figure 1. Then the lexicographic max-product of them, that is $G_1[G_2]^L_{max}$ and $G_2[G_1]^L_{max}$ are in Figure 5, where the values of vertices and edges of the vague graph $G_1[G_2]^L_{max}$ are given by Tables 3 and 4.

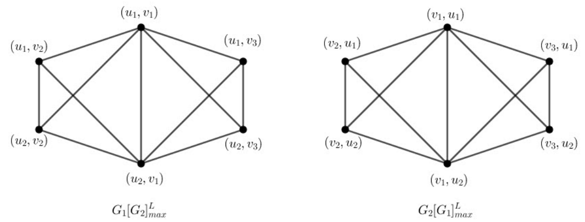


Figure 5: Vague graphs $G_1[G_2]^L_{max}$ and $G_2[G_1]^L_{max}$

| Vertices of $G_1[G_2]^L_{max}$ | $(t_A, f_A)$ |
|---|---|
| $(u_1, v_2)$ | $(0.2, 0.5)$ |
| $(u_1, v_1)$ | $(0.3, 0.4)$ |
| $(u_1, v_3)$ | $(0.4, 0.3)$ |
| $(u_2, v_2)$ | $(0.2, 0.4)$ |
| $(u_2, v_1)$ | $(0.3, 0.4)$ |
| $(u_2, v_3)$ | $(0.4, 0.3)$ |

| Vertices of $G_2[G_1]^L_{max}$ | $(t_A, f_A)$ |
|---|---|
| $(v_2, u_1)$ | $(0.2, 0.5)$ |
| $(v_1, u_1)$ | $(0.3, 0.4)$ |
| $(v_3, u_1)$ | $(0.4, 0.3)$ |
| $(v_2, u_2)$ | $(0.2, 0.4)$ |
| $(v_1, u_2)$ | $(0.3, 0.4)$ |
| $(v_3, u_2)$ | $(0.4, 0.3)$ |

Table 3: Value of vertices of vague graph $G_1[G_2]^L_{max}$ and $G_2[G_1]^L_{max}$

| Edges of $G_1[G_2]^L_{max}$ | $(t_B, f_B)$ |
|---|---|
| $(u_1, v_1)(u_1, v_2)$ | $(0.2, 0.5)$ |
| $(u_1, v_1)(u_2, v_2)$ | $(0.2, 0.5)$ |
| $(u_1, v_1)(u_2, v_1)$ | $(0.3, 0.4)$ |
| $(u_1, v_1)(u_2, v_3)$ | $(0.3, 0.4)$ |
| $(u_1, v_1)(u_1, v_3)$ | $(0.3, 0.4)$ |
| $(u_1, v_2)(u_2, v_2)$ | $(0.2, 0.5)$ |
| $(u_1, v_2)(u_2, v_1)$ | $(0.2, 0.5)$ |
| $(u_1, v_3)(u_2, v_1)$ | $(0.3, 0.4)$ |
| $(u_1, v_3)(u_2, v_3)$ | $(0.4, 0.3)$ |
| $(u_2, v_1)(u_2, v_2)$ | $(0.2, 0.4)$ |
| $(u_2, v_1)(u_2, v_3)$ | $(0.3, 0.4)$ |

| Edges of $G_2[G_1]^L_{max}$ | $(t_B, f_B)$ |
|---|---|
| $(v_1, u_1)(v_2, u_1)$ | $(0.2, 0.5)$ |
| $(v_1, u_1)(v_2, u_2)$ | $(0.2, 0.5)$ |
| $(v_1, u_1)(v_1, u_2)$ | $(0.3, 0.4)$ |
| $(v_1, u_1)(v_3, u_2)$ | $(0.3, 0.4)$ |
| $(v_1, u_1)(v_3, u_1)$ | $(0.3, 0.4)$ |
| $(v_2, u_1)(v_2, u_2)$ | $(0.2, 0.5)$ |
| $(v_2, u_1)(v_1, u_2)$ | $(0.2, 0.5)$ |
| $(v_3, u_1)(v_1, u_2)$ | $(0.3, 0.4)$ |
| $(v_3, u_1)(v_3, u_2)$ | $(0.4, 0.3)$ |
| $(v_1, u_2)(v_2, u_2)$ | $(0.2, 0.4)$ |
| $(v_1, u_2)(v_3, u_2)$ | $(0.3, 0.4)$ |

Table 4: Value of edges of vague graphs $G_1[G_2]^L_{max}$ and $G_2[G_1]^L_{max}$

Remark 2.17. The lexicographic max-product of two vague graphs is not commutative. That is $G_1[G_2]^L_{max}$ is different from $G_2[G_1]^L_{max}$, in general (see Figure 5.)

Similar to the proof of Proposition 2.4 and Theorems 2.8, 2.10, we prove the following theorem.

Theorem 2.18. *(i) The lexicographic max-product of two vague graphs $G_1$ and $G_2$ is a vague graph, too.*

*(ii) The lexicographic max-product $G_1[G_2]^L_{max}$ is a connected vague graph if and only if vague graph $G_1$ is connected.*

*(iii) The number of connected components in the lexicographic max-product $G_1[G_2]^L_{max}$ is equal to the number of connected components in vague graph $G_1$.*

REMARK 2.19. The lexicographic max-product of two strong (complete) vague graphs need not be strong (complete) vague graph, in general.

EXAMPLE 2.20. In the Example 2.16, $G_1$ and $G_2$ are strong (complete) vague graphs but neither $G_1[G_2]^L_{max}$ nor $G_2[G_1]^L_{max}$ is a strong (complete) vague graph.

DEFINITION 2.21. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two vague graphs on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then the degree of vertex $(u, v)$ in $G_1[G_2]^L_{max}$ is defined by $d_{G_1[G_2]^L_{max}}(u, v) = (s_1, s_2)$, where

$$s_1 = \sum_{\substack{uu_k\in E_1,\\ vv_l\in E_2}} t_{B_1}(uu_k) \vee t_{B_2}(vv_l) + \sum_{\substack{u=u_k\in V_1,\\ vv_l\in E_2}} t_{A_1}(u) \vee t_{B_2}(vv_l) + \sum_{\substack{v=v_l\in V_2,\\ uu_k\in E_1}} t_{A_2}(v) \vee t_{B_1}(uu_k),$$

$$s_2 = \sum_{\substack{uu_k\in E_1^*,\\ vv_l\in E_2^*}} f_{B_1}(uu_k) \wedge f_{B_2}(vv_l) + \sum_{\substack{u=u_k\in V_1^*,\\ vv_l\in E_2^*}} f_{A_1}(u) \wedge f_{B_2}(vv_l) + \sum_{\substack{v=v_l\in V_2^*,\\ uu_k\in E_1^*}} f_{A_2}(v) \wedge f_{B_1}(uu_k).$$

EXAMPLE 2.22. In Example 2.16, for $G_1[G_2]^v_{max}$ we have

$d_{G_1[G_2]^v_{max}}(u_1, v_1) = (0.2 + 0.2 + 0.3 + 0.3 + 0.3, 0.5 + 0.5 + 0.4 + 0.4 + 0.4) = (1.3, 2.2)$

$d_{G_1[G_2]^v_{max}}(u_1, v_2) = (0.2 + 0.2 + 0.2, 0.5 + 0.5 + 0.5) = (0.6, 1.5)$

$d_{G_1[G_2]^v_{max}}(u_2, v_1) = (0.3 + 0.2 + 0.3 + 0.2 + 0.3, 0.4 + 0.5 + 0.4 + 0.4 + 0.4) = (1.3, 2.1)$

$d_{G_1[G_2]^v_{max}}(u_2, v_2) = (0.2 + 0.2 + 0.2, 0.5 + 0.5 + 0.4) = (0.6, 1.4)$

$d_{G_1[G_2]^v_{max}}(u_1, v_3) = (0.3 + 0.3 + 0.4, 0.4 + 0.4 + 0.3) = (1, 1.1)$

$d_{G_1[G_2]^v_{max}}(u_2, v_3) = (0.3 + 0.4 + 0.3, 0.4 + 0.3 + 0.4) = (1, 1.1)$

DEFINITION 2.23. The lexicographic max-product of two vague graphs $G_1$ and $G_2$, that is $G = G_1[G_2]^L_{max}$, is called $(k_1, k_2)$-*regular* if $d_{G_1[G_2]^L_{max}}(u, v) = (k_1, k_2)$, for all $(u, v) \in V_1 \times V_2$. Moreover, $G_1[G_2]^L_{max}$ is called *regular vague graph of degree* $(k_1, k_2)$.

If $G_1$ and $G_2$ are two regular vague graphs, then the lexicographic max-product of $G_1$ and $G_2$ is not a regular vague graph, in general.

Now, in what follows we get the relationship between lexicographic min-products and max-products.

DEFINITION 2.24. If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are two vague graphs on $G_1^* = G_2^* = (V, E)$. Then we say that $G_1$ is a spanning vague subgraph of $G_2$ if $t_{A_1}(u) \leqslant t_{A_2}(u)$ and $f_{A_1}(u) \geqslant f_{A_2}(u)$ for all $u \in V$ and $t_{B_1}(uv) \leqslant t_{B_2}(uv)$ and $f_{B_1}(uv) \geqslant f_{B_2}(uv)$ for all $uv \in E$.

THEOREM 2.25. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two vague graphs with the same underlying crisp graphs. Then $G_1[G_2]^L_{min}$ is a spanning vague graph of $G_1[G_2]^L_{max}$.*

*Proof.* Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two vague graphs on $G_1^* = G_2^* = (V, E)$. Then $G_1[G_2]^L_{max} = (A, B)$ and $G_1[G_2]^L_{min} = (A', B')$ are two vague graphs with the same underlying crisp graph $G^* = (V, E)$ where $V = V \times V$ and $E = \{(u_1, v_1)(u_2, v_2) \mid (u_1u_2 \in E, v_1v_2 \in E)$ or $(u_1 = u_2 \in V, v_1v_2 \in E)$ or $(v_1 = v_2 \in V$ and $u_1u_2 \in E)\}$. From the definition, it is clear that $t_{A'}(u_1, u_2) \leqslant t_A(u_1, u_2)$ and $f_{A'}(u_1, u_2) \geqslant f_A(u_1, u_2)$ for all $(u_1, u_2) \in V$ and $t_{B'}((u_1u_2)(v_1v_2)) \leqslant t_B((u_1u_2)(v_1v_2))$ and $f_{B'}((u_1u_2)(v_1v_2)) \geqslant f_B((u_1, v_1)(u_2, v_2))$ for all $((u_1, v_1)(u_2, v_2)) \in E$. Hence the lexicographic min-product vague graph is a spanning vague subgraph of the lexicographic min-product vague graph. □

We have defined the concepts of lexicographic min-product and lexicographic max-product and we will compare them in the next section.

## 3. Application

In this section, applications of lexicographic (min) max-product of two vague graphs are demonstrated and accordingly these two products on vague graphs are compared in order to optimize the project organization. The project includes a set of complex and unique operations, which consist of logical and related activities which are executed under the supervision of a specific management and organization in order to achieve the specified objectives within the framework of a predetermined time schedule and budget. The duration of a project in the public sector is often more than one year and less than three years. However, the duration of the project in the private sector is shorter than this. Of course, the feature of the duration of the implementation is not accurate and commonly used to judge the appropriateness of the title or project assignment to the set of actions and operations. An employer is a device which executes on behalf of the executive in order to contract with the consultant and the contractor and pursues all stages of the execution until completing the task. The employer pledges to pay the contractor a prepayment to strengthen the contractors financial resources. Further, the contractor assesses the status of the work performed from the beginning of the work to the date measured according to the execution plans and the form of the meetings at the end of each month, and calculates the amount of the status item according to the regular price list and submits it to the consultant. Then, the consultant will send it to the employer after the evaluation and approval, and accordingly the employer will pay the contractor after deducting the legal fees and the amount of previous interim payments. This payment path remains unchanged until the end of the project. A contractor is a true or legal person to the other party signing the contract and undertakes the implementation of the treaty subject to the treaty documents. The contractor is one of the main pillars for creating or implementing a project. One of the major tasks of any contractor is the implementation of the project based on the approved timetable. In fact, the contractor will undertake to provide a detailed program for implementing the work according to the consultant engineer, who prepares a general plan and submits it to the consultant to notify the contractor after notifying and approving the employer based on the existing plans. After communicating the timetable, the contractor undertakes to complete the project based on the timetable and, submit a report on the progress of the project to the consultant at the end of each month. The report should include the amount and percentage of the performed activities, the amount of progress or delay in relation to the detailed timetable, the problems and barriers on the way to the execution of the project, and other relevant information. The legal personality of the signatory is regarded as another party to the contract with the employer, whose basic duties are as follows:

- Preliminary studies.
- Preparation of plans and executive plans.
- Tender and supervision.

Supervising the implementation of the work is considered as a key task for the consultant. To this end, it should be done all the time by his employees who are regularly visiting all of the work and monitor the compliance of the contractor's operations with regular specifications to the contract. In addition, they confirm that the work has been performed according to specifications, and a request for the consideration of the status of the contractors contract is processed and certified after evaluating the work done and maintaining the booklet. Project management is a process which plans and directs the project life cycle through the most convenient way, along with the best results in achieving the objectives of the project. The project management process consists of three planning and implementation tasks. After planning the beginning and end of implementing the project, we should always evaluate and control the performance and compare actual performances with the predicted program. In this regard, the following factors should be taken into consideration:

- Which activities have already been done?
- How much is the physical progress?
- How much is the kernel's backend?
- How much delay have been created in the project? How much is the deviation from the program ?
- How much is the actual cost have been spent on the project? Does the project have any profit?

The supervisor of a real personal workshop with expertise and experience is the person whose con-tractor will nominate to the consultant engineer to supervise the execution of the contract in the workshop. The head of the workshop is directly or indirectly responsible for some tasks such as skilled manpower management, provision of equipment for letterheads, payments, preparation of situation form, reporting work accidents to the management, participation in meetings, and the like. However, doing something is regarded as the most important task of the supervisor in the decision-making workshop, when dealing with inevitable drawbacks. In order to obtain a project, the contractor is acting in various ways, the most important of which is participating in tenders by calling the executive agencies. After preparing the plans and specifications of the project, which is usually provided by the consulting engineers, the contractors selection process is as follows:

- Duplicate bidding.
- Inviting the contractors to bid.
- Selling the offer price and guaranteeing the tender by the contractor.
- Choosing a contractor with the best bid.
- Contract with the contractor.
- Delivering goodwill to fulfill the obligations by the contractor.

After making the initial decision for implementing a project, the way the project should be implemented with regard to the selection of the consultant should be designed and evaluated. Different approaches are adopted in choosing a consultant, depending on the type of project and the provincial or nationality. Since the advisor actually acts as the employers arm in the supervisory dimension as his principal agent, the employers have more options in selecting the consultant. Accordingly, the following steps should be taken:

- Inquiring from the Management and Planning Organization and obtaining the names of eligible consultants.
- Setting Tender Documents and Targeting Objectives for Negotiates and Investigators.
- Receiving offers and completed documents from applicants.
- Meeting with the committee members of the tender.
- Selecting consultant.
- Contracting.

Let $G_1 = (V_1, E_1)$ be Project factors, where $V_1 = \{\text{Employer}(E), \text{Contractor}(CTT), \text{Consultant}(CST)\}$ so that the Employer is responsible for the financial commitment and support of the project, the Contractor is re-responsible for the quality and timing of the project and Consultant should monitor the project carefully as illustrated in Figure 6. Graph $G_2 = (V_2, E_2)$ is the Contractor's key operating factors, where $V_2 = \{\text{Project manager}(PM), \text{Site manager}(SM)\}$ so that the Project manager is responsible for monitoring and controlling the project path and the Site manager plays the role of moderator for the project plans and contractor (Figure 6). Further, $E_1$ and $E_2$ are regarded as their relations.

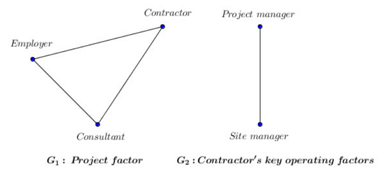


Figure 6: Vague graphs $G_1$ and $G_2$

| **Name** | **Financial commitments and support** |
|---|---|
| $t_{A_1}$ | Financial commitments and support on time |
| $f_{A_1}$ | Financial commitments and irregular support |
| **Name** | **Quality and timing** |
| $t_{A_1}$ | Quality and timing appropriate |
| $f_{A_1}$ | Quality and timing inappropriate |
| **Name** | **Careful monitoring** |
| $t_{A_1}$ | Regular and monthly monitoring |
| $f_{A_1}$ | Irregular and unplanned monitoring |

Table 5: Abbreviation of $t_{A_1}$ and $f_{A_1}$ for vertices of vague graph $G_1$

| **Name** | **Relations between them** |
|---|---|
| $t_{B_1}$ | The relations between precise and principled |
| $f_{B_1}$ | Relationships between irregular and non-standardized |

Table 6: Abbreviation of $t_{B_1}$ and $f_{B_1}$ for edges of vague graph $G_1$

| **Name** | **Monitoring and controlling the project path** |
|---|---|
| $t_{A_2}$ | Precise monitoring and control over project implementation |
| $f_{A_2}$ | Irregular monitoring and control over project implementation |
| **Name** | **Moderator of project plans and contractor** |
| $t_{A_2}$ | Exact execution according to the plan and standards in the subject |
| $f_{A_2}$ | Run irregular and out of the standard and framework |

Table 7: Abbreviation of $t_{A_2}$ and $f_{A_2}$ for vertices of vague graph $G_2$

| **Name** | **Relations between them** |
|---|---|
| $t_{B_2}$ | The relations between precise and principled |
| $f_{B_2}$ | Relationships between irregular and non-standardized |

Table 8: Abbreviation of $t_{B_2}$ and $f_{B_2}$ for edges of vague graph $G_2$

| | **E** | **CTT** | **CST** |
|---|---|---|---|
| $t_{A_1}$ | 1 | 0.3 | 0.45 |
| $f_{A_1}$ | 0 | 0.52 | 0.49 |

| | **(E,CTT)** | **(CTT,CST)** | **(E,CST)** |
|---|---|---|---|
| $t_{B_1}$ | 0.82 | 0.93 | 0.62 |
| $f_{B_1}$ | 0.03 | 0.04 | 0.31 |

| | **PM** | **SM** |
|---|---|---|
| $t_{A_2}$ | 0.72 | 0.67 |
| $f_{A_2}$ | 0.21 | 0.28 |

| | **(PM,SM)** |
|---|---|
| $t_{B_2}$ | 0.89 |
| $f_{B_2}$ | 0.08 |

Table 9: Value of vertices and edges of vague graphs $G_1$ and $G_2$

Now we consider the lexicographic (min) max-product of vague graph $G_1$ with $G_2$.

| min | **(CTT,PM)** | **(CST,PM)** | **(E,PM)** | **(CTT,SM)** | **(E,SM)** | **(CST,SM)** |
|---|---|---|---|---|---|---|
| $(t_A, f_A)$ | $(0.3, 0.52)$ | $(0.45, 0.49)$ | $(0.72, 0.21)$ | $(0.3, 0.52)$ | $(0.67, 0.28)$ | $(0.45, 0.49)$ |
| max | **(CTT,PM)** | **(CST,PM)** | **(E,PM)** | **(CTT,SM)** | **(E,SM)** | **(CST,SM)** |
| $(t_A, f_A)$ | $(0.72, 0.21)$ | $(0.72, 0.21)$ | $(1, 0)$ | $(0.67, 0.28)$ | $(1, 0)$ | $(0.67, 0.28)$ |

Table 10: Degree of all vertices in $G_1[G_2]^v_{min}$ and $G_1[G_2]^v_{max}$

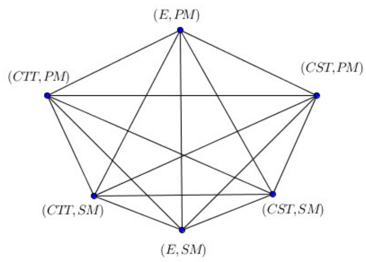


Figure 7: Vague graph $G_1[G_2]^v_{min}$ and $G_1[G_2]^v_{max}$

| Number | | $G_1[G_2]^v_{min}$ | $G_1[G_2]^v_{max}$ |
|---|---|---|---|
| 1 | $(t_B, f_B)((CTT, PM), (CTT, SM))$ | $(0.3, 0.52)$ | $(0.89, 0.08)$ |
| 2 | $(t_B, f_B)((CTT, PM), (E, SM))$ | $(0.82, 0.08)$ | $(0.89, 0.03)$ |
| 3 | $(t_B, f_B)((CTT, PM), (CTT, SM))$ | $(0.89, 0.08)$ | $(0.93, 0.04)$ |
| 4 | $(t_B, f_B)((CTT, PM), (E, PM))$ | $(0.72, 0.21)$ | $(0.82, 0.03)$ |
| 5 | $(t_B, f_B)((CTT, PM), (CST, PM))$ | $(0.72, 0.21)$ | $(0.93, 0.04)$ |
| 6 | $(t_B, f_B)((CTT, SM), (CST, PM))$ | $(0.89, 0.08)$ | $(0.93, 0.04)$ |
| 7 | $(t_B, f_B)((CTT, SM), (E, PM))$ | $(0.82, 0.08)$ | $(0.82, 0.03)$ |
| 8 | $(t_B, f_B)((CTT, SM), (CST, SM))$ | $(0.67, 0.28)$ | $(0.93, 0.04)$ |
| 9 | $(t_B, f_B)((CTT, SM), (E, SM))$ | $(0.67, 0.28)$ | $(0.82, 0.03)$ |
| 10 | $(t_B, f_B)((E, SM), (CST, SM))$ | $(0.62, 0.31)$ | $(0.67, 0.28)$ |
| 11 | $(t_B, f_B)((E, SM), (CST, PM))$ | $(0.62, 0.31)$ | $(0.89, 0.08)$ |
| 12 | $(t_B, f_B)((E, SM), (E, PM))$ | $(0.89, 0.08)$ | $(1, 0)$ |
| 13 | $(t_B, f_B)((CST, SM), (E, PM))$ | $(0.62, 0.31)$ | $(0.89, 0.08)$ |
| 14 | $(t_B, f_B)((CST, SM), (CST, PM))$ | $(0.45, 0.49)$ | $(0.89, 0.08)$ |
| 15 | $(t_B, f_B)((CST, PM), (E, PM))$ | $(0.62, 0.31)$ | $(0.72, 0.21)$ |

Table 11: Degree of all edges in $G_1[G_2]^v_{min}$ and $G_1[G_2]^v_{max}$

Therefore, as shown in Table 10, the cooperation of Employer and Project manager at lexicographic min-product is considered as the best condition. In addition, it is in the best condition for the lexicographic max-product. Therefore, the cooperation of capital between Employer and Project manager is the best way for both operators. Further, the relation between Employer and Project manager, as well as between the Employer and Site manager is in the most ideal state of the project for lexicographic (min) max-product of two vague graphs $G_1$ and $G_2$ as indicated in Table 11. Thus, the cooperation of Employer and Project manager is regarded as the most important part of the project and relationship between Employer and Project manager with the Employer and Site manager is the most important link to make the project work as much as possible.

## References


**[1]** M. Akram, N.Ganji, A. Borumand Saeid, *Vague hyper graphs*, Journal of International and Fuzzy Systems, **26** (2014), 647–653.

**[2]** R. A. Borzooei, H. Rashmanlou, S. Samanta, Madhumangal Pal, *Regularity of vague graphs*, Journal of Intelligent and Fuzzy Systems, **30** (2016), 3681–3689.

**[3]** W. L. Gau, D. J. Buehrer, *Vague sets*, IEEE Transactions on 501 Systems Man and Cybernetics, **23(2)** (1993), 610–614.

**[4]** A. Kauffman, *Introduction a la theories des sous-emsembles 503 flous*, Masson et Cie 1 (1973).

**[5]** K. Radha, S. Arumugam, *Research article on maximal product of two fuzzy graphs*, International Journal of Current Research, **7(1)** (2015), 11508–11515.

**[6]** K. Radha, S. Arumugam *On lexicographic produt of two fuzzy graphs*, International Journal of Fuzzy Mathematical Archive, **7(2)** (2015), 169–176.

**[7]** N. Ramakrishna, *Vague graphs*, International Journal of Computational Cognition, **7** (2009), 51–58. Mathematical Archive, **7(2)** (2015), 169–176.

**[8]** H. Rashmanlou, R.A. Borzooei, *Vague graphs with application*, Journal of Intelligent and Fuzzy Systems, **30** (2016), 3291–3299.

**[9]** H. Rashmanlou, R.A. Borzooei, *Product vague graphs and its applications*, Journal of Intelligent and Fuzzy Systems, **30** (2016), 371–382.

**[10]** A. Rosenfeld, *Fuzzy graphs, Fuzzy Sets and their Applications 513 (L. A. Zadeh, K. S. Fu, and M. Shimura. Eds.)*, Academic Press, New York, 1975, pp. 77–95.

**[11]** M. Suzuki, *Group theory I*, Springer-Verlag, New York, 1982.

**[12]** L. A. Zadeh, *Fuzzy sets*, Information and control, **8** (1965), 338–353.

**[13]** L. A. Zadeh, *Similarity relations and fuzzy ordering*, Information Sciences, **3** (1971), 177–200.

**[14]** L. A. Zadeh, *Is there a need for fuzzy logical*, Information Sciences, **178** (2008), 2751–2779.


(received 27.07.2018; in revised form 05.04.2019; available online 26.07.2019)


Department of Mathematics, Shahid Beheshti University, Tehran, Iran

*E-mail*: borzooei@sbu.ac.ir

Department of Mathematics, Shahid Beheshti University, Tehran, Iran

*E-mail*: b_sheikhhoseini@sbu.ac.ir

Department of Mathematics, Shahid Beheshti University, Tehran, Iran

Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

*E-mail*: skywine@gmail.com