MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 16.12.2018

research paper
оригинални научни рад

A NEW APPROACH TO SOME FIXED POINT THEOREMS FOR MULTIVALUED NONLINEAR F-CONTRACTIVE MAPS

Rahmatollah Lashkaripour, Hamid Baghani and Zahra Ahmadi

Abstract. In this article, by introducing a new operator, we give a new generalized contraction condition for multivalued maps. Moreover, without assumption of lower semi-continuity, we prove some fixed point theorems in incomplete metric spaces. Our results are extension of the corresponding results of I. Altun et al. (Nonlinear Analysis: Modeling and control, 2016, Vol. 21, No. 2, 201–210). Also, we provide some examples to show that our main theorem is a generalization of some previous results.

1. Introduction

Throughout this paper $C(X)$ indicates the family of all nonempty, closed subsets of X, $CB(X)$ indicates the family of all nonempty, closed and bounded subsets of X, and $K(X)$ indicates the family of all compact subsets of X. Also, the Hausdorff metric H on $C(X)$ is defined by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\},$$

for $A, B \in C(X)$ and $d(x,A) = \inf\{d(x,y) : y \in A\}$. In recent decades, several fixed point theorems for multivalued mappings were obtained by a large number of researchers [2,5,6,8–10,13]. Among them, Feng and Liu [8] proved the following one.

THEOREM 1.1. *Let (X,d) be a complete metric space and $T : X \to C(X)$. Assume that the following conditions hold:*
(i) the map $x \mapsto d(x,Tx)$ is lower semi-continuous;
(ii) there exist $b, c \in (0,1)$ with $b < c$ such that for any $x \in X$, there is $y \in I_b^x$ satisfying $d(y,Ty) \le cd(x,y)$, where $I_b^x = \{y \in Tx : bd(x,y) \le d(x,Tx)\}$.
Then T has a fixed point.

2010 Mathematics Subject Classification: 54H25, 47H10
Keywords and phrases: Fixed point; Hausdorff metric; multivalued contraction; *SO*-complete metric space; *SO*-lower semi-continuous.

Also, Klim and Wardowski [10] obtained a new result by proving the following theorem.

THEOREM 1.2. *Let (X, d) be a complete metric space and $T : X \to C(X)$. Assume that the following conditions hold:*
(i) the map $x \mapsto d(x, Tx)$ is lower semi-continuous;

(ii) there exist $b \in (0, 1)$ and $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\limsup_{t \to s^+} \varphi(t) < b$ for $s \geq 0$, and for any $x \in X$, there is $y \in I_b^x$ satisfying $d(y, Ty) \leq \varphi(d(x, y))d(x, y)$.
Then T has a fixed point.

In recent years, Minak et al. [11], using the concept of F-contraction, introduced by Wardowski [15] for the first time, proved the following fixed point theorems.

DEFINITION 1.3 ([15]). Let $F : (0, +\infty) \to \mathbb{R}$ be a function such that
(F_1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

(F_2) For all sets $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F_3) There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$;

(F_4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.
We denote the family of all $F : (0, +\infty) \to \mathbb{R}$ which satisfy conditions (F_1)–(F_3) and (F_1)–(F_4) by $\mathcal{F}$ and $\mathcal{F}_*$, respectively.

THEOREM 1.4 ([11]). *Let (X, d) be a complete metric space, $T : X \to K(X)$ and $F \in \mathcal{F}$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying $\tau + F(d(y, Ty)) \leq F(d(x, y))$, where $F_\sigma^x = \{y \in Tx : F(d(x, y)) \leq F(d(x, Tx)) + \sigma\}$ and $\sigma < \tau$, then T has a fixed point in X provided $x \mapsto d(x, Tx)$ is lower semi-continuous.*

REMARK 1.5 ([11]). If F satisfies (F_1) then it satisfies (F_4) if and only if it is right continuous.

If $T : X \to K(X)$, then for all $\sigma \geq 0$ and $x \in X$ with $d(x, Tx) > 0$, we have $F_\sigma^x \neq \emptyset$.

If $T : X \to C(X)$, then F_σ^x may be empty for some $x \in X$ and $\sigma \geq 0$. But if $F \in \mathcal{F}_*$, then for all $\sigma \geq 0$ and $x \in X$ with $d(x, Tx) > 0$, we have $F_\sigma^x \neq \emptyset$.

Obviously, the following theorem is an extension of Theorem 1.1.

THEOREM 1.6 ([11]). *Let $(X.d)$ be a complete metric space, $T : X \to C(X)$ and $F \in \mathcal{F}_*$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying $\tau + F(d(y, Ty)) \leq F(d(x, y))$, then T has a fixed point in X provided $\sigma < \tau$ and $x \mapsto d(x, Tx)$ is lower semi-continuous.*

Recently, Altun et al. [3] introduced a new class of multivalued maps and by applying them, they obtained an extension of Theorem 1.2.

THEOREM 1.7 ([3]). *Let (X, d) be a complete metric space, $T : X \to C(X)$ and $F \in \mathcal{F}_*$. Assume that the following conditions hold:*

(i) the map $x \to d(x, Tx)$ is lower semi-continuous;

(ii) there exists $\sigma > 0$ and a function $\varphi : (0, \infty) \to (\sigma, \infty)$ such that $\liminf_{t \to s^+} \tau(t) > \sigma$ for $s \geq 0$, and for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying $\tau(d(x, y)) + F(d(y, Ty)) \leq F(d(x, y))$.

Then T has a fixed point.

THEOREM 1.8 ([3]). *Let (X, d) be a complete metric space, $T : X \to K(X)$ and $F \in \mathcal{F}$. Assume that the conditions (i) and (ii) of Theorem 1.7 hold. Then, T has a fixed point.*

In this paper, we address the following questions.

(Q_1) Is it possible to generalize the contraction (ii) of Theorems 1.7 and 1.8?

(Q_2) Is it possible to remove the completeness of the space in Theorems 1.7 and 1.8?

(Q_3) Is it possible to remove the lower semi-continuity condition of the mapping $x \mapsto d(x, Tx)$ in Theorems 1.7 and 1.8?

To answer the above questions, we first introduce a function $\psi(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}_+$ with several properties as follows and use it to improve and generalize the condition *(ii)* of Theorems 1.7 and 1.8 in the next section.

DEFINITION 1.9. Let Δ be the class of those functions $\psi : \mathbb{R}_+^5 \to \mathbb{R}_+$ satisfying the following:

(Δ_1) ψ is increasing in t_2, t_3, t_4 and t_5;

(Δ_2) For all positive sequences $\{t_n\}$, $t_{n+1} < \psi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0)$ implies that $t_{n+1} < t_n$;

(Δ_3) $\phi(u, u, u, 2u, 0) \leq u$, for each $u \in \mathbb{R}^+ = [0, +\infty)$.

EXAMPLE 1.10. Let $\psi : \mathbb{R}_+^5 \to \mathbb{R}_+$ be defined by $\psi(t_1, t_2, t_3, t_4, t_5) = \dfrac{t_2^2 + 3t_3}{t_1 + 3}$. Then, it is clear that (Δ_1) holds. To show (Δ_2), let $\{t_n\}$ be a positive sequence such that

$$t_{n+1} < \psi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0) = \frac{t_n^2 + 3t_{n+1}}{t_n + 3}.$$

Therefore, $(t_{n+1}t_n) + 3t_{n+1} < t_n^2 + 3t_{n+1}$. This implies that $t_{n+1} < t_n$. It is obvious that the property (Δ_3) holds for this function, and so $\psi \in \Delta$.

In the sequel, we show that Theorems 1.7 and 1.8 hold whenever X is an incomplete metric space. For this purpose, we apply the notation of orthogonal set (see [1]). Indeed, we introduce the concept of *SO*-lower semi-continuous mapping and prove that lower semi-continuity assumptions of the mappings $x \to d(x, Tx)$ is not necessary. At first, we recall some important definitions.

DEFINITION 1.11 ([7]). Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If "$\perp$" satisfies the following condition: $\exists x_0$*:* $(\forall y, y \perp x_0)$ or $\forall y, x_0 \perp y)$, then "$\perp$" is called a strong orthogonality relation and the pair $(X, \perp)$ is called an orthogonal set (briefly *SO*-set).

Note that in the above definition, we say that x_0 is an orthogonal element. Also, we say that elements $x, y \in X$ are $\perp$-comparable if either $x \perp y$ or $y \perp x$.

Definition 1.12 ([14]). 1. Let $(X, \perp)$ be an SO-set. A sequence $\{x_n\}$ is called a strongly orthogonal sequence (briefly, SO-sequence) if $(\forall n, k;\ x_n \perp x_{n+k})$ or $(\forall n, k;\ x_{n+k} \perp x_n)$.

2. Let $(X, \perp, d)$ be an orthogonal metric space. X is said to be strongly orthogonal complete (briefly, SO-complete) if every Cauchy SO-sequence is convergent.

3. Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $f : X \to X$ is strongly orthogonal continuous (briefly, SO-continuous) in $x \in X$ if for each SO-sequence $\{x_n\}$ in X if $x_n \to x$, then $f(x_n) \to f(x)$. Also, f is SO-continuous on X if f is SO-continuous in each $x \in X$.

Definition 1.13 ([4]). 1. Let $(X, \perp, d)$ be an orthogonal metric space. For each $A, B \in CB(X)$, we have $A \perp B$ if and only if $a \perp b$, for all $a \in A$ and $b \in B$.

2. Let $(X, \perp)$ be an SO-set. A mapping $T : X \to CB(X)$ is said to be $\perp$-preserving if $x \perp y$ implies $T(x) \perp T(y)$.

Here, we define the concept of SO-lower semi-continuity that is weaker than lower semi-continuity and SO-continuity.

Definition 1.14. Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is strongly orthogonal lower semi-continuous (briefly, SO-lower semi-continuous) in $x \in X$ if for each SO-sequence $\{x_n\}$ in X with $x_n \to x$, we have $\liminf_{n\to\infty} f(x_n) \geq f(x)$.

It is easy to see that every lower semi-continuous mapping is SO-lower semi-continuous. The following example shows that the converse is not true in general.

Example 1.15. Let $X = [1, \infty)$ with the Euclidean metric. Set an orthogonal relation "$\perp$" by $x \perp y \ \Leftrightarrow\ xy \in \{x, y\}$. Define $f : X \to X$ by

$$f(x) = \begin{cases} 4 & x = 1 \\ 2x & x > 1. \end{cases}$$

We show that this function is not lower semi-continuous, but it is SO-lower semi-continuous. We consider the sequence $\left\{\frac{n+1}{n}\right\}$, $n \in \mathbb{N}$, and we have $\frac{n+1}{n} \to 1$, but $2 = \liminf_{n\to\infty} f\left(\frac{n+1}{n}\right) < f(1) = 4$. Therefore, f is not lower semi-continuous. On the other hand, since the only SO-sequence is $\{1\}$ and $1 = \liminf_{n\to\infty} f(1) = f(1) = 1$, then f is a SO-lower semi-continuous function.

Clearly, every SO-continuous function is SO-lower semi-continuous function. In the next example f is SO-lower semi-continuous but it is not SO-continuous and lower semi-continuous.

EXAMPLE 1.16. Let $X = [0, \infty)$ with the Euclidean metric. Set orthogonal relation "$\perp$" as $x \perp y \iff x = 0$ or $0 < x \leq y \leq 1$. Define $f : X \to X$ by

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & 0 < x \leq 1 \\ \dfrac{x}{2} & x > 1. \end{cases}$$

Clearly, f is not SO-continuous because the SO-sequence $\left\{\dfrac{1}{n}\right\}$ is convergent to zero but $1 = f\left(\frac{1}{n}\right) \nrightarrow f(0) = 0$. Also, f is not lower semi-continuous by considering the sequence $\left\{\dfrac{n+1}{n}\right\}$. But, we can show that f is SO-lower semi-continuous. For this purpose, if $x_n \to 0$, then $1 = \liminf_{n\to\infty}(x_n) > f(0) = 0$. Otherwise, for each sequence $\{x_n\}$ with $x_n \to x$ such that $0 < x \leq 1$, we have $1 = \liminf_{n\to\infty} f(x_n) = f(x) = 1$. Then, f is a SO-lower semi-continuous function.

2. Fixed point problems

The following theorem gives a partial answer to Q_1, Q_2 and Q_3.

THEOREM 2.1. *n Let $(X, d, \perp)$ be a SO-complete (not necessarily complete) metric space with orthogonal element x_0, $T : X \to C(X)$ be $\perp$-preserving and $F \in \mathcal{F}$. Suppose that T is compact or F is continuous from the right. Assume that the following conditions hold:*
(i) the map $x \mapsto d(x, Tx)$ is SO-lower semi-continuous;

(ii) there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ with $\liminf_{n\to s^+} \tau(t) > \sigma$ for all $s \geq 0$, such that for any $\perp$-comparable x, y in X with $d(x, Tx) > 0$ and $y \in F^x_\sigma$, we have $\tau(d(x, y)) + F(d(y, Ty)) \leqslant F(\psi(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)))$, where $\psi \in \Delta$ and F^x_σ is the same as in Theorem 1.4. Then T has a fixed point in X.

Proof. Aiming for contradiction, assume that T has no fixed points. Then, for all $x \in X$, $d(x, Tx) > 0$, and so $d(x_0, Tx_0) > 0$. Since T is compact or F is continuous from the right and $Tx_0 \in C(X)$, recalling Remark 1.5, we observe that the set $F^{x_0}_\sigma$ is nonempty for any $\sigma > 0$. Therefore, there exists $x_1 \in F^{x_0}_\sigma$. On the other hand, since x_0 is an orthogonal element, then x_0 and x_1 are $\perp$-comparable. Applying (ii), we have

$$\begin{aligned} &\tau(d(x_0, x_1)) + F(d(x_1, Tx_1)) \\ &\quad \leq F(\psi(d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), d(x_0, Tx_1), d(x_1, Tx_0))). \end{aligned}$$

Similarly, for $x_1 \in X$, since T is $\perp$-preserving, there exists $x_2 \in F^{x_1}_\sigma$, $\perp$-comparable with x_1 such that

$$\begin{aligned} &\tau(d(x_1, x_2)) + F(d(x_2, Tx_2)) \\ &\quad \leq F(\psi(d(x_1, x_2), d(x_1, Tx_2), d(x_2, Tx_2), d(x_1, Tx_2), d(x_2, Tx_1))). \end{aligned}$$

Proceed by induction to obtain an SO-sequence $\{x_n\}$, where $x_{n+1} \in F_\sigma^{x_n}$ and

$$\tau(d(x_n, x_{n+1})) + F(d(x_{n+1}, Tx_{n+1})) \leq F(\psi(d(x_n, x_{n+1}), d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_{n+1}), d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n))). \quad (1)$$

We will show that $\{x_n\}$ is Cauchy SO-sequence. Since $x_{n+2} \in F_\sigma^{x_{n+1}}$, thus

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_{n+1}, Tx_{n+1})) + \sigma. \quad (2)$$

In view of (1), (2), (F_1) and (Δ_1), we have

$$\begin{aligned} &\tau(d(x_n, x_{n+1})) - \sigma + F(d(x_{n+1}, x_{n+2})) \\ &\leq F(\psi(d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n))) \\ &\leq F(\psi(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1}))) \\ &\leq F(\psi(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})+d(x_{n+1}, x_{n+2}), 0)). \quad (3) \end{aligned}$$

Let $a_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$; we have $a_n > 0$. Then, (3) implies that $F(a_{n+1}) \leq F(\psi(a_n, a_n, a_{n+1}, a_n + a_{n+1}, 0))$. Bearing (F_1) and (Δ_2) in mind, we have

$$a_{n+1} \leq a_n \quad \text{for all } n \in \mathbb{N}. \quad (4)$$

That is, $\{a_n\}$ is a decreasing sequence. Thus, there exists $\delta \geq 0$ such that $\lim_{n\to\infty} a_n = \delta$. Let $\delta > 0$. Applying (4), (3) and (Δ_3), we obtain

$$\begin{aligned} F(a_{n+1}) &\leq F(\psi(a_n, a_n, a_{n+1}, a_n + a_{n+1}, 0)) + \sigma - \tau(a_n) \\ &\leq F(\psi(a_n, a_n, a_n, 2a_n, 0)) + \sigma - \tau(a_n) \leq F(a_n) + \sigma - \tau(a_n). \quad (5) \end{aligned}$$

By (5), we can write

$$\begin{aligned} F(a_{n+1}) &\leq F(a_n) + \sigma - \tau(a_n) \leq F(a_{n-1}) + 2\sigma - \tau(a_n) - \tau(a_{n-1}) \\ &\leq \ldots \leq F(a_0) + n\sigma - \tau(a_n) - \tau(a_{n-1}) - \ldots - \tau(a_0). \quad (6) \end{aligned}$$

Let $\tau(a_{p_n}) = \min\{\tau(a_0), \tau(a_1), \ldots, \tau(a_n)\}$ for all $n \in \mathbb{N}$. Applying (6), we have

$$F(a_{n+1}) \leq F(a_0) + n(\sigma - \tau(a_{p_n})). \quad (7)$$

Here, we consider two cases for the sequence $\{\tau(a_{p_n})\}$ as follows:

Case 1. For each $n \in \mathbb{N}$, there is $m > n$ such that $\tau(a_{p_n}) > \tau(a_{p_m})$. Then we obtain a subsequence $\{a_{p_{n_k}}\}$ of $\{a_{p_n}\}$ with $\tau(a_{p_{n_k}}) > \tau(a_{p_{n_{k+1}}})$ for all k. Since $a_{p_{n_k}} \to \delta$, this implies that $\liminf_{k\to\infty} \tau(a_{p_{n_k}}) > \sigma$. Therefore, $F(a_{n_k}) \leq F(a_0)+n_k(\sigma-\tau(a_{p_{n_k}}))$ for all k, and so $\liminf_{k\to\infty} F(a_{n_k}) = -\infty$. Recalling (F_2), we observe $\lim_{k\to\infty}(a_{p_{n_k}}) = 0$, which contradicts the fact $\lim_{n\to\infty}(a_{p_n}) > 0$.

Case 2. There is $n_0 \in \mathbb{N}$ such that $\tau(a_{p_{n_0}}) = \tau(a_{p_m})$ for all $m > n_0$. Therefore, $F(a_m) \leq F(a_0) + m(\sigma - \tau(a_{p_{n_0}}))$ for all $m > n_0$, and so $\liminf_{m\to\infty} F(a_m)=-\infty$. In view of (F_2), we conclude that $\lim_{m\to\infty} a_m = 0$, which contradicts the fact $\lim_{m\to\infty} a_m > 0$. Thus, $\lim_{n\to\infty} a_n = 0$. Applying (F_3), there exists $k \in (0,1)$ such that $\lim_{n\to\infty} a_n^k F(a_n) = 0$. Using (7), we obtain $a_n^k F(a_n) - a_n^k F(a_0) \leq a_n^k n(\sigma - \tau(a_{p_n}) \leq 0$. Letting $n\to\infty$ in the above inequality, we deduce that $\lim_{n\to\infty} n a_n^k = 0$. So, there exists $n_0 \in \mathbb{N}$ such that $n a_n^k \leq 1$ for all $n \geq n_0$. Hence, for all $n \geq n_0$, we have

$$a_n \leq \frac{1}{n^{1/k}}. \quad (8)$$

Let $m, n \in \mathbb{N}$ be such that $m > n \geq n_1$. Applying triangular inequality and (8), we can write

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leqslant \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy SO-sequence. Since $(X, \perp, d)$ is a SO-complete metric space, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. On the other hand, since $x_{n+1} \in F_\sigma^{x_n}$, we have

$$F(d(x_n, x_{n+1})) \leq F(d(x_n, Tx_n)) + \sigma. \tag{9}$$

Applying (1), (4), (F_1), (Δ_1), (Δ_2) and (9), we deduce that

$$\begin{aligned} &\tau(d(x_n, x_{n+1})) + F(d(x_{n+1}, Tx_{n+1})) \\ &\quad \leq F(\psi(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}), 0)) \\ &\quad \leq F(\psi(d(x_n, x_{n+1}), d(x_n, x_{n+1})), d(x_n, x_{n+1})), 2d(x_n, x_{n+1}), 0)) \\ &\quad = F(d(x_n, x_{n+1})) \leq F(d(x_n, Tx_n)) + \sigma. \end{aligned}$$

So, we can write

$$F(d(x_{n+1}, Tx_{n+1})) \leq F(d(x_n, Tx_n)) + \sigma - \tau(d(x_n, x_{n+1})). \tag{10}$$

Similarly to (6), we can conclude from (10) that

$$F(d(x_{n+1}, Tx_{n+1})) \leq F(d(x_0, Tx_0)) + n(\sigma - \tau(a_{p_n})). \tag{11}$$

Now, (11) and (F_2) imply that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since $x \mapsto d(x, Tx)$ is SO-lower semi-continuous, therefore $0 < d(z, Tz) \leq \liminf_{n\to\infty} d(x_n Tx_n) = 0$. This is a contradiction. Hence, T has a fixed point. $\square$

In the following, by setting $\psi(t_1, t_2, t_3, t_4, t_5) = t_1$, we have a generalization of Theorems 1.7 and 1.8 again.

Corollary 2.2. *Let $(X, \perp, d)$ be an SO-complete metric space. Let*
(i) $T : X \to C(X)$ be $\perp$-preserving and $F \in \mathcal{F}$, or

(ii) $T : X \to K(X)$ be $\perp$-preserving and $F \in \mathcal{F}_*$.
If there exists $\tau > 0$ such that for any $\perp$-comparable $x, y \in X$ with $d(x, Tx) > 0$ and $y \in F_\sigma^x$, we have $\tau + F(d(y, Ty)) \leqslant F(\psi(d(x, y))$, then T has a fixed point in X provided $\sigma < \tau$ and $x \mapsto d(x, Tx)$ is SO-lower semi-continuous.

We can also extend Theorems 1.1 and 1.2 as follows.

Corollary 2.3. *Let $(X, \perp, d)$ be an SO-complete metric space. Let $T : X \to K(X)$ be $\perp$-preserving. Assume that the following conditions hold:*
(i) the map $x \mapsto d(x, Tx)$ is SO-lower semi-continuous;

(ii) there exist constants $c, b \in (0, 1)$ with $b < c$ such that for any $\perp$-comparable $x, y \in X$ with $y \in I_b^x$, we have $d(y, Ty) \leq c d(x, y)$.
Then T has a fixed point.

COROLLARY 2.4. *Let $(X, \perp, d)$ be an SO-complete metric space with orthogonal element x_0. Suppose that $T : X \to K(X)$ is $\perp$-preserving. Assume that the following conditions hold:*
(i) the map $x \to d(x, Tx)$ is SO-lower semi-continuous;
(ii) there exist constant $b \in (0, 1)$ and $\varphi : [0, \infty) \to [0, b)$ satisfying $\limsup_{t \to s^+} \varphi(t) < b$ for $s \geq 0$, and for any $\perp$-comparable $x, y \in X$ with $y \in I_b^x$, we have $d(y, Ty) \leq cd(x, y)$. Then T has a fixed point.

As a direct consequence of [12, Theorem 2.7], we obtain the following corollaries.

COROLLARY 2.5. *Let $(X, \perp, d)$ be an SO-complete metric space with orthogonal element x_0. Suppose that $T : X \to P(X)$ is $\perp$-preserving, $F \in \mathcal{F}_*$. Suppose there exists $\tau > 0$ such that for any $\perp$-comparable $x, y \in X$ with $d(x, Tx) > 0$ and $y \in F_\sigma^x$ satisfying $d(y, Ty) > 0$, $\tau + F(d(y, Ty)) \leq F(d(x, y))$ holds. If $d(x_0, Tx_0) > 0$ and for all convergent SO-sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$, we have that $T(\lim x_n)$ is closed, then T has a fixed point in X provided $\sigma < \tau$ and $x \longrightarrow d(x, Tx)$ is SO-lower semi-continuous.*

COROLLARY 2.6. *Let $(X, \perp, d)$ be an SO-complete metric space with orthogonal element x_0. Let $T : X \to P(X)$ be $\perp$-preserving. Suppose there exists $c \in (0, 1)$ such that for any $\perp$-comparable $x, y \in X$ with $d(x, Tx) > 0$ and $y \in I^x(b \in (0, 1))$, $0 < d(y, Ty) \leq cd(x, y)$ holds. If $d(x_0, Tx_0) > 0$ and for all convergent SO-sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$, we have that $T(\lim x_n)$ is closed, then T has a fixed point in X provided $c < b$ and $x \longrightarrow d(x, Tx)$ is SO-lower semi-continuous.*

3. Some examples

Now, the main result can be illustrated by the following examples.

EXAMPLE 3.1. Let $X = \{x_n = n(n+1)/2, 1 \leq n \leq 11\}$ and $d(x, y) = |x - y|$. Define a mapping $T : X \to C(X)$ as

$$T(x) = \begin{cases} \{x_n\} & x = x_n,\ n \in \{1, 2, 9, 10, 11\} \\ \{x_{n+1}, x_{n+2}\} & x = x_n,\ 3 \leq n \leq 8. \end{cases}$$

Clearly, $x \mapsto d(x, Tx)$ is continuous. Set $x \perp y \Leftrightarrow (x, y) \in X \times X$. Then T is $\perp$-preserving. Taking $F(x) = \ln(x)$, $\tau(t) = \ln \dfrac{t+9}{4}$, $\sigma = \ln \dfrac{9}{4}$ and $\phi(t_1, t_2, t_3, t_4, t_5) = \dfrac{t_4 + 12t_5}{2}$, clearly, for all $x \in X$, with $d(x, Tx) > 0$, there exists $y \in F_{\perp\sigma}^x$ such that the condition (ii) of Theorem 2.1 is satisfied. Applying Theorem 2.1, T has a fixed point in X.

It is simple to verify that Theorem 1.7 cannot be applied to our example. In fact, it is just enough to show that the condition (ii) of Theorem 1.7 is not satisfied. Note that if $d(x, y) > 0$, then $x = x_n$ for $3 \leq n \leq 8$. In this case $Tx_n = \{x_{n+1}, x_{n+2}\}$. If we set

$x = x_3$, we have two cases for y. If $y = x_4 \in F_\sigma^{x_3}$, we have $5 = d(y, Ty) > d(x, y) = 4$, and so the condition (ii) of Theorem 1.7 for each $F \in \mathcal{F}^*$ is not satisfied. Otherwise, let $y = x_5 \in F_\sigma^{x_3}$. Since $d(y, Ty) = 6$, $d(x, y) = 9$ and $d(x, Tx) = 4$, for each $F \in \mathcal{F}^*$ with $F(d(x, y)) \le F(d(x, Tx)) + \sigma$, we should have $\sigma \ge F(9) - F(4)$. Therefore, since $\liminf_{t\to s^+} \tau(t) > \sigma$ for all $s \ge 0$, we have $\tau(d(x, y)) + F(d(y, Ty)) = \tau(9) + F(6) \ge F(9) - F(4) + F(6) \ge F(9) = F(d(x, y))$. Hence, the condition (ii) of Theorem 2.1 is not satisfied and we cannot apply Theorem 1.7 for this example.

Example 3.2. Let $X = [0, 2)$. Set $x \perp y \;\Leftrightarrow\; x = 0$. Then $(X, \perp)$ is an SO-set with orthogonal element $x_0 = 0$. Clearly, X with the Euclidean metric is not a complete metric space but it is an SO-complete metric space. Define a mapping $T : X \to C(X)$ as

$$T(x) = \begin{cases} \left\{\dfrac{n-1}{2(n+1)}, \dfrac{3n+1}{2(n+1)}, 0\right\} & x = \dfrac{n}{n+1},\ n \in \mathbb{N} \\ \{\dfrac{1}{3}, 0\} & x = 1 \\ x & \text{otherwise.} \end{cases}$$

Clearly, T is $\perp$-preserving. It is easy to compute that

$$d(x, Tx) = \begin{cases} \dfrac{1}{2} & x = \dfrac{n}{n+1},\ n \in \mathbb{N} \\ \dfrac{2}{3} & x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $x \mapsto d(x, Tx)$ is not lower semi-continuous, because $\{x_n\} = \{\frac{n}{n+1}\}$ is convergent to 1 and we have $\frac{1}{2} = \liminf_{n\to\infty} d(\frac{n}{n+1}, T(\frac{n}{n+1})) < d(1, T(1)) = \frac{2}{3}$. Therefore condition (i) of Theorem 1.7 is not satisfied, but we will show that condition (i) of Theorem 2.1 is satisfied. Note that $x \mapsto d(x, Tx)$ is SO-lower semi-continuous, because our only SO-sequence is zero and we have $0 = \liminf_{n\to\infty} d(0, T(0)) \ge d(0, T(0)) = 0$. Now, let $F(x) = \ln(x)$. If $d(x, Tx) > 0$, then $\left(x = \frac{n}{n+1},\ n \in \mathbb{N}\right)$ or $(x = 1)$.

If $x = \frac{n}{n+1}$, $n \in \mathbb{N}$, then for $y = 0 \in Tx$, we have $d(x, y) = \frac{n}{n+1}$, $d(x, Tx) = \frac{1}{2}$, $d(y, Ty) = 0$, and so $\ln(d(x, y)) - \ln(d(x, Tx)) = \ln(\frac{n}{n+1}) - \ln(\frac{1}{2}) < \ln(1) + \ln(2) = \ln(2)$, and $-\infty = \frac{n}{n+1} + \ln(3) + \ln(0) \le \ln(\frac{n}{n+1})$. Therefore $y = 0 \in F_\sigma^x$, and $\tau(d(x, y)) + F(d(y, Ty)) \le F(d(x, y))$ is satisfied for $\sigma = \ln(2)$ and $\tau(t) = t + \ln(3)$. If $x = 1$, then for $y = 0 \in Tx$, we have $d(x, y) = 1$, $d(x, Tx) = \frac{2}{3}$, $d(y, Ty) = 0$, and so $\ln(d(x, y)) - \ln(d(x, Tx)) = \ln(1) - \ln(\frac{2}{3}) = \ln(\frac{3}{2})$, and $-\infty = 1 + \ln(\frac{3}{2}) - \infty \le \ln(1) = 0$. Hence, $y = 0 \in F_\sigma^x$, and $\tau(d(x, y)) + F(d(y, Ty)) \le F(d(x, y))$ is satisfied by taking $\sigma = \ln(\frac{3}{2})$ and $\tau(t) = t + \ln(2)$. Then, T has a fixed point according to Theorem 2.1.

References

[1] Z. Ahmadi, R. Lashkaripour, H. Baghani, *A fixed point problem with constraint inequalities via a contraction in incomplete metric spaces,* Filomat, to appear.

[2] I. Altun, G. Minak, H. Dag, *Multivalued F-contractions on complete metric space,* J. Nonlinear Convex Anal. **16** (2015), 659–666.

[3] I. Altun, G. Minak, M. Olgun, *Fixed point of mutivalued nonlinear F-contractions on complete metric spaces,* Nonlinear Anal. Model. Control **21** (2016) 201–210.

[4] H. Baghani, M. Ramezani, *A fixed point theorem for a new class of set-valued mappings in R-complete(not necessarily complete) metric spaces,* Filomat **31** (2017), 3875–3884.

[5] V. Berinde, M. Pacurar, *The role of the Pompeiu-Hausdorff metric in fixed point theory,* Creat. Math. Inform. **2** (2013), 35–42.

[6] Lj. B. Ćirić, *Multi-valued nonlinear contraction mappings,* Nonlinear Anal. **71** (2009), 2716–2723.

[7] M. Eshaghi Gordji, M. Ramezani, M. De La Sen and Y.J. Cho, *On orthogonal sets and Banach Fixed Point theorem,* Fixed Point Theory, **18** (2017), 569–578.

[8] Y. Feng, S. Liu, *Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings,* J. Math. Anal. Appl. **317** (2006), 103–112.

[9] T. Kamran, Q. Kiran, *Fixed point theorems for multi-valued mappings obtained by altering distances,* Math. Comput. Modelling, **54** (2011), 2772–2777.

[10] D. Klim, D. Wardowski, *Fixed point theorems for set-valued contractions in complete metric spaces,* J. Math. Anal. Appl. **334** (2007), 132–139.

[11] G. Minak, I. Altun, *Some new generalizations of Mizoguchi-Takahashi type fixed point theorem,* J. Ineq. Appl. (2013), 2013–493.

[12] G. Minak, M. Olgun, I, Altun, *A new approach to fixed point theorems for multivalued contractive maps,* Carpathian J. Math. **31** (2015), 241–248.

[13] S.B. Nadler, *Multi-valued contraction mappings,* Pac. J. Math. **30** (1969), 475–488.

[14] M. Ramezani, H. Baghani, *The Meir-Keeler fixed point theorem in incomplete modular spaces with application,* J. Fixed Point Theory and Appl. **19** (2017), 2369–2382.

[15] D. Wardowski, *Fixed point of a new type of contractive mappings in complete metric spaces,* Fixed Point Theory Appl. **2012** (2012), 1–6.

(received 27.08.2018; in revised form 04.10.2018; available online 16.12.2018)

Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

E-mail: lashkari@hamoon.usb.ac.ir

Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

E-mail: h.baghani@gmail.com, h.baghani@math.usb.ac.ir

Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

E-mail: Z.ahmadiz@yahoo.com