MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
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# NONLOCAL THREE-POINT MULTI-TERM MULTIVALUED FRACTIONAL-ORDER BOUNDARY VALUE PROBLEMS

**Ahmed Alsaedi, Najla Alghamdi, Bashir Ahmad and Sotiris K. Ntouyas**

**Abstract**. In this paper we study a new kind of boundary value problems of multi-term fractional differential inclusions and three-point nonlocal boundary conditions. The existence of solutions is established for convex and non-convex multivalued maps by using standard theorems from the fixed point theory. We also construct some examples for demonstrating the application of the main results.

## 1. Introduction

In recent years, the study of fractional-order boundary value problems received much attention in view of their occurrence in several diverse disciplines. Now one can find a variety of results involving different kinds of boundary conditions in the related literature. Such problems can be categorized as single-valued and multivalued problems. For recent development on the topic, we refer the reader to a series of papers [2, 3, 9] and the references cited therein. Fractional-order models are regarded more realistic than their integer-order counterparts, for instance, see [14, 20, 22]. This fact has added worth to the topic of boundary value problems of fractional-order. In particular, multivalued (inclusions) problems are found to be of special significance in studying dynamical systems and stochastic processes, see e.g. [17, 21] for the analysis of control problems.

Equations containing more than one differential operator are termed as multi-term differential equations, for example, see [17–19, 23, 24]. In a recent work [1], the authors obtained some existence results for multi-term fractional differential equations supplemented with nonlocal boundary conditions.

Let $\eta \in (0, 1)$ be a fixed number. In this paper, motivated by [1], we investigate a new boundary value problem of multi-term fractional differential inclusions and nonlocal three-point boundary conditions given by

$$(a_2\ {}^cD^{q+2} + a_1\ {}^cD^{q+1} + a_0\ {}^cD^q)x(t) \in F(t, x(t)), \quad 0 < q < 1, \quad 0 < t < 1, \qquad (1)$$

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$$x(0) = 0, \quad x(\eta) = 0, \quad x(1) = 0, \tag{2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order $q$, $a_i$ $(i = 0, 1, 2)$ are real constants and $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

The paper is organized as follows. In Section 2 we recall some preliminary concepts from multi-valued analysis and fractional calculus related to our work. Section 3 contains the main results. The first result involving convex valued maps is based on the nonlinear alternative of Leray-Schauder type. The second result dealing with non-convex valued maps relies on a fixed point theorem for contractive multivalued maps due to Covitz and Nadler, while in the third result, we combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. The methods used in our analysis are standard, however their exposition in the framework of problem (1)-(2) is new and worth-contributing to the literature on fractional-order multivalued problems.

## 2. Preliminaries

### 2.1 Basic material from fractional calculus

We begin this subsection with some definitions [12].

DEFINITION 2.1. The Riemann-Liouville fractional integral of order $\tau > 0$ of a function $h : (0, \infty) \to \mathbb{R}$ is defined by

$$I^\tau h(u) = \int_0^u \frac{(u - v)^{\tau - 1}}{\Gamma(\tau)} h(v)\, dv, \quad u > 0,$$

provided the right-hand side is point-wise defined on $(0, \infty)$, where $\Gamma$ is the Gamma function.

DEFINITION 2.2. The Caputo derivative of order $\tau$ for a function $h : [0, \infty) \to \mathbb{R}$ with $h \in C^n[0, \infty)$ is defined by

$${}^cD^\tau h(u) = \frac{1}{\Gamma(n - \tau)} \int_0^u \frac{h^{(n)}(v)}{(u - v)^{\tau + 1 - n}}\, dv = I^{n-\tau} h^{(n)}(u),\ t > 0,\ n - 1 < \tau < n,$$

PROPERTY 2.3. *With the given notation, the following equality holds:*

$$I^\tau({}^cD^\tau h(u)) = h(u) - c_0 - c_1 u - \ldots - c_{n-1} u^{n-1},\ u > 0,\ n - 1 < \tau < n,$$

*where* $c_i = \frac{h^{(i)}(0)}{i!},\ i = 1, \ldots, n - 1.$

The following known result [1] facilitates the transformation of the problem (1)-(2) into a fixed point problem.

LEMMA 2.4. *For any* $y \in C([0, 1], \mathbb{R})$*, the solution of linear multi-term fractional differential equation*

$$(a_2\, {}^cD^{q+2} + a_1\, {}^cD^{q+1} + a_0\, {}^cD^q)x(t) = y(t), \quad 0 < q < 1, \quad 0 < t < 1,$$

*supplemented with the boundary conditions* (2) *is given by*

$$\begin{aligned}
\text{(i)} \quad x(t) =& \frac{1}{a_2(m_2-m_1)}\bigg\{\int_0^t\int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\ ds \\
&+\sigma_1(t)\int_0^1\int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\ ds \\
&+\sigma_2(t)\int_0^\eta\int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\ ds\bigg\}, \quad \textit{if} \quad a_1^2-4a_0a_2>0,
\end{aligned}$$

$$\begin{aligned}
\textit{where} \quad \Phi(\kappa) =& e^{m_2(\kappa-s)}-e^{m_1(\kappa-s)}, \quad \kappa=t,\ 1,\ \textit{and}\ \eta, \\
m_1 =& \frac{-a_1-\sqrt{a_1^2-4a_0a_2}}{2a_2}, \quad m_2=\frac{-a_1+\sqrt{a_1^2-4a_0a_2}}{2a_2}, \\
\sigma_1(t) =& \frac{\gamma_2\rho_2(t)-\gamma_4\rho_1(t)}{\mu}, \quad \sigma_2(t)=\frac{\gamma_3\rho_1(t)-\gamma_1\rho_2(t)}{\mu}, \mu=\gamma_1\gamma_4-\gamma_2\gamma_3\neq 0, \\
\rho_1(t) =& \frac{m_2(1-e^{m_1t})-m_1(1-e^{m_2t})}{a_2m_1m_2(m_2-m_1)}, \quad \rho_2(t)=e^{m_1t}-e^{m_2t}, \\
\gamma_1 =& \frac{m_2(1-e^{m_1})-m_1(1-e^{m_2})}{a_2m_1m_2(m_2-m_1)}, \quad \gamma_2=\frac{m_2(1-e^{m_1\eta})-m_1(1-e^{m_2\eta})}{a_2m_1m_2(m_2-m_1)}, \qquad (3) \\
\gamma_3 =& e^{m_1}-e^{m_2}, \quad \gamma_4=e^{m_1\eta}-e^{m_2\eta},
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad x(t) =& \frac{1}{a_2}\bigg\{\int_0^t\int_0^s \Psi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\ ds \\
&+\psi_1(t)\int_0^1\int_0^s \Psi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\ ds \\
&+\psi_2(t)\int_0^\eta\int_0^s \Psi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\ ds\bigg\}, \quad \textit{if} \quad a_1^2-4a_0a_2=0,
\end{aligned}$$

$$\begin{aligned}
\textit{where} \quad \Psi(\kappa) =& (\kappa-s)e^{m(\kappa-s)}, \quad \kappa=t,\ 1,\ \textit{and}\ \eta, \\
\psi_1(t) =& \frac{(t-\eta)e^{m(t+\eta)}-te^{mt}+\eta e^{m\eta}}{\Lambda}, \quad \psi_2(t)=\frac{(1-t)e^{m(t+1)}+te^{mt}-e^m}{\Lambda}, \\
\Lambda =& (\eta-1)e^{m(\eta+1)}-\eta e^{m\eta}+e^m\neq 0,\ m=\frac{-a_1}{2a_2},
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad x(t) =& \frac{1}{a_2\beta}\bigg\{\int_0^t\int_0^s \Omega(t)\frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\ ds \\
&+\varphi_1(t)\int_0^1\int_0^s \Omega(1)\frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\ ds \\
&+\varphi_2(t)\int_0^\eta\int_0^s \Omega(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\ ds\bigg\}, \quad \textit{if} \quad a_1^2-4a_0a_2<0,
\end{aligned}$$

$$\textit{where} \quad \Omega(\kappa) = e^{-\alpha(\kappa-s)}\sin\beta(\kappa-s), \quad \kappa=t,\ 1,\ \textit{and}\ \eta,$$

$$\begin{aligned}
\alpha =&\frac{a_1}{2a_2}, \quad \beta = \frac{\sqrt{4a_0a_2 - a_1{}^2}}{2a_2},\\
\varphi_1(t) =&\frac{\omega_4\varrho_1(t) - \omega_2\varrho_2(t)}{\Omega}, \quad \varphi_2(t) = \frac{\omega_1\varrho_2(t) - \omega_3\varrho_1(t)}{\Omega},\\
\varrho_1(t) =&\frac{\beta - \beta e^{-\alpha t}\cos\beta t - \alpha e^{-\alpha t}\sin\beta t}{\alpha^2+\beta^2}, \quad \varrho_2(t) = a_2\beta e^{-\alpha t}\sin\beta t,\\
\omega_1 =&\frac{\beta - \beta e^{-\alpha}\cos\beta - \alpha e^{-\alpha}\sin\beta}{\alpha^2+\beta^2}, \quad \omega_2 = \frac{\beta - \beta e^{-\alpha\eta}\cos\beta\eta - \alpha e^{-\alpha\eta}\sin\beta\eta}{\alpha^2+\beta^2},\\
\omega_3 =&a_2\beta e^{-\alpha}\sin\beta, \quad \omega_4 = a_2\beta e^{-\alpha\eta}\sin\beta\eta, \quad \Omega = \omega_2\omega_3 - \omega_1\omega_4 \neq 0.
\end{aligned}$$

### 2.2 Basic material for multivalued maps

Let $\mathcal{C} := C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)|,\ t \in [0,1]\}$. Also by $L^1([0,1],\mathbb{R})$ we denote the space of functions $x : [0,1] \to \mathbb{R}$ such that $\|x\|_{L^1} = \int_0^1 |x(t)|\,dt$.

For each $y \in \mathcal{C}$, define the set of selections of $F$ by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,y(t)) \quad \text{a.e. on} \quad [0,1]\}.$$

We define the graph of $G$ to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall a result for closed graphs and upper-semicontinuity.

Lemma 2.5 ( [7, Proposition 1.2]). *If $G : X \to \mathcal{P}_{cl}(Y)$ ($\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y$ is closed$\}$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if $G$ is completely continuous and has a closed graph, then it is upper semi-continuous.*

The following lemma will be used in the sequel.

Lemma 2.6 ([16]). *Let $X$ be a separable Banach space. Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ ($\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y$ is compact and convex$\}$) be an $L^1$-Carathéodory multivalued map and let $\Theta$ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator $\Theta \circ S_F : C(J,X) \to \mathcal{P}_{cp,c}(C(J,X))$, $x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(J,X) \times C(J,X)$.*

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps.

Lemma 2.7 ( [10, Nonlinear alternative for Kakutani maps]). *Let $E$ be a Banach space, $C$ a closed convex subset of $E$, $U$ an open subset of $C$ and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is an upper semicontinuous compact map. Then either*

*(i) $F$ has a fixed point in $\overline{U}$, or*

*(ii) there are $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

For some basic concepts about multivalued analysis, we refer the reader to the books [7, 11].

## 3. Existence results for the case $a_1^2 - 4a_0a_2 > 0$

### 3.1 The Carathéodory case

In this subsection we consider the case when $F$ has convex values and prove an existence result based on nonlinear alternative of Leray-Schauder type, assuming that $F$ is Carathéodory.

Definition 3.1. Let $a_1^2 - 4a_0a_2 > 0$. A function $x \in \mathcal{C}$, possessing a Caputo derivative of order at most $q+2$, is a solution of the problem (1)-(2) if $x(0) = 0$, $x(\eta) = 0$, $x(1) = 0$, and there exists a function $v \in L^1([0,1], \mathbb{R})$ such that $v(t) \in F(t, x(t))$ a.e. on $[0,1]$ and

$$x(t) = \frac{1}{a_2(m_2 - m_1)} \Bigg\{ \int_0^t \int_0^s \Phi(t) \frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\, du\, ds \tag{4}$$
$$+ \sigma_1(t) \int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\, du\, ds + \sigma_2(t) \int_0^\eta \int_0^s \Phi(\eta) \frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\, du\, ds \Bigg\}.$$

Theorem 3.2. *Assume that:*

*($H_1$) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is $L^1$-Carathéodory;*

*($H_2$) there exists a continuous nondecreasing function $Q : [0, \infty) \to (0, \infty)$ and a function $g \in C([0,1], \mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le g(t) Q(\|x\|) \quad \text{for each} \quad (t,x) \in [0,1] \times \mathbb{R};$$

*($H_3$) there exists a constant $M > 0$ such that $\frac{M}{\frac{\|g\| Q(M)}{\Gamma(q+1)} \left\{ \varepsilon + \widehat{\sigma}_1 \gamma_1 + \eta^q \widehat{\sigma}_2 \gamma_2 \right\}} > 1$, where*

$$\widehat{\sigma}_1 = \max_{t \in [0,1]} |\sigma_1(t)|, \quad \widehat{\sigma}_2 = \max_{t \in [0,1]} |\sigma_2(t)|, \quad \varepsilon = \max_{t \in [0,1]} \left| \frac{m_2(1 - e^{m_1 t}) - m_1(1 - e^{m_2 t})}{a_2 m_1 m_2 (m_2 - m_1)} \right|,$$

*and $\gamma_1, \gamma_2$ are defined in (3).*

*Then the boundary value problem (1)-(2), with $a_1^2 - 4a_0a_2 > 0$, has at least one solution on $[0,1]$.*

*Proof.* To transform the problem (1)-(2) into a fixed point problem, we define an operator $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{C})$ by

$$\mathcal{F}(x) = \left\{ \begin{array}{c} h \in C([0,1], \mathbb{R}) : \\ h(t) = \left\{ \begin{array}{c} \frac{1}{a_2(m_2 - m_1)} \left\{ \int_0^t \int_0^s \Phi(t) \frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\, du\, ds \right. \\ + \sigma_1(t) \int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\, du\, ds \\ \left. + \sigma_2(t) \int_0^\eta \int_0^s \Phi(\eta) \frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\, du\, ds \right\}, \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,x}$. It is obvious that the fixed points of $\mathcal{F}$ are solutions of the boundary value problem (1)-(2).

We will show that $\mathcal{F}$ satisfies the assumptions of Leray-Schauder nonlinear alternative (Lemma 2.7). The proof consists of several steps.

**Step 1.** $\mathcal{F}(x)$ is convex for each $x \in \mathcal{C}$.

This step is obvious since $S_{F,x}$ is convex ($F$ has convex values), and therefore we omit the proof.

**Step 2.** $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $\mathcal{C}$.

For a positive number $r$, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $\mathcal{C}$. Then, for each $h \in \mathcal{F}(x)$, $x \in B_r$, there exists $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{a_2(m_2-m_1)}\Bigg\{\int_0^t\int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds$$
$$+\sigma_1(t)\int_0^1\int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds + \sigma_2(t)\int_0^\eta\int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds\Bigg\}.$$

Then, for $t \in [0,1]$, we have

$$|h(t)| \le \frac{1}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{\int_0^t\int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}|v(u)|\,du\,ds$$
$$+|\sigma_1(t)|\int_0^1\int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|v(u)|\,du\,ds + |\sigma_2(t)|\int_0^\eta\int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|v(u)|\,du\,ds\Bigg\}$$
$$\le \frac{\|g\|Q(r)}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{\int_0^t\Big(e^{m_2(t-s)}-e^{m_1(t-s)}\Big)\frac{s^q}{\Gamma(q+1)}ds$$
$$+|\sigma_1(t)|\int_0^1\Big(e^{m_2(1-s)}-e^{m_1(1-s)}\Big)\frac{s^q}{\Gamma(q+1)}ds + |\sigma_2(t)|\int_0^\eta\Big(e^{m_2(\eta-s)}-e^{m_1(\eta-s)}\Big)\frac{s^q}{\Gamma(q+1)}ds\Bigg\}$$
$$\le \frac{\|g\|Q(r)}{\Gamma(q+1)}\big\{\varepsilon + \widehat{\sigma}_1\gamma_1 + \eta^q\widehat{\sigma}_2\gamma_2\big\},$$

which yields $\|h\| \le \frac{\|g\|Q(r)}{\Gamma(q+1)}\big\{\varepsilon + \widehat{\sigma}_1\gamma_1 + \eta^q\widehat{\sigma}_2\gamma_2\big\}$.

**Step 3.** $\mathcal{F}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$.

Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. Then, for each $h \in \mathcal{B}(x)$, we obtain

$$|h(t_2)-h(t_1)| \le \frac{1}{a_2(m_2-m_1)}\Bigg\{\Bigg|\int_0^{t_1}\int_0^s\Big[\Phi(t_2)-\Phi(t_1)\Big]\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds$$
$$+\int_{t_1}^{t_2}\int_0^s \Phi(t_2)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds\Bigg| + |\sigma_1(t_2)-\sigma_1(t_1)|\int_0^1\int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|v(u)|\,du\,ds$$
$$+|\sigma_2(t_2)-\sigma_2(t_1)|\int_0^\eta\int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|v(u)|\,du\,ds\Bigg\}$$
$$\le \frac{\|g\|Q(r)}{a_2m_1m_2(m_2-m_1)\Gamma(q+1)}\Big\{\Big(t_1^q-t_2^q\Big)\Big(m_1(1-e^{m_2(t_2-t_1)})-m_2(1-e^{m_1(t_2-t_1)})\Big)$$
$$+t_1^q\Big(m_1(e^{m_2t_2}-e^{m_2t_1})-m_2(e^{m_1t_2}-e^{m_1t_1})\Big)$$

$$+ |\sigma_1(t_2) - \sigma_1(t_1)|(m_2(1 - e^{m_1}) - m_1(1 - e^{m_2}))$$
$$+ |\sigma_1(t_2) - \sigma_1(t_1)|\eta^q(m_2(1 - e^{m_1\eta}) - m_1(1 - e^{m_2\eta}))\Big\}.$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{F} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is completely continuous.

Since $\mathcal{F}$ is completely continuous, in order to prove that it is u.s.c. it is enough to prove that it has a closed graph.

**Step 4.** $\mathcal{F}$ has a closed graph.

Let $x_n \to x_*, h_n \in \mathcal{F}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{F}(x_*)$. Associated with $h_n \in \mathcal{F}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$h_n(t) = \frac{1}{a_2(m_2 - m_1)}\Bigg\{ \int_0^t \int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)} v_n(u)\,du\,ds$$
$$+ \sigma_1(t)\int_0^1 \int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)} v_n(u)\,du\,ds + \sigma_2(t)\int_0^\eta \int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)} v_n(u)\,du\,ds\Bigg\}.$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$h_*(t) = \frac{1}{a_2(m_2 - m_1)}\Bigg\{ \int_0^t \int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)} v_*(u)\,du\,ds$$
$$+ \sigma_1(t)\int_0^1 \int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)} v_*(u)\,du\,ds + \sigma_2(t)\int_0^\eta \int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)} v_*(u)\,du\,ds\Bigg\}.$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to \mathcal{C}$ given by

$$v \mapsto \Theta(v)(t) = \frac{1}{a_2(m_2 - m_1)}\Bigg\{ \int_0^t \int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\,du\,ds$$
$$+ \sigma_1(t)\int_0^1 \int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\,du\,ds + \sigma_2(t)\int_0^\eta \int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\,du\,ds\Bigg\}.$$

Observe that $\|h_n(t) - h_*(t)\| \to 0$, as $n \to \infty$, and thus, it follows by Lemma 2.6 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{1}{a_2(m_2 - m_1)}\Bigg\{ \int_0^t \int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)} v_*(u)\,du\,ds$$
$$+ \sigma_1(t)\int_0^1 \int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)} v_*(u)\,du\,ds + \sigma_2(t)\int_0^\eta \int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)} v_*(u)\,du\,ds\Bigg\},$$

for some $v_* \in S_{F,x_*}$.

**Step 5.** We show that there exists an open set $U \subseteq \mathcal{C}$ with $x \notin \theta\mathcal{F}(x)$ for any $\theta \in (0,1)$ and $x \in \partial U$.

Let $\theta \in (0,1)$ and $x \in \theta\mathcal{F}(x)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$x(t) = \theta\frac{1}{a_2(m_2-m_1)}\left\{\int_0^t\int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds\right.$$

$$\left.+\sigma_1(t)\int_0^1\int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds+\sigma_2(t)\int_0^\eta\int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds\right\}.$$

Then, by using the computations from **Step 2.**, we get

$$|x(t)| \le \frac{\|g\|Q(\|x\|)}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\left\{\int_0^t\left(e^{m_2(t-s)}-e^{m_1(t-s)}\right)\frac{s^q}{\Gamma(q+1)}ds\right.$$

$$+|\sigma_1(t)|\int_0^1\left(e^{m_2(1-s)}-e^{m_1(1-s)}\right)\frac{s^q}{\Gamma(q+1)}ds$$

$$\left.+|\sigma_2(t)|\int_0^\eta\left(e^{m_2(\eta-s)}-e^{m_1(\eta-s)}\right)\frac{s^q}{\Gamma(q+1)}ds\right\}$$

$$\le \frac{\|g\|Q(\|x\|)}{\Gamma(q+1)}\{\varepsilon+\widehat{\sigma}_1\gamma_1+\eta^q\widehat{\sigma}_2\gamma_2\},$$

which implies that $\frac{\|x\|}{\frac{\|g\|Q(\|x\|)}{\Gamma(q+1)}\{\varepsilon+\widehat{\sigma}_1\gamma_1+\eta^q\widehat{\sigma}_2\gamma_2\}} \le 1.$

In view of $(H_3)$, there exists $M$ such that $\|x\| \neq M$. Let us set $U = \{x \in \mathcal{C} : \|x\| < M\}$. Note that the operator $\mathcal{F} : \overline{U} \to \mathcal{P}(\mathcal{C})$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of $U$, there is no $x \in \partial U$ such that $x \in \theta\mathcal{F}(x)$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 2.7), we deduce that $\mathcal{F}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1)-(2). This completes the proof. □

### 3.2 The Lipschitz case

In this subsection we prove the existence of solutions for the problem (1)-(2) with a not necessary nonconvex valued right-hand side, by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [6].

Let $(X,d)$ be a metric space induced from the normed space $(X;\|\cdot\|)$. Let $H_d : \mathcal{P}(X)\times\mathcal{P}(X) \to \mathbb{R}\cup\{\infty\}$ be defined by $H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$, where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space (see [13]).

Definition 3.3. Let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called $(i)$ $\gamma$-Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(N(x),N(y)) \le \gamma d(x,y)$ for each $x,y \in X$; and $(ii)$ a contraction if and only if it is $\gamma$-Lipschitz with $\gamma < 1$.

Lemma 3.4 ([6]). *Let $(X,d)$ be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Theorem 3.5. *Assume that:*
*($A_1$) $F:[0,1]\times\mathbb{R}\to\mathcal{P}_{cp}(\mathbb{R})$ ($\mathcal{P}_{cp}(\mathbb{R})=\{Y\in\mathcal{P}(\mathbb{R}): Y$ is compact$\}$) is such that $F(\cdot,x):[0,1]\to\mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x\in\mathbb{R}$;*

*($A_2$) $H_d(F(t,x),F(t,\bar{x}))\le m(t)|x-\bar{x}|$ for almost all $t\in[0,1]$ and $x,\bar{x}\in\mathbb{R}$ with $m\in C([0,1],\mathbb{R}^+)$ and $d(0,F(t,0))\le m(t)$ for almost all $t\in[0,1]$.*
*Then the boundary value problem (1)-(2), with $a_1^2-4a_0a_2>0$, has at least one solution on $[0,1]$ if $\frac{\|m\|}{\Gamma(q+1)}\{\varepsilon+\widehat{\sigma}_1\gamma_1+\eta^q\widehat{\sigma}_2\gamma_2\}<1$.*

*Proof.* Consider the operator $\mathcal{F}$ defined at the beginning of the proof of Theorem 3.2. Observe that the set $S_{F,x}$ is nonempty for each $x\in\mathcal{C}$ by the assumption ($A_1$), so $F$ has a measurable selection (see [5, Theorem III.6]). Now we show that the operator $\mathcal{F}$ satisfies the assumptions of Lemma 3.4. We show that $\mathcal{F}(x)\in\mathcal{P}_{cl}(\mathcal{C})$ for each $x\in C([0,1],\mathbb{R})$. Let $\{u_n\}_{n\ge 0}\in\mathcal{F}(x)$ be such that $u_n\to u$ $(n\to\infty)$ in $\mathcal{C}$. Then $u\in\mathcal{C}$ and there exists $v_n\in S_{F,x_n}$ such that, for each $t\in[0,1]$,

$$u_n(t)=\frac{1}{a_2(m_2-m_1)}\Bigg\{\int_0^t\int_0^s\Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}v_n(u)\,du\,ds$$
$$+\sigma_1(t)\int_0^1\int_0^s\Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}v_n(u)\,du\,ds+\sigma_2(t)\int_0^\eta\int_0^s\Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}v_n(u)\,du\,ds\Bigg\}.$$

As $F$ has compact values, we pass onto a subsequence (if necessary) to obtain that $v_n$ converges to $v$ in $L^1([0,1],\mathbb{R})$. Thus, $v\in S_{F,x}$ and for each $t\in[0,1]$, we have

$$u_n(t)\to v(t)=\frac{1}{a_2(m_2-m_1)}\Bigg\{\int_0^t\int_0^s\Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds$$
$$+\sigma_1(t)\int_0^1\int_0^s\Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds+\sigma_2(t)\int_0^\eta\int_0^s\Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds\Bigg\}.$$

Hence, $u\in\mathcal{F}(x)$.

Next we show that there exists $\delta<1$ $\left(\delta:=\frac{\|m\|}{\Gamma(q+1)}\{\varepsilon+\widehat{\sigma}_1\gamma_1+\eta^q\widehat{\sigma}_2\gamma_2\}\right)$ such that $H_d(\mathcal{F}(x),\mathcal{F}(\bar{x}))\le\delta\|x-\bar{x}\|$ for each $x,\bar{x}\in\mathcal{C}$. Let $x,\bar{x}\in\mathcal{C}$ and $h_1\in\mathcal{F}(x)$. Then there exists $v_1(t)\in F(t,x(t))$ such that, for each $t\in[0,1]$,

$$h_1(t)=\frac{1}{a_2(m_2-m_1)}\Bigg\{\int_0^t\int_0^s\Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}v_1(u)\,du\,ds$$
$$+\sigma_1(t)\int_0^1\int_0^s\Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}v_1(u)\,du\,ds+\sigma_2(t)\int_0^\eta\int_0^s\Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}v_1(u)\,du\,ds\Bigg\}.$$

By 2, we have $H_d(F(t,x),F(t,\bar{x}))\le m(t)|x(t)-\bar{x}(t)|$. So, there exists $w\in F(t,\bar{x}(t))$ such that $|v_1(t)-w|\le m(t)|x(t)-\bar{x}(t)|$, $t\in[0,1]$.

Define $U:[0,1]\to\mathcal{P}(\mathbb{R})$ by $U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\le m(t)|x(t)-\bar{x}(t)|\}$. Since the multivalued operator $U(t)\cap F(t,\bar{x}(t))$ is measurable ( [5, Proposition III.4]), there exists a function $v_2(t)$ which is a measurable selection for $U$. So $v_2(t)\in F(t,\bar{x}(t))$ and for each $t\in[0,1]$, we have $|v_1(t)-v_2(t)|\le m(t)|x(t)-\bar{x}(t)|$.

For each $t \in [0,1]$, let us define

$$h_2(t) = \frac{1}{a_2(m_2-m_1)}\Bigg\{ \int_0^t \int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)} v_2(u)\, du\; ds$$

$$+ \sigma_1(t)\int_0^1 \int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)} v_2(u)\, du\; ds + \sigma_2(t)\int_0^\eta \int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)} v_2(u)\, du\; ds\Bigg\}.$$

Thus,

$$|h_1(t)-h_2(t)| \leq \frac{1}{a_2(m_2-m_1)}\Bigg\{ \int_0^t \int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)} |v_1(u)-v_2(u)|\, du\; ds$$

$$+ \sigma_1(t)\int_0^1 \int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)} |v_1(u)-v_2(u)|\, du\; ds$$

$$+ \sigma_2(t)\int_0^\eta \int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)} |v_1(u)-v_2(u)|\, du\; ds\Bigg\}$$

$$\leq \frac{\|m\|}{a_2(m_2-m_1)\Gamma(q+1)} \sup_{t\in[0,1]} \Bigg\{ \int_0^t \Big(e^{m_2(t-s)} - e^{m_1(t-s)}\Big) ds$$

$$+ |\sigma_1(t)|\int_0^1 \Big(e^{m_2(1-s)} - e^{m_1(1-s)}\Big) ds + \eta^q|\sigma_2(t)|\int_0^\eta \Big(e^{m_2(\eta-s)} - e^{m_1(\eta-s)}\Big) ds\Bigg\}\|x-\bar{x}\|$$

$$\leq \frac{\|m\|}{\Gamma(q+1)}\big\{\varepsilon + \widehat{\sigma}_1\gamma_1 + \eta^q\widehat{\sigma}_2\gamma_2\big\}\|x-\bar{x}\|.$$

Hence, $\|h_1-h_2\| \leq \frac{\|m\|}{\Gamma(q+1)}\big\{\varepsilon + \widehat{\sigma}_1\gamma_1 + \eta^q\widehat{\sigma}_2\gamma_2\big\}\|x-\bar{x}\|$.

Analogously, interchanging the roles of $x$ and $\overline{x}$, we obtain $H_d(\mathcal{F}(x),\mathcal{F}(\bar{x})) \leq \frac{\|m\|}{\Gamma(q+1)}\big\{\varepsilon + \widehat{\sigma}_1\gamma_1 + \eta^q\widehat{\sigma}_2\gamma_2\big\}\|x-\bar{x}\|$.

So $\mathcal{F}$ is a contraction. Therefore, it follows by Lemma 3.4 that $\mathcal{F}$ has a fixed point $x$ which is a solution of (1)-(2). □

### 3.3 The lower semicontinuous case

In the next result, $F$ is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [4] for lower semi-continuous maps with decomposable values.

Let $X$ be a nonempty closed subset of a Banach space $E$ and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. $G$ is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y)\cap B \neq \emptyset\}$ is open for any open set $B$ in $E$. Let $A$ be a subset of $[0,1]\times\mathbb{R}$. $A$ is $\mathcal{L}\otimes\mathcal{B}$ measurable if $A$ belongs to the $\sigma$-algebra generated by all sets of the form $\mathcal{J}\times\mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J}\subset[0,1]=J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 3.6. Let $Y$ be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say that $N$ has a property (BC) if $N$ is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with $F$ as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with $F$.

Definition 3.7. Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say that $F$ is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 3.8 ([8]). *Let $Y$ be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then $N$ has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 3.9. *Assume that ($H_2$), ($H_3$) and the following condition holds:*
*($H_4$) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*

*(a) $(t,x) \longmapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*

*(b) $x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,1]$;*

*Then the boundary value problem* (1)-(2), *with* $a_1^2 - 4a_0a_2 > 0$, *has at least one solution on* $[0,1]$.

*Proof.* It follows from $(H_2)$ and $(H_4)$ that $F$ is of l.s.c. type. Then from Lemma 3.8, there exists a continuous function $f : \mathcal{C} \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in \mathcal{C}$.

Consider the problem

$$\begin{cases} (a_2\ {}^cD^{q+2} + a_1\ {}^cD^{q+1} + a_0\ {}^cD^q)x(t) = f(x(t)), \quad 0 < q < 1, \quad 0 < t < 1, \\ x(0) = 0, \quad x(\eta) = 0, \quad x(1) = 0, \quad 0 < \eta < 1. \end{cases} \tag{5}$$

Observe that if $x \in \mathcal{C}$ is a solution of (5) in the sense of Definition 3.1, then $x$ is a solution to the problem (1)-(2). In order to transform the problem (5) into a fixed point problem, we define an operator $\overline{\mathcal{F}}$ by

$$\overline{\mathcal{F}}x(t) = \frac{1}{a_2(m_2 - m_1)} \left\{ \int_0^t \int_0^s \Phi(t) \frac{(s-u)^{q-1}}{\Gamma(q)} f(x(u))\, du\, ds \right.$$
$$\left. + \sigma_1(t) \int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{q-1}}{\Gamma(q)} f(x(u))\, du\, ds + \sigma_2(t) \int_0^\eta \int_0^s \Phi(\eta) \frac{(s-u)^{q-1}}{\Gamma(q)} f(x(u))\, du\, ds \right\}.$$

It can be easily shown that $\overline{\mathcal{F}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.2. □

## 4. Existence results for the case $a_1^2 - 4a_0a_2 = 0$.

In this section, we discuss the existence of solutions for the problem (1)-(2) when $a_1^2 - 4a_0a_2 = 0$. Let us first define a solution of the problem (1)-(2) in this case.

DEFINITION 4.1. Let $a_1^2 - 4a_0a_2 = 0$. A function $x \in \mathcal{C}$, possessing a Caputo derivative of at most order $q+2$, is a solution of the problem (1)-(2) if $x(0) = 0$, $x(\eta) = 0$, $x(1) = 0$, and there exists function $v \in L^1([0,1], \mathbb{R})$ such that $v(t) \in F(t, x(t))$ a.e. on $[0,1]$ and

$$x(t) = \frac{1}{a_2}\Bigg\{ \int_0^t \int_0^s \Psi(t) \frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\, du\; ds + \psi_1(t) \int_0^1 \int_0^s \Psi(1) \frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\, du\; ds$$
$$+ \psi_2(t) \int_0^\eta \int_0^s \Psi(\eta) \frac{(s-u)^{q-1}}{\Gamma(q)} v(u)\, du\; ds \Bigg\}. \tag{6}$$

Analogously to Theorems 3.2, 3.5, and 3.9 proved in the last section, we present the existence results for the problem (1)-(2) when $a_1^2 - 4a_0a_2 = 0$. We do not provide the proofs of the proposed results as those are similar to the ones for Theorems 3.2, 3.5, and 3.9.

THEOREM 4.2. *Assume that* $(H_1)$*,* $(H_2)$ *are satisfied. In addition we assume that:*
$(H_3')$ *There exists a constant* $M_1 > 0$ *such that:* $\frac{M_1}{\|g\| Q(M_1)\mu} > 1$,
*where*

$$\widehat{\psi}_1 = \max_{t\in[0,1]} |\psi_1(t)|, \quad \widehat{\psi}_2 = \max_{t\in[0,1]} |\psi_2(t)|,$$
$$\mu = \frac{1}{a_2 m^2 \Gamma(q+1)} \Big\{ (1+\widehat{\psi}_1)\Big((m-1)e^m + 1\Big) + \widehat{\psi}_2 \eta^q \Big((m\eta - 1)e^{m\eta} + 1\Big) \Big\}. \tag{7}$$

*Then the boundary value problem* (1)-(2)*, with* $a_1^2 - 4a_0a_2 = 0$*, has at least one solution on* $[0,1]$.

THEOREM 4.3. *Assume that* $(A_1)$*,* $(A_2)$ *are satisfied. Then the boundary value problem* (1)-(2)*, with* $a_1^2 - 4a_0a_2 = 0$*, has at least one solution on* $[0,1]$ *if* $\|m\|\mu < 1$.

THEOREM 4.4. *Assume that* $(H_2)$*,* $(H_3')$ *and* $(H_4)$ *are satisfied. Then the boundary value problem* (1)-(2)*, with* $a_1^2 - 4a_0a_2 = 0$*, has at least one solution on* $[0,1]$.

## 5. Existence results for the case $a_1^2 - 4a_0a_2 < 0$.

This section is devoted to the existence results for the problem (1)-(2) with $a_1^2 - 4a_0a_2 < 0$. Before presenting the main results, we define a solution of the problem (1)-(2) in this case.

DEFINITION 5.1. Let $a_1^2 - 4a_0a_2 < 0$. A function $x \in \mathcal{C}$, possessing a Caputo derivative of at most order $q+2$, is a solution of the problem (1)-(2) if $x(0) = 0$, $x(\eta) = 0$, $x(1) =$

0, and there exists a function $v \in L^1([0,1],\mathbb{R})$ such that $v(t) \in F(t,x(t))$ a.e. on $[0,1]$ and

$$x(t) = \frac{1}{a_2\beta}\Bigg\{\int_0^t\int_0^s \Omega(t)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds$$
$$+\varphi_1(t)\int_0^1\int_0^s \Omega(1)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds + \varphi_2(t)\int_0^\eta\int_0^s \Omega(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)\,du\,ds\Bigg\}.$$

Now we give the existence results for the problem (1)-(2) in case of $a_1^2 - 4a_0a_2 < 0$ without proof. One can prove these results following the strategy employed in the proofs of Theorems 3.2, 3.5, and 3.9.

Theorem 5.2. *Assume that* ($H_1$), ($H_2$) *are satisfied. In addition we assume that:* ($H_3''$) *There exists a constant* $M_2 > 0$ *such that:* $\frac{M_2}{\|g\|Q(M_2)\rho} > 1$, *where*

$$\widehat{\varphi}_1 = \max_{t\in[0,1]}|\varphi_1(t)|, \quad \widehat{\varphi}_2 = \max_{t\in[0,1]}|\varphi_2(t)|,$$
$$\rho = \frac{1}{a_2(\alpha^2+\beta^2)\Gamma(q+1)}\Big\{(1+\widehat{\varphi}_1)\Big(1 - e^{-\alpha}\cos\beta - (\alpha/\beta)e^{-\alpha}\sin\beta\Big)$$
$$+ \widehat{\varphi}_2\eta^q\Big(1 - e^{-\alpha\eta}\cos\beta\eta - (\alpha/\beta)e^{-\alpha\eta}\sin\beta\eta\Big)\Big\}.$$

*Then the boundary value problem* (1)-(2), *with* $a_1^2 - 4a_0a_2 < 0$, *has at least one solution on* $[0,1]$.

Theorem 5.3. *Assume that* ($A_1$), ($A_2$) *are satisfied. Then the boundary value problem* (1)-(2), *with* $a_1^2 - 4a_0a_2 < 0$, *has at least one solution on* $[0,1]$ *if* $\|m\|\rho < 1$.

Theorem 5.4. *Assume that* ($H_2$), ($H_3''$) *and* ($H_4$) *are satisfied. Then the boundary value problem* (1)-(2), *with* $a_1^2 - 4a_0a_2 < 0$, *has at least one solution on* $[0,1]$.

## 6. Examples

Example 6.1. Consider the following boundary value problem for fractional differential inclusions

$$\begin{cases} ({}^cD^{5/2} + 5\,{}^cD^{3/2} + 4\,{}^cD^{1/2})x(t) \in F(t,x(t)), \quad 0<q<1, \quad 0<t<1, \\ x(0)=0, \quad x(4/5)=0, \quad x(1)=0, \end{cases} \tag{8}$$

where $F(t,x(t)) = \left[\frac{2}{\sqrt{t^2+64}}\Big(\frac{|x(t)|}{3}\Big(\frac{|x(t)|}{|x(t)|+1}+2\Big)+1\Big), \quad \frac{e^{-t}}{9+t}\Big(\sin x(t) + \frac{1}{80}\Big)\right]$.

Here $q = 1/2$, $\eta = 4/5$, $a_1^2 - 4a_0a_2 = 9 > 0$. Clearly $|F(t,x(t))| \le g(t)Q(\|x\|)$, where $g(t) = \frac{2}{\sqrt{t^2+64}}$ and $Q(\|x\|) = \|x\|+1$. Using the value $\frac{\varepsilon+\widehat{\sigma}_1\gamma_1+\eta^q\widehat{\sigma}_2\gamma_2}{\Gamma(q+1)} \approx 0.75241$, we find that $M > 0.231682$. Since the hypotheses of Theorem 3.2 are satisfied, the problem (8) has at least one solution on $[0,1]$.

Example 6.2. Consider the following boundary value problem for fractional differential inclusions

$$\begin{cases} ({}^cD^{7/3} + 6\,{}^cD^{4/3} + 5\,{}^cD^{1/3})x(t) \in F(t, x(t)), \quad 0 < q < 1, \quad 0 < t < 1, \\ x(0) = 0, \quad x(3/4) = 0, \quad x(1) = 0, \end{cases} \tag{9}$$

where $F(t, x(t)) = \left[ \frac{1}{\sqrt{81+t^2}}, \quad \frac{\sin x(t)}{(2-t)^2} + \frac{2}{49} \right]$.

Here $q = 1/3$, $\eta = 3/4$, $a_1^2 - 4a_0a_2 = 16 > 0$. Clearly $H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x - \bar{x}|$, where $m(t) = \frac{1}{(2-t)^2}$. We find that $\frac{\|m\|}{\Gamma(q+1)}\left\{\varepsilon + \widehat{\sigma}_1\gamma_1 + \eta^q\widehat{\sigma}_2\gamma_2\right\} \approx 0.123565 < 1$. Since the hypotheses of Theorem 3.5 are satisfied, the problem (9) has at least one solution on $[0, 1]$.

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Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Saudi Arabia

*E-mail*: aalsaedi@hotmail.com

Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Saudi Arabia

Department of Mathematics, Faculty of Science, University of Jeddah, Saudi Arabia

*E-mail*: njl-ghamdi@hotmail.com

Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Saudi Arabia

*E-mail*: bashirahmad_qau@yahoo.com

Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Saudi Arabia

Department of Mathematics, University of Ioannina, Greece

*E-mail*: sntouyas@uoi.gr