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ON A CLASS OF ELLIPTIC NAVIER BOUNDARY VALUE PROBLEMS INVOLVING THE $(p_1(\cdot), p_2(\cdot))$-BIHARMONIC OPERATOR

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Abstract. In this article, we study the existence and multiplicity of weak solutions for a class of elliptic Navier boundary value problems involving the $(p_1(\cdot), p_2(\cdot))$-biharmonic operator. Our technical approach is based on variational methods and the theory of the variable exponent Lebesgue spaces. We establish the existence of at least one solution and infinitely many solutions of this problem, respectively.

1. Introduction

In recent years, the study of differential equations and variational problems with $p(\cdot)$-growth conditions was an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics. In that context we refer the reader to Ruzicka [14], Zhikov [20] and the reference therein; see also [6, 7].

Fourth-order equations appear in various contexts. Some of these problems come from different areas of applied mathematics and physics such as micro electro-mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells (see [9]). In addition, this type of equations can describe the static from change of beam or the sport of rigid body.

In this work, we consider the problem

$$\begin{cases} \Delta(|\Delta u|^{p_1(x)-2}\Delta u) + \Delta(|\Delta u|^{p_2(x)-2}\Delta u) = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $N \geq 1$, $\Delta^2_{p_i(x)} u := \Delta(|\Delta u|^{p_i(x)-2}\Delta u)$, is the $p_i(\cdot)$-biharmonic operator with $p_i(\cdot)$ for $i = 1, 2$ are continuous functions on $\overline{\Omega}$ with $p_i^- = \inf_{x\in\overline{\Omega}} p_i(x) > N$ and $p_i^+ = \max_{x\in\overline{\Omega}} p_i(x) > N$ for $i = 1, 2$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathèodorey function.

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We point out that elliptic equations involving the $p(\cdot)$-biharmonic equations are not trivial generalizations of similar problems studied in the constant case since the $p(\cdot)$-biharmonic operator is not homogeneous and, thus, some techniques which can be applied in the case of the p-biharmonic operators will fail in that new situation, such as the Lagrange Multiplier Theorem.

The class of $p(\cdot)$-biharmonic equations was considered by many authors in recent years. Many researchers have investigated into beam equations under various boundary conditions and through different approaches.

For example, in [3] the authors studied a class of $p(\cdot)$-biharmonic of the form

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda |u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $N \geq 1$, $\lambda \geq 0$, by using variational methods based on the Mountain Pass Lemma and Ekeland's Variational Principle, they established several existence criteria for eigenvalues.

By an approach based mainly on an adequate variational techniques, in [15], the authors studied some problems with indefinite weight under Neumann boundary conditions and Navier boundary conditions.

In the case where $p(\cdot) = q(\cdot)$, the authors in [2] established the existence of infinitely many eigenvalue sequences for the problem (2) by using the Ljusternik-Schnirelmann theory on C^1-manifolds.

In [11], Lin Li et al. considered the above problem and using variational methods, by some assumptions on the Carathéodory function f, using the mountain pass theorem, fountain theorem, local linking theorem and symmetric mountain pass theorem, they have established the existence of at least one solution and infinitely many solutions of this problem, respectively.

In another direction, the authors in [5] have considered the fourth-order quasilinear elliptic equation and using variational methods, by some assumptions on the Carathéodory function f, they have established the existence of three solutions for the problem of the form

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) + a(x)|u|^{p(x)-2}u = f(x,u) + \lambda g(x,u) \quad \text{in } \Omega,$$
$$Bu = Tu = 0 \quad \text{on } \partial\Omega,$$

where $Bu = Tu = 0$ denotes the following boundary conditions:

1. $B = B_1$, $T = T_1$, Navier boundary condition, i.e. $B_1 u = \Delta u = 0$ and $T_1 u = u = 0$ on $\partial\Omega$,
2. $B = B_2$, $T = T_2$, Neumann boundary condition, i.e $B_2 u = \frac{\partial u}{\partial \nu} = 0$ and $T_2 u = \frac{\partial}{\partial \nu}\left(|\Delta u|^{p(x)-2}u\right) = 0$ where ν is the outward unit normal.

Inspired by the above references, based on the use of Kransnoselskii genus and Mountain Pass Theorem, the aim of this article is to establish, respectively, the existence of at least one solution and infinitely many solutions of problem (1) under the following assumptions:

($\boldsymbol{F}_0$) $f : \Omega \times \mathbb{R}$ satisfies $\mid f(x,t) \mid \leq c_1(1 + |t|^{r(x)-1})$, for all $(x,t) \in \Omega \times \mathbb{R}$, where

$r(x) \in C_+(\overline{\Omega})$, $r(x) < p^*(x)$ for all $x \in \overline{\Omega}$ $p_i^- > r^- := \inf_{x\in\overline{\Omega}} r(x)$ for $i = 1,2$ and c_1 is a positive constant,

($\boldsymbol{F}_1$) $f(x,t) = o(|t|^{r(x)-1})$ as $t \to 0$ and for all $x \in \overline{\Omega}$,

($\boldsymbol{F}_2$) $f(x,t) \geq c_2|t|^{\alpha(x)-1}$ as $t \to 0$ and for all $x \in \overline{\Omega}$, where $\alpha^+ = \sup_{x\in\overline{\Omega}} \alpha(x) < \min_{i=1,2} p_i^-$,

($\boldsymbol{F}_3$) $\limsup_{|t|\to\infty} \dfrac{F(x,t)}{|t|^{\theta(x)}} \leq a(x)$ such that $\theta \in C_+(\overline{\Omega})$, with $\theta^+ = \sup_{x\in\overline{\Omega}} \theta(x) < \min_{i=1,2} p_i^-$, where $F(x,t) = \int_0^t f(x,s)\,ds$ and $a \in L^\infty(\Omega)$,

($\boldsymbol{F}_4$) there exists $M > 0$, $\beta > \max_{i=1,2} p_i^+$ such that for all $x \in \Omega$ and all $t \in \mathbb{R}$ with $|t| > M$, $0 < \beta F(x,t) < f(x,t)t$,

($\boldsymbol{F}_5$) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$.

The main result of this paper is expressed by the following theorems.

THEOREM 1.1. *Assume that (*$\boldsymbol{F}_0$*) and (*$\boldsymbol{F}_3$*) hold. Then the problem* (1) *has a weak solution.*

THEOREM 1.2. *If (*$\boldsymbol{F}_0$*), (*$\boldsymbol{F}_1$*) and (*$\boldsymbol{F}_4$*) hold, then the problem* (1) *has a nontrivial weak solution.*

THEOREM 1.3. *If (*$\boldsymbol{F}_0$*)-(*$\boldsymbol{F}_2$*) and (*$\boldsymbol{F}_4$*)-(*$\boldsymbol{F}_5$*) hold., then the problem* (1) *has infinitely many weak solutions.*

This paper is organized in three sections. In Section 2, we recall some basic properties of the variable exponent Lebegue-Sobolev spaces. In Section 3, we give the proof of the main result.

2. Preliminaries

For the sake of completeness of this paper, we need to recall some results on the variable exponent spaces $L^{p(\cdot)}(\Omega)$ and $W^{k,p(\cdot)}(\Omega)$, and some properties which we use later. Let Ω be a bounded domain of $\mathbb{R}^N$ and denote

$$C_+(\overline{\Omega}) = \Big\{h(x): \quad h(x) \in C(\overline{\Omega}), \quad h(x) > 1, \quad \forall x \in \overline{\Omega}\Big\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \max\Big\{h(x):\ x \in \overline{\Omega}\Big\}, \quad h^- = \min\Big\{h(x):\ x \in \overline{\Omega}\Big\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\},$$

endowed with the so-called *Luxemburg norm* $|u|_{p(\cdot)} = \inf\Big\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(\cdot)}\,dx \leq 1\Big\}$. Then $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a Banach space.

Proposition 2.1 ([8]). *The space* $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ *is separable, uniformly convex, reflexive and its conjugate space is* $L^{q(\cdot)}(\Omega)$ *where* $q(\cdot)$ *is the conjugate function of* $p(\cdot)$, *i.e.,* $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$. *For* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{q(\cdot)}(\Omega)$, *we have*

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(\cdot))} |v|_{q(\cdot)} \leq 2|u|_{p(\cdot)}|v|_{q(\cdot)}.$$

The Sobolev space with variable exponent $W^{k,p(\cdot)}(\Omega)$ is defined as

$$W^{k,p(\cdot)}(\Omega) = \Big\{ u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \leq k \Big\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}} u$, with multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$ and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(\cdot)}(\Omega)$ equipped with the norm $\|u\|_{k,p(\cdot)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(\cdot)}$, also becomes a separable and reflexive Banach space. For more details, we refer the reader to [8, 12, 18]. For any $k \geq 1$, denote

$$p_k^*(\cdot) = \begin{cases} \frac{Np(\cdot)}{N - kp(\cdot)}, & \text{if} \quad kp(\cdot) < N, \\ +\infty, & \text{if} \quad kp(\cdot) \geq N. \end{cases}$$

Proposition 2.2 ([8]). *For* $p, r \in C_+(\overline{\Omega})$ *such that* $r(\cdot) \leq p_k^*(\cdot)$, *there is a continuous embedding* $W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$. *If we replace* $\leq$ *with* $<$, *the embedding is compact.*

We denote by $W_0^{k,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$. Note that the weak solutions of problem (1) are considered in the generalized Sobolev space $X = X_1 \cap X_2$ equipped with the norm $\|u\| = \|u\|_{p_1} + \|u\|_{p_2}$, where

$$\|u\|_r = \inf \left\{ \mu > 0 : \int_\Omega \Big(\Big| \frac{\Delta u(x)}{\mu} \Big|^{r(\cdot)} \leq 1 \right\}$$

and

$$X_i = \Big(W^{2,p_i(\cdot)}(\Omega) \cap W_0^{1,p_i(\cdot)}(\Omega) \Big), \quad i = 1, 2,$$

equipped with the norm $\|u\| = \|u\|_{p_i}$.

Remark 2.3. According to [19, Theorem 4.4.], the norm $\|\cdot\|_{2,p(\cdot)}$ is equivalent to the norm $|\cdot|_{p(\cdot)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(\cdot)}$, $\|\cdot\|$ and $|\cdot|_{p(\cdot)}$ are equivalent.

Proposition 2.4 ([4]). *If we denote* $\rho(u) = \int_\Omega |\Delta u|^{p(\cdot)}\, dx$, *then for* $u, u_n \in X$, *we have*

(i) $\|u\|_p < 1$ *(respectively=1;* > 1*)* $\iff \rho(u) < 1$ *(respectively* $= 1$; > 1*);*

(ii) $\|u\|_p \leq 1 \Rightarrow \|u\|_p^{p^+} \leq \rho(u) \leq \|u\|_p^{p^-}$;

(iii) $\|u\|_p \geq 1 \Rightarrow \|u\|_p^{p^-} \leq \rho(u) \leq \|u\|_p^{p^+}$;

(iv) $\|u_n\|_p \to 0$ *(respectively* $\to \infty$*)* $\iff \rho(u_n) \to 0$ *(respectively* $\to \infty$*).*

Let us define the functional

$$J(u) = \int_\Omega \frac{1}{p_1(x)} |\Delta u|^{p_1(x)}\, dx + \int_\Omega \frac{1}{p_2(x)} |\Delta u|^{p_2(x)}\, dx, \quad \forall u \in X.$$

Using the same arguments as in [3, Proposition 2.5] and [1, Proposition 1.6], we can show the following lemma.

LEMMA 2.5. *$J \in C^1(X,\mathbb{R})$ and derivative operator J' of J is*

$$J'(u)v = \int_\Omega |\Delta u|^{p_1(x)-2}\Delta u \Delta v\, dx + \int_\Omega |\Delta u|^{p_2(x)-2}\Delta u \Delta v\, dx, \forall u,v \in X.$$

and we have
(i) $J' : X \to X^*$ is a bounded homeomorphism and strictly monotone operator,

(ii) J' is a mapping of type (S_+), namely $u_n \rightharpoonup u$ and $\limsup_{n\to\infty} J'(u_n)(u_n - u) \le 0$ implies $u_n \to u$.

The main tool used in proving Theorem 1.2 is the well known Mountain Pass Theorem. It remains to show that there exists an $e \in X$ with $\|e\| > \rho$ and $I(e) \le 0$. Further details can be found in [13, 16].

In order to establish the existence of infinitely many solutions for the problem (1), we will use the Kransnoselskii genus and more information on this subject may be found [10].

Let E be a real Banach space and denote by $\sum$ the class of closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

DEFINITION 2.6. Let $A \in \sum$. The Kransnoselskii genus $\gamma(A)$ is defined as being the least positive integer n such that there is an odd mapping $\varphi \in C(A, \mathbb{R}^n - \{0\})$. When such number does not exist, we consider $\gamma(A) = +\infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.

THEOREM 2.7 ([10]). *Let $E = \mathbb{R}^\mathbb{N}$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^\mathbb{N}$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

PROPOSITION 2.8 ([10]). *Let $A, B \in \sum$. Then:*
(i) if there exists an odd map $f \in C(A,B)$, then $\gamma(A) \le \gamma(B)$. Consequently, if there exists an odd homeomorphism $f : A \to B$, then $\gamma(A) = \gamma(B)$.

(ii) if $A \subset B$,then $\gamma(A) \le \gamma(B)$.

(iii) $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$.

THEOREM 2.9 ([10]). *Let $I \in C^1(E,\mathbb{R})$ be a functional satisfying the Palais-Smale condition. Also suppose that:*
(i) I is bounded from below and even;

(ii) there is a compact set $K \in \sum$ such that $\gamma(K) = k$ and $\sup_{x\in K} I(x) < I(0)$. Then I possesses at least k pairs of distinct critical points and their corresponding critical values are less than $I(0)$.

3. Existence and muliplicity of weak solutions

In this section, we well state and prove our main result.

DEFINITION 3.1. We say that $u \in X$ is a weak solution of (1) if

$$\int_\Omega |\Delta u|^{p_1(x)-2}\Delta u \Delta v\, dx + \int_\Omega |\Delta u|^{p_2(x)-2}\Delta u \Delta v\, dx = \int_\Omega f(x,u)v\, dx,$$

for all $v \in X$.

The functional associated to (1) is given by

$$I(u) = \int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}\, dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}\, dx - \int_\Omega F(x,u)\, dx. \tag{3}$$

Note that under the condition $(\boldsymbol{F}_0)$ the functional I is of class $C^1(X,\mathbb{R})$ and

$$I'(u)v = \int_\Omega |\Delta u|^{p_1(x)-2}\Delta u \Delta v\, dx + \int_\Omega |\Delta u|^{p_2(x)-2}\Delta u \Delta v\, dx - \int_\Omega f(x,u)v\, dx, \forall u, v \in X.$$

Then, we know that the weak solution of (1) corresponds to the critical point of the functional I.

Proof of Theorem 1.1

LEMMA 3.2. *Under the assumptions $(\boldsymbol{F}_1)$ and $(\boldsymbol{F}_3)$, I is sequentially weakly lower semi continuous and coercive.*

Proof. From the continuity of F and assumption $(\boldsymbol{F}_3)$ we deduce that there exists a positive constant c_3 such that $F(x,u) \le a(x)|u|^{\theta(x)} + c_3$, $\forall u \in \mathbb{R}\ \forall x \in \Omega$. We have for $\|u\| > 1$ that

$$\begin{aligned} I(u) =& \int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}\, dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}\, dx - \int_\Omega F(x,u)\, dx \\ \ge& \frac{1}{p_1^+}\int_\Omega |\Delta u|^{p_1(x)}\, dx + \frac{1}{p_2^+}\int_\Omega |\Delta u|^{p_2(x)}\, dx - \int_\Omega \left(a(x)|u|^{\theta(x)} + c_3\right)\, dx \\ \ge& \frac{1}{p_1^+}\int_\Omega |\Delta u|^{p_1(x)}\, dx + \frac{1}{p_2^+}\int_\Omega |\Delta u|^{p_2(x)}\, dx - |a|_\infty \int_\Omega |u|^{\theta(x)}\, dx - c_3|\Omega|. \end{aligned}$$

Since $p_1^- > N$ and $p_2^- > N$, the embeddings $W_0^{1,p_1(x)}(\Omega) \hookrightarrow C(\overline{\Omega})$ and $W_0^{1,p_2(x)}(\Omega) \hookrightarrow C(\overline{\Omega})$ are continuous, so there exist positive constants c_4 and c_5 such that

$$\int_\Omega |u|^{\theta(x)}\, dx \le \int_\Omega |u|_\infty^{\theta(x)}\, dx \le c_4 \int_\Omega \|u\|_{p_1}^{\theta(x)}\, dx \le c_4\|u\|_{p_1}^{\theta^+}$$

and

$$\int_\Omega |u|^{\theta(x)}\, dx \le \int_\Omega |u|_\infty^{\theta(x)}\, dx \le c_5 \int_\Omega \|u\|_{p_2}^{\theta(x)}\, dx \le c_5\|u\|_{p_2}^{\theta^+}.$$

This implies that

$$\begin{aligned} I(u) \ge& \frac{1}{p_1^+}\int_\Omega |\Delta u|^{p_1(x)}\, dx + \frac{1}{p_2^+}\int_\Omega |\Delta u|^{p_2(x)}\, dx - |a|_\infty \int_\Omega |u|^{\theta(x)}\, dx - c_3|\Omega| \\ \ge& \frac{1}{p_1^+}\parallel u \parallel_{p_1}^{p_1^-} + \frac{1}{p_2^+}\|u\|_{p_2}^{p_2^-} - |a|_\infty\left(c_5\|u\|_{p_2}^{\theta^+} + c_4\|u\|_{p_1}^{\theta^+}\right) - c_3|\Omega| \\ \ge& \begin{cases} \frac{2}{p_1^+}\|u\|^{p_1^-} - |a|_\infty \max(c_4,c_5)\|u\|^{\theta^+} - c_3|\Omega|, & \text{if } p_1^- \ge p_2^+ \\ \frac{2}{p_2^+}\|u\|^{p_2^-} - |a|_\infty \max(c_4,c_5)\|u\|^{\theta^+} - c_3|\Omega|, & \text{if } p_1^+ \le p_2^-. \end{cases} \end{aligned}$$

Since $\theta^+ \leq \min_{i=1,2} p_i^-$ then I is coercive. As the function $u \mapsto \int_\Omega F(x,u)\,dx$ is weakly lower semi-continuous and J is convex uniformly, we deduce that I is weakly lower semi-continuous. Therefore I has a global minimum point $u \in X$, that is a weak solution to problem (1). □

Proof of Theorem 1.2

We first start with the following lemma.

LEMMA 3.3. *If ($\boldsymbol{F}_0$) and ($\boldsymbol{F}_4$) hold, then I satisfies the Palais-smale condition (PS) in X, namely, if any sequence (u_n) such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to +\infty$, has convergent subsequence.*

Proof. Suppose that

$$I(u_n) \text{ is bounded and } I'(u_n) \to 0 \text{ as } n \to +\infty. \tag{4}$$

Since $p_i^- > r^-$ for $i = 1, 2$, then

$$\int_\Omega |u|\,dx \leq c_6 \|u\|_{p_1} \text{ and } \int_\Omega |u|\,dx \leq c_7 \|u\|_{p_2}, \tag{5}$$

$$\int_\Omega |u|^{r(x)}\,dx \leq c_8 \|u\|_r^{r^+} + c_9 \|u\|_r^{r^-}. \tag{6}$$

By (5), (6), ($\boldsymbol{F}_4$) and Proposition 2.4 we have

$$\begin{aligned}
&I(u_n) - \frac{1}{\beta} I'(u_n)u_n \\
&= \int_\Omega \frac{1}{p_1(x)} |\Delta u_n|^{p_1(x)}\,dx + \int_\Omega \frac{1}{p_2(x)} |\Delta u_n|^{p_2(x)}\,dx - \int_\Omega F(x,u_n)\,dx \\
&\quad - \frac{1}{\beta}\left(\int_\Omega |\Delta u_n|^{p_1(x)}\,dx + \int_\Omega |\Delta u_n|^{p_2(x)}\,dx - \int_\Omega f(x,u_n)u_n\,dx\right) \\
&\geq \left(\frac{1}{p_1^+} - \frac{1}{\beta}\right) \max(\|u_n\|_{p_1}^{p_1^+}, \|u_n\|_{p_1}^{p_1^-}) + \left(\frac{1}{p_2^+} - \frac{1}{\beta}\right) \max(\|u_n\|_{p_2}^{p_2^+}, \|u_n\|_{p_2}^{p_2^-}) \\
&\quad + \int_\Omega \left[\frac{1}{\beta} f(x,u_n)u_n - F(x,u_n)\right] dx \\
&\geq \left(\frac{1}{p_1^+} - \frac{1}{\beta}\right) \max(\|u_n\|_{p_1}^{p_1^+}, \|u_n\|_{p_1}^{p_1^-}) + \left(\frac{1}{p_2^+} - \frac{1}{\beta}\right) \max(\|u_n\|_{p_2}^{p_2^+}, \|u_n\|_{p_2}^{p_2^-}).
\end{aligned}$$

Since $p_i^- > r^-$ for $i = 1, 2$, then the sequence (u_n) is bounded. Thus, passing to a subsequence if necessary, there exists $u \in X$ such that $u_n \rightharpoonup u$ weakly in X. Thanks to the compact embedding $X \hookrightarrow L^{r(x)}(\Omega)$, we get $u_n \to u$ *in* $L^{r(x)}(\Omega)$, $u_n(x) \to u(x)$ a. e. $x \in \Omega$. Since (u_n) is bounded sequence and by (4), we have

$$\begin{aligned}
I'(u_n)(u_n - u) &= \int_\Omega |\Delta u_n|^{p_1(x)-2} \Delta u_n \Delta(u_n - u)\,dx + \int_\Omega |\Delta u_n|^{p_2(x)-2} \Delta u_n \Delta(u_n - u)\,dx \\
&\quad - \int_\Omega f(x,u_n)(u_n - u)\,dx \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Using the condition ($\boldsymbol{F}_0$) and the Hölder inequality, we deduce that

$$\begin{aligned}\left|\int_\Omega f(x,u_n)(u_n-u)\,dx\right| &\le \int_\Omega |f(x,u_n)||(u_n-u)|\,dx\\ &\le c_1\int_\Omega |(1+|u_n|^{r(x)-1})||(u_n-u)|\,dx\\ &\le C_1\left(|1|_{\frac{r(x)}{r(x)-1}}+\left||u_n|^{r(x)-1}\right|_{\frac{r(x)}{r(x)-1}}\right)|(u_n-u)|_{r(x)}\to 0\,\text{as}\,n\to\infty,\end{aligned} \tag{7}$$

which yields

$$\int_\Omega f(x,u_n)(u_n-u)\,dx\to 0\,\text{as}\,n\to+\infty. \tag{8}$$

By (7) and (8), we have

$$\int_\Omega |\Delta u_n|^{p_1(x)-2}\Delta u_n\Delta(u_n-u)\,dx+\int_\Omega |\Delta u_n|^{p_2(x)-2}\Delta u_n\Delta(u_n-u)\,dx\to 0,$$

According to Lemma 2.5, the functional J' is of type (S_+). Thus $u_n\to u$ strongly in X as $n\to+\infty$ and the functional I satisfies the (PS) condition. □

To prove Theorem 1.2, we need to check that I satisfies the conditions of the Mountain Pass Theorem.

Since $r^+<p_1^-$ and $r^+<p_2^-$, $X_1\hookrightarrow L^{r^+}(\Omega)$ and $X_2\hookrightarrow L^{r^+}(\Omega)$ then there exists c_9 and c_{10} such that $|u|_{r^+}\le c_{10}\|u\|_{p_1}$ and $|u|_{r^+}\le c_{11}\|u\|_{p_2}$ for all $u\in X$. Let $\epsilon>0$ be small enough such that $\epsilon c_{10}^{r^+}<\frac{1}{2r^+}$. By assumptions ($\boldsymbol{F}_0$) and ($\boldsymbol{F}_1$), we have

$$F(x,t)\le\epsilon|t|^{r^+}+c(\epsilon)|t|^{r(x)}. \tag{9}$$

In view of (9) with $\|u\|<1$ and by (3), we have

$$\begin{aligned}I(u)&=\int_\Omega\frac{1}{p_1(x)}|\Delta u|^{p_1(x)}\,dx+\int_\Omega\frac{1}{p_2(x)}|\Delta u|^{p_2(x)}\,dx-\int_\Omega F(x,u)\,dx\\ &\ge\frac{1}{p_1^+}\|u\|_{p_1}^{p_1^+}+\frac{1}{p_2^+}\|u\|_{p_2}^{p_2^+}-\epsilon\int_\Omega|u|^{r^+}\,dx-c(\epsilon)\int_\Omega|u|^{r(x)}\,dx\\ &\ge\begin{cases}\frac{2}{p_1^+}\|u\|^{p_1^+}-\epsilon c_{10}^{r^+}\|u\|^{r^+}-c(\epsilon)\|u\|^{r^-}, & \text{if } p_1^+\le p_2^+\\ \frac{2}{p_2^+}\|u\|^{p_2^+}-\epsilon c_{10}^{r^+}\|u\|^{r^+}-c(\epsilon)\|u\|^{r^-}, & \text{if } p_1^+\ge p_2^+.\end{cases}\end{aligned} \tag{10}$$

Therefore, since $r^+<p_1^-$ and $r^+<p_2^-$, there exist $\rho>0$ and $\delta>0$ such that $I(u)\ge\delta>0$ for every $\|u\|=\rho$.

From ($\boldsymbol{F}_4$) we have $F(x,t)>c_{12}|t|^\beta-c_{13}$, for all $x\in\Omega$ and $|t|>M$. For $\omega\in X\setminus\{0\}$ and $t>1$ we have

$$I(t\omega)=\int_\Omega\frac{1}{p_1(x)}|\Delta t\omega|^{p_1(x)}\,dx+\int_\Omega\frac{1}{p_2(x)}|\Delta t\omega|^{p_2(x)}\,dx-\int_\Omega F(x,t\omega)\,dx$$

$$\le\frac{|t|^{p_1^+}}{p_1^-}\int_\Omega|\Delta\omega|^{p_1(x)}\,dx+\frac{|t|^{p_2^+}}{p_2^-}\int_\Omega|\Delta\omega|^{p_2(x)}\,dx-c_{12}|t|^\beta\int_\Omega|\omega|^\beta\,dx-c_{13}|\Omega|.$$

Due to $\beta > \max\left(p_1^+, p_2^+\right)$, we have

$$I(t\omega) \to -\infty \quad \text{as} \quad t \to +\infty. \tag{11}$$

Since $I(0) = 0$, it follows from (10) and (11) that I satisfies the conditions of Mountain Pass Theorem. By Lemma (3.3), I satisfies (PS) condition in X. Therefore I admits at least one nontrivial critical point.

Proof of Theorem 1.3

For the proof of Theorem (1.3) we will need the following steps.
Step 1. By Lemma (3.3) I satisfies the (PS) condition.

Step 2. I is bounded from below. Indeed, by (10) and $(\boldsymbol{F}_5)$ we have that I is bounded from below and even.

Step 3. We notice that $X_{p_1^+} := \left(W^{2,p_1^+}(\Omega) \cap W_0^{1,p_1^+}(\Omega)\right) \subset X_1$ and $X_{p_2^+} := \left(W^{2,p_2^+}(\Omega) \cap W_0^{1,p_2^+}(\Omega)\right) \subset X_2$.

Consider $\{e_1, e_2, \ldots\}$, a Schauder basis of the space $X_{p_1^+} \cap X_{p_2^+}$, and for each $k \in \mathbb{N}$, consider, X_k the subspace of $X_{p_1^+} \cap X_{p_2^+}$ generated by k vectors $\{e_1, e_2, \ldots, e_k\}$.

It is clear that X_k is a subspace of X. So we notice that $X_k \subset L^{r(x)}(\Omega)$ because $X_{p_1^+} \cap X_{p_2^+} \subset L^{r(x)}$. Thus, the norms $\|\cdot\|$ and $|\cdot|_r$ are equivalent on a finite dimensional space X_k. Consequently, there exists a positive constant C_k such that $-|u|_r \leq -C_k\|u\|$ for all $u \in X_k$. By $(\boldsymbol{F}_2)$ we obtain $-F(x,u) \leq -\frac{c_2}{\alpha(x)}|u|^{\alpha(x)} - c_{14}$. Then we have

$$\int_\Omega -F(x,u)\,dx \leq -\int_\Omega \frac{c_2}{\alpha(x)}|u|^{\alpha(x)}\,dx - c_{14}|\Omega|,$$

so that

$$\int_\Omega -F(x,u)\,dx \leq -\frac{c_2}{\alpha^+}C_k\|u\|^{\alpha^+} - c_{14}|\Omega|. \tag{12}$$

Therefore by (12) and (3) we have

$$I(u) \leq \begin{cases} \frac{1}{p_1^-}\parallel u \parallel_{p_1}^{p_1^-} + \frac{1}{p_2^-}\|u\|_{p_2}^{p_2^-} - \frac{c_2}{\alpha^+}C_k\|u\|^{\alpha^+} - c_{14}|\Omega|, & \text{if } \|u\|_{p_1} \leq 1 \text{ and } \|u\|_{p_2} \leq 1 \\ \frac{1}{p_1^-}\parallel u \parallel_{p_1}^{p_1^-} + \frac{1}{p_2^-}\|u\|_{p_2}^{p_2^+} - \frac{c_2}{\alpha^+}C_k\|u\|^{\alpha^+} - c_{14}|\Omega|, & \text{if } \|u\|_{p_1} \leq 1 \text{ and } \|u\|_{p_2} \geq 1 \\ \frac{1}{p_1^-}\parallel u \parallel_{p_1}^{p_1^+} + \frac{1}{p_2^-}\|u\|_{p_2}^{p_2^+} - \frac{c_2}{\alpha^+}C_k\|u\|^{\alpha^+} - c_{14}|\Omega|, & \text{if } \|u\|_{p_1} \geq 1 \text{ and } \|u\|_{p_2} \geq 1 \\ \frac{1}{p_1^-}\parallel u \parallel_{p_1}^{p_1^+} + \frac{1}{p_2^-}\|u\|_{p_2}^{p_2^-} - \frac{c_2}{\alpha^+}C_k\|u\|^{\alpha^+} - c_{14}|\Omega|, & \text{if } \|u\|_{p_1} \geq 1 \text{ and } \|u\|_{p_2} \leq 1. \end{cases}$$

In case $\|u\|_{p_1} \leq 1$ and $\|u\|_{p_2} \leq 1$, we choose $R > 0$ small enough such that $\frac{1}{p_1^-}R^{p_1^- - \alpha^+} + \frac{1}{p_2^-}R^{p_2^- - \alpha^+} < \frac{c_2}{\alpha^+}C_k$. Thus, for $0 < \varrho < R$, we consider the set $K = \{u \in X_k : \|u\| = \varrho\}$. For all $u \in K$, we have

$$\begin{aligned} I(u) \leq & \varrho^{\alpha^+}\left(\frac{1}{p_1^-}\varrho^{p_1^- - \alpha^+} + \frac{1}{p_2^-}\varrho^{p_2^- - \alpha^+} - \frac{c_2}{\alpha^+}C_k\right) \\ \leq & R^{\alpha^+}\left(\frac{1}{p_1^-}R^{p_1^- - \alpha^+} + \frac{1}{p_2^-}R^{p_2^- - \alpha^+} - \frac{c_2}{\alpha^+}C_k\right) \leq I(0). \end{aligned}$$

Since

$$I(u) \leq \begin{cases} \|u\|^{\alpha^+}\left(\frac{1}{p_1^-}\parallel u \parallel_{p_1}^{p_1^- -\alpha^+} + \frac{1}{p_2^-}\|u\|_{p_2}^{p_2^- -\alpha^+} - \frac{c_2}{\alpha^+}C_k\right) & \text{if } \|u\|_{p_1} \leq 1 \text{ and } \|u\|_{p_2} \leq 1 \\ \|u\|^{\alpha^+}\left(\frac{1}{p_1^-}\parallel u \parallel_{p_1}^{p_1^- -\alpha^+} + \frac{1}{p_2^-}\|u\|_{p_2}^{p_2^+ -\alpha^+} - \frac{c_2}{\alpha^+}C_k\right) & \text{if } \|u\|_{p_1} \leq 1 \text{ and } \|u\|_{p_2} \geq 1 \\ \|u\|^{\alpha^+}\left(\frac{1}{p_1^-}\parallel u \parallel_{p_1}^{p_1^+ -\alpha^+} + \frac{1}{p_2^-}\|u\|_{p_2}^{p_2^+ -\alpha^+} - \frac{c_2}{\alpha^+}C_k\right) & \text{if } \|u\|_{p_1} \geq 1 \text{ and } \|u\|_{p_2} \geq 1 \\ \|u\|^{\alpha^+}\left(\frac{1}{p_1^-}\parallel u \parallel_{p_1}^{p_1^+ -\alpha^+} + \frac{1}{p_2^-}\|u\|_{p_2}^{p_2^- -\alpha^+} - \frac{c_2}{\alpha^+}C_k\right) & \text{if } \|u\|_{p_1} \geq 1 \text{ and } \|u\|_{p_2} \leq 1, \end{cases}$$

applying similar reasoning to the other cases, we conclude that $I(u) < 0 = I(0)$.

We can considere the odd homeomorphism $g : K \to S^{k-1}$ defined by $g(u) = (\xi_1, \xi_2, \ldots, \xi_k)$, where S^{k-1} is the sphere in $\mathbb{R}^k$. From Theorem (2.7) and Proposition (2.8) we conclude that $\gamma(k) = k$, thanks to Theorem (2.9), I has at least k pairs of different critical points. Since k is arbitrary, we obtain infinitely many critical points of I.

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