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# ON THE PARTIAL NORMALITY OF A CLASS OF BOUNDED OPERATORS

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**Abstract**. In this paper, some various partial normality classes of weighted conditional expectation type operators on $L^2(\Sigma)$ are investigated. For a weakly hyponormal weighted conditional expectation type operator $M_wEM_u$, we show that the conditional Cauchy-Schwartz inequality for u and w becomes an equality. Assuming this equality, we then show that the joint point spectrum is equal to the point spectrum of $M_wEM_u$. Also, we compute the approximate point spectrum of $M_wEM_u$ and we prove that under a mild condition the approximate point spectrum and the spectrum of $M_wEM_u$ are the same.

## 1. Introduction

The notion of conditional expectation is rightfully thought of as belonging to the theory of probability. In that context, it is set against a background of a probability space $(\Omega, \mathcal{F}, P)$ and $\sigma$-subalgebra ($\sigma$-field as it is commonly called in probability texts) $\mathcal{G}$ of $\mathcal{F}$. If $X$ denotes an integrable random variable, then the conditional expected value of $X$ given $\mathcal{G}$ is the random variable $E[X|\mathcal{G}]$ such that
1. $E[X|\mathcal{G}]$ is $\mathcal{G}$-measurable,
2. $E[X|\mathcal{G}]$ satisfies the functional relation $\int_G E[X|\mathcal{G}]\, dP = \int_G X\, dP, \quad \forall G \in \mathcal{G}$.
A number of standard texts will illustrate concisely the probabilistic formulation and interpretation of the function $E[X|\mathcal{G}]$, and the reader is invited to consult for example reference [2]. Our main interests, however, reside in the view of conditional expectation as an operator between the $L^p$-spaces, specially between $L^2$-spaces.

Among the earlier investigations along these lines is that of Shu-Teh Chen Moy in his seminal 1954 paper [10]. Set within the familiar framework of a probability space $(\Omega, \mathcal{F}, P)$, Moy obtains necessary and sufficient conditions for a linear transformation $T$ between function spaces to be of the form $TX = E[gX|\mathcal{G}]$, where $\mathcal{G} \subset \mathcal{F}$ is a

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$\sigma$-subalgebra and $g$ is a nonnegative measurable function with bounded conditional expected value. The function $E[gX|\mathcal{G}]$ can best be described as the weighted conditional expected value of $X$. Moreover, conditional expectations have been studied in an operator theoretic setting, by, for example, R. G. Douglas, [4], de Pagter and Grobler [7], P.G. Dodds, C.B. Huijsmans and B. De Pagter, [3], J. Herron, [8], Alan Lambert [9] and Rao [11, 12], as positive operators acting on $L^p$-spaces or Banach function spaces. The combination of conditional expectation and multiplication operators appears more often as a tool in the study of other operators rather than being, in themselves, the object of the study.

In [5], we investigated some classic properties of multiplication conditional expectation operators $M_wEM_u$ on $L^p$ spaces. We continue in this paper our study of properties of multiplication conditional expectation operators. Here we will be concerned with characterizing weighted conditional expectation type operators on $L^2(\Sigma)$ in terms of membership in the various partial normality classes and some applications of them in spectral theory.

## 2. Preliminaries

Let $(X, \Sigma, \mu)$ be a complete $\sigma$-finite measure space. For any $\sigma$-subalgebra $\mathcal{A} \subseteq \Sigma$, the $L^2$-space $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated by $L^2(\mathcal{A})$, and its norm is denoted by $\|.\|_2$. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set. The support of a measurable functions $f$ is defined as $S(f) = \{x \in X; f(x) \neq 0\}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable functions on $X$ by $L^0(\Sigma)$.

For a $\sigma$-subalgebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \mapsto E^{\mathcal{A}}f$, defined for all non-negative, measurable functions $f$ as well as for all $f \in L^2(\Sigma)$, where $E^{\mathcal{A}}f$, by the Radon-Nikodym theorem, is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f\, d\mu = \int_A E^{\mathcal{A}} f\, d\mu, \quad \forall A \in \mathcal{A}.$$

As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^2(\Sigma)) = L^2(\mathcal{A})$. If there is no possibility of confusion, we write $E(f)$ in place of $E^{\mathcal{A}}(f)$. This operator will play a major role in our work and we list here some of its useful properties:

(i) If $g$ is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$.

(ii) $|E(f)|^2 \leq E(|f|^2)$.

(iii) If $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.

(iv) $|E(fg)| \leq (E(|f|^2))^{\frac{1}{2}}(E(|g|^2))^{\frac{1}{2}}$, (Hölder inequality).

(v) For each $f \geq 0$, $S(f) \subseteq S(E(f))$.

A detailed discussion and verification of most of these properties may be found in [13].

Let $f \in L^0(\Sigma)$; then $f$ is said to be conditionable with respect to $E$ if $f \in \mathcal{D}(E) := \{g \in L^0(\Sigma) : E(|g|) \in L^0(\mathcal{A})\}$. Throughout this paper we take $u$ and $w$ in $\mathcal{D}(E)$.

Every operator $T$ on a Hilbert space $\mathcal{H}$ can be decomposed into $T = U|T|$ with a partial isometry $U$, where $|T| = (T^*T)^{\frac{1}{2}}$ . $U$ is determined uniquely by the kernel condition $\mathcal{N}(U) = \mathcal{N}(|T|)$. This decomposition is called the polar decomposition. The Aluthge transformation of $T$ is the operator $\hat{T}$ given by $\hat{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.

The plan for the remainder of this paper is to present characterizations of weighted conditional expectation type operators in some various normality classes. Here is a brief review of what constitutes membership for an operator $T$ on a Hilbert space in some classes:

(i) $T$ is normal if $T^*T = TT^*$.

(ii) $T$ is hyponormal if $T^*T \geq TT^*$.

(iii) For $0 < p < \infty$, $T$ is $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$.

(iv) $T$ is $\infty$-hyponormal if it is $p$-hyponormal for all $p$.

(v) $T$ is $p$-quasihyponormal if $T^*(T^*T)^pT \geq T^*(TT^*)^pT$.

(vi) $T$ is weakly hyponormal if $|\hat{T}| \geq |T| \geq |\hat{T}^*|$.

(vii) $T$ is normaloid if $\|T\|^n = \|T^n\|$ for all $n \in \mathbb{N}$.

## 3. Some classes of weighted conditional expectation type operators

We first recall some theorems that we have proved in [5].

THEOREM 3.1. *The operator $T = M_wEM_u$ is bounded on $L^2(\Sigma)$ if and only if $(E|w|^2)^{\frac{1}{2}}(E|u|^2)^{\frac{1}{2}} \in L^\infty(\mathcal{A})$, and in this case its norm is given by*

$$\|T\| = \|(E(|w|^2))^{\frac{1}{2}}(E(|u|^2))^{\frac{1}{2}}\|_\infty.$$

LEMMA 3.2. *Let $T = M_wEM_u$ be a bounded operator on $L^2(\Sigma)$ and $p \in (0, \infty)$. Then $(T^*T)^p = M_{\bar{u}(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p}EM_u$ and $(TT^*)^p = M_{w(E(|w|^2))^{p-1}\chi_G(E(|u|^2))^p}EM_{\bar{w}}$, where $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$.*

THEOREM 3.3. *The unique polar decomposition of bounded operator $T = M_wEM_u$ is $U|T|$, where $|T|(f) = \left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}}\chi_S\bar{u}E(uf)$ and $U(f) = \left(\frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}}wE(uf)$, for all $f \in L^2(\Sigma)$.*

THEOREM 3.4. *The Aluthge transformation of $T = M_wEM_u$ is $\widehat{T}(f) = \frac{\chi_S E(uw)}{E(|u|^2)}\bar{u}E(uf)$, $f \in L^2(\Sigma)$.*

From now on, we consider the operators $M_wEM_u$ and $EM_u$ to be bounded operators on $L^2(\Sigma)$. In the sequel some necessary and sufficient conditions for normality, hyponormality, $p$-hyponormality, etc. will be presented.

THEOREM 3.5. *Let $T = M_wEM_u$, then*

*(a) If* $(E(|u|^2))^{\frac{1}{2}}\bar{w} = u(E(|w|^2))^{\frac{1}{2}}$, *then* $T$ *is normal.*

*(b) If* $T$ *is normal, then* $|E(u)|^2E(|w|^2) = |E(w)|^2E(|u|^2)$.

*Proof.* (a) Applying Lemma 3.2 we have $T^*T-TT^* = M_{\bar{u}E(|w|^2)}EM_u - M_{wE(|u|^2)}EM_{\bar{w}}$. So for every $f \in L^2(\Sigma)$,

$$\begin{aligned}\langle T^*T - TT^*(f), f\rangle &= \int_X E(|w|^2)E(uf)\overline{uf} - E(|u|^2)E(\bar{w}f)w\bar{f}\,d\mu \\ &= \int_X |E(u(E(|w|^2))^{\frac{1}{2}}f)|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w}f)|^2\,d\mu.\end{aligned}$$

This implies that if $(E(|u|^2))^{\frac{1}{2}}\bar{w} = u(E(|w|^2))^{\frac{1}{2}}$, then for all $f \in L^2(\Sigma)$, $\langle T^*T - TT^*(f), f\rangle = 0$, thus $T^*T = TT^*$.

(b) Suppose that $T$ is normal. For all $f \in L^2(\Sigma)$ we have

$$\int_X |E(u(E(|w|^2))^{\frac{1}{2}}f)|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w}f)|^2\,d\mu = 0.$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing $f$ to $\chi_A$, we have

$$\int_A |E(u(E(|w|^2))^{\frac{1}{2}})|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w})|^2\,d\mu = 0$$

and so

$$\int_A |E(u)|^2E(|w|^2) - |E(w)|^2E(|u|^2)\,d\mu = 0.$$

Since $A \in \mathcal{A}$ is arbitrary, then $|E(u)|^2E(|w|^2) = |E(w)|^2E(|u|^2)$. □

Corollary 3.6. *The operator* $EM_u$ *is normal if and only if* $u \in L^\infty(\mathcal{A})$.

Theorem 3.7. *Let* $T = M_wEM_u$ *and let* $p \in (0, \infty)$.

*(a) The following statements are equivalent:*

$$T \text{ is hyponormal} \iff T \text{ is } p\text{-hyponormal} \iff T \text{ is } \infty\text{-hyponormal}$$

*(b) If* $|E(uf)|^2 \geq E(|f|^2)E(|u|^2)$ *on* $G$ *for all* $f \in L^2(\Sigma)$, *then* $T$ *is hyponormal.*

*(c) If* $T$ *is hyponormal, then* $|E(u)|^2E(|w|^2) - |E(w)|^2E(|u|^2) \geq 0$.

*Proof.* (a) Applying Lemma 3.2 we obtain that $(T^*T)^p \geq (TT^*)^p$ if and only if

$$M_{\chi_{S\cap G}(E(|u|^2))^{p-1}(E(|w|^2))^{p-1}}(M_{\bar{u}E(|w|^2)}EM_u - M_{wE(|u|^2)}EM_{\bar{w}}) \geq 0.$$

This inequality holds if and only if $T^*T-TT^* = M_{\bar{u}E(|w|^2)}EM_u - M_{wE(|u|^2)}EM_{\bar{w}} \geq 0$, where we have used the fact that $T_1T_2 \geq 0$ if $T_1 \geq 0$, $T_2 \geq 0$ and $T_1T_2 = T_2T_1$ for all $T_i \in \mathcal{B}(\mathcal{H})$, the set of all bounded linear operators on Hilbert space $\mathcal{H}$. Since $0 < p < \infty$ is arbitrary, all equivalencies hold.

(b) By Lemma 3.2, we have $T^*T - TT^* = M_{\bar{u}E(|w|^2)}EM_u - M_{wE(|u|^2)}EM_{\bar{w}}$. So for every $f \in L^2(\Sigma)$,

$$\begin{aligned}\langle T^*T - TT^*(f), f\rangle &= \int_X E(|w|^2)|E(uf)|^2 - E(|u|^2)|E(\bar{w}f)|^2\,d\mu \\ &\geq \int_X E(|w|^2)(|E(uf)|^2 - E(|f|^2)E(|u|^2))\,d\mu.\end{aligned}$$

This implies that, if $|E(uf)|^2 \geq E(|f|^2)E(|u|^2)$ on $G$, then $T$ is hyponormal.

(c) Let $T$ be hyponormal. For all $f \in L^2(\Sigma)$ we have

$$\int_X |E(u(E(|w|^2))^{\frac{1}{2}} f)|^2 - |E((E(|u|^2))^{\frac{1}{2}} \bar{w} f)|^2 \, d\mu \geq 0.$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing $f$ by $\chi_A$, we have

$$\int_A |E(u(E(|w|^2))^{\frac{1}{2}})|^2 - |E((E(|u|^2))^{\frac{1}{2}} \bar{w})|^2 \, d\mu \geq 0$$

and so

$$\int_A |E(u)|^2 E(|w|^2) - |E(w)|^2 E(|u|^2) \, d\mu \geq 0.$$

Since $A \in \mathcal{A}$ is arbitrary, then $|E(u)|^2 E(|w|^2) \geq |E(w)|^2 E(|u|^2)$. □

THEOREM 3.8. *Let $T = M_w E M_u$, then $T$ is p-quasihyponormal if and only if $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$ .*

*Proof.* By Lemma 3.2, it is easy to check that

$$T^*(T^*T)^p T = M_{\bar{u}(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p |E(uw)|^2} E M_u;$$

$$T^*(TT^*)^p T = M_{\bar{u}(E(|w|^2))^{p+1}(E(|u|^2))^p} E M_u.$$

It follows that $T^*(T^*T)^p T \geq T^*(TT^*)^p T$ if

$$M_{(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p} M_{(|E(uw)|^2 - E(|w|^2)E(|u|^2))} M_{\bar{u}} E M_u \geq 0. \quad (1)$$

By the same argument as in Theorem 3.7, (1) holds if $M_{(|E(uw)|^2 - E(|w|^2)E(|u|^2))} \geq 0$ i.e. $|E(uw)|^2 - E(|w|^2)E(|u|^2) \geq 0$.

Conversely, suppose that $T$ is $p$-quasihyponormal. Then for all $f \in L^2(\mathcal{A})$, we have

$$\langle T^*(T^*T)^p T - T^*(TT^*)^p T f, f \rangle$$
$$= \int_X (E(|u|^2))^{p-1} \chi_S (E(|w|^2))^p (|E(uw)|^2 - E(|w|^2)E(|u|^2)) |E(u)|^2 |f|^2 \, d\mu \geq 0.$$

Thus $(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p(|E(uw)|^2 - E(|w|^2)E(|u|^2))|E(u)|^2 \geq 0$, and hence we obtain $|E(uw)|^2 \geq E(|w|^2)E(|u|^2)$. □

So we have the following corollary.

COROLLARY 3.9. *Let $T = EM_u$ and $p \in (0, \infty)$. Then the following statements are equivalent.*

*(i) $T$ is normal.*

*(ii) $T$ is hyponormal.*

*(iii) $T$ is p-hyponormal.*

*(iv) $T$ is $\infty$-hyponormal.*

*(v) $T$ is p-quasihyponormal.*

*(vi) $u \in L^\infty(\mathcal{A})$.*

THEOREM 3.10. *Let $T = M_w E M_u$; then $T$ is weakly hyponormal if and only if $|E(uw)|^2 = E(|u|^2)E(|w|^2)$.*

*Proof.* For every $f \in L^2(\Sigma)$, by Theorems 3.3 and 3.4, we have $|\widehat{T}|(f) = |(\widehat{T})^*|(f) = |E(uw)|\chi_S(E(|u|^2))^{-1}\bar{u}E(uf)$, where $S = S(E(|u|^2))$.

So, $T$ is weakly hyponormal if and only if $|T| = |\widehat{T}|$. For every $f \in L^2(\Sigma)$,

$$\langle |T|(f) - |\widehat{T}|(f), f\rangle = \int_X \left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}} \chi_S \overline{uf} E(uf) - |E(uw)|\chi_S (E(|u|^2))^{-1} \overline{uf} E(uf)\, d\mu$$

$$\int_X \left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}} \chi_S |E(uf)|^2 - |E(uw)|\chi_S (E(|u|^2))^{-1} |E(uf)|^2\, d\mu;$$

this implies that if $|E(uw)|^2 = E(|u|^2)E(|w|^2)$, then $|T| = |\widehat{T}|$.

Conversely, if $|T| = |\widehat{T}|$, then for all $f \in L^2(\Sigma)$ we have

$$\int_X \left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}} \chi_S |E(uf)|^2 - |E(uw)|\chi_S (E(|u|^2))^{-1} |E(uf)|^2\, d\mu = 0.$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing $f$ by $\chi_A$, we have

$$\int_A \left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}} \chi_S |E(u)|^2 - |E(uw)|\chi_S (E(|u|^2))^{-1} |E(u)|^2\, d\mu = 0.$$

Since $A \in \mathcal{A}$ is arbitrary, then

$$\left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}} \chi_S |E(u)|^2 - |E(uw)|\chi_S (E(|u|^2))^{-1} |E(u)|^2 = 0.$$

Hence $|E(uw)|^2 = E(|u|^2)E(|w|^2)$. □

Theorems 3.7, 3.10 and [1, Theorem 1.3] imply that if $u(E(|w|^2))^{\frac{1}{2}} - (E(|u|^2))^{\frac{1}{2}}\bar{w} \geq 0$, then $|E(uw)|^2 = E(|u|^2)E(|w|^2)$.

Corollary 3.11. *(a) If $T = EM_u$, then $T$ is weakly hyponormal if and only if $u \in L^\infty(\mathcal{A})$.*

*(b) If $T = M_w E$, then $T$ is weakly hyponormal if and only if $w \in L^\infty(\mathcal{A})$.*

Theorem 3.12. *If $T = M_w E M_u$ is weakly hyponormal with $\ker T \subset \ker T^*$, then $T = \widehat{T}$.*

*Proof.* Direct computations show that $\widehat{T}$ is normal and by [1, Theorem 2.6] $T = \widehat{T}$. □

Here we give some examples of conditional expectation operators.

Example 3.13. (a) Let $X = \mathbb{N}$, $\mathcal{G} = 2^{\mathbb{N}}$ and let $\mu(\{x\}) = pq^{x-1}$, for each $x \in X$, $0 \leq p \leq 1$ and $q = 1-p$. Elementary calculations show that $\mu$ is a probability measure on $\mathcal{G}$. Let $\mathcal{A}$ be the $\sigma$-algebra generated by the partition $B = \{X_1 = \{3n : n \geq 1\}, X_1^c\}$ of $X$. So, for every $f \in \mathcal{D}(E^{\mathcal{A}})$, $E(f) = \alpha_1 \chi_{X_1} + \alpha_2 \chi_{X_1^c}$ and direct computations show that

$$\alpha_1 = \frac{\sum_{n\geq 1} f(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{3n-1}} \quad \text{and} \quad \alpha_2 = \frac{\sum_{n\geq 1} f(n)pq^{n-1} - \sum_{n\geq 1} f(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{n-1} - \sum_{n\geq 1} pq^{3n-1}}.$$

If we set $f(x) = x$, then we have $\alpha_1 = \frac{3}{1-q^3}$, $\alpha_2 = \frac{1+q^6-3q^4+4q^3-3q^2}{(1-q^2)(1-q^3)}$.

(b) Let $\Omega = [-\pi, \pi]$, $d\mu = \frac{1}{2}dx$ and $\mathcal{A} = \langle\{(-a, a) : 0 \leq a \leq \pi\}\rangle$ (Sigma algebra generated by symmetric intervals). Then $E^{\mathcal{A}}(f)(x) = \frac{1}{2}(f(x) + f(-x))$, $x \in \Omega$, where $E^{\mathcal{A}}(f)$ is defined. Let $J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)}(\frac{x}{2})^{2m+n}$, for each $n \in \mathbb{N}$ and $-\pi \leq x \leq \pi$ where $\Gamma(z)$ is the Gamma function, be Bessel functions of the first kind. Then for every $n \in \mathbb{N}$, $E(J_{2n-1}) = 0$ and $E(J_{2n}) = J_{2n}$. And so, $\{J_{2n-1} : n \in \mathbb{N}\} \subseteq \{f \in L^2([-\pi, \pi]) : E(f) = 0, a.e\}$.

Also, $\{J_{2n}\}_{n\in\mathbb{N}} \subseteq \mathcal{R}(E)$. Thus, the null space and the range of conditional expectation $E$ contains infinite number of special functions.

## 4. Some applications

In this section, we shall denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_{jp}(T)$, $\sigma_a(T)$, $r(T)$ the spectrum of $T$, the point spectrum of $T$, the joint point spectrum of $T$, the approximate point spectrum, the spectral radius of $T$, respectively. The spectrum of an operator $T$ is the set $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not invertible}\}$. A complex number $\lambda \in \mathbb{C}$ is said to be in the point spectrum $\sigma_p(T)$ of the operator $T$, if there is a unit vector $x$ satisfying $(T - \lambda)x = 0$. If in addition, $(T^* - \bar{\lambda})x = 0$, then $\lambda$ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of $T$. The approximate point spectrum of $T$ is the set of all $\lambda$ such that $T - \lambda I$ is not an isomorphism onto a closed subspace of the space [6].

Also, the spectral radius of $T$ is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.

For each natural number $n$, we define $\triangle_n(T) = \widehat{\triangle_{n-1}T} \quad \triangle_1(T) = \triangle(T) = \widehat{T}$. We call $\triangle_n(T)$ the $n$-th Aluthge transformation of $T$. It is proved in [15] that $r(T) = \lim_{n\to\infty} \|\triangle_n(T)\|$.

THEOREM 4.1. *Let $T = M_wEM_u$. Then*
*(a) $\widehat{T}$ is normaloid.*

*(b) $T$ is normaloid if and only if $\|E(uw)\|_\infty = \|(E(|u|^2))^{\frac{1}{2}}(E(|w|^2))^{\frac{1}{2}}\|_\infty$.*

*Proof.* (a) By Theorem 3.1 we have $\|\widehat{T}\| = \|E(uw)\|_\infty$. By Theorem 3.4 we conclude that for every natural number $n$ we have $\triangle_n(T) = \triangle(T) = \widehat{T}$. Hence $r(\widehat{T}) = r(T) = \|\widehat{T}\| = \|E(uw)\|_\infty$. So $\widehat{T}$ is normaloid.

(b) By conditional type Hölder inequality, boundedness of $T$ and Theorem 3.1 we have $r(T) = \|E(uw)\|_\infty \leq \|(E(|u|^2))^{\frac{1}{2}}(E(|w|^2))^{\frac{1}{2}}\|_\infty = \|T\|$. Hence $T$ is normaloid if and only if $\|E(uw)\|_\infty = \|(E(|u|^2))^{\frac{1}{2}}(E(|w|^2))^{\frac{1}{2}}\|_\infty$. Theorems 4.1 and 3.10 show that if $T$ is weakly hyponormal, then $T$ is normaloid. Also, Theorem 1.3 of [1] and Theorem 3.7 imply that if $u(E(|w|^2))^{\frac{1}{2}} - (E(|u|^2))^{\frac{1}{2}}\bar{w} \geq 0$, then $T$ is normaloid. □

It can be proved that (see [6, 14]): $\sigma(M_wEM_u) \setminus \{0\} = \text{ess range}\,(E(uw)) \setminus \{0\}$ and $\sigma_p(M_wEM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} : \mu(A_{\lambda,w}) > 0\} \setminus \{0\}$, where $A_{\lambda,w} = \{x \in X : E(uw)(x) = \lambda\}$ and $\text{ess range}\,(E(uw)) = \{\lambda \in \mathbb{C} : \forall\varepsilon > 0, \mu(\{x \in X : |E(uw)(x) - \lambda| < \varepsilon\}) > 0\}$. Furthermore, for every bounded operator $S$ on a Hilbert space $\mathcal{H}$ we have $\sigma(S) = \sigma_a(S) \cup \overline{\sigma_p(S^*)}$ [16]. By these facts we get that $\sigma_a(M_wEM_u) \setminus \{0\} = \text{ess range}\,(E(uw)) \setminus (\{\lambda \in \mathbb{C} : \mu(A_{\lambda,w}) > 0\} \cup \{0\})$.

THEOREM 4.2. *If $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$, then $\sigma_p(M_wEM_u) = \sigma_{jp}(M_wEM_u)$.*

So, by Theorems 3.8 and 3.10 we have the next corollary.

COROLLARY 4.3. *If* $T = M_w E M_u$ *is weakly hyponormal or p-quasihyponormal, then* $\sigma_p(M_w E M_u) = \sigma_{jp}(M_w E M_u)$.

THEOREM 4.4. *If* $u(E(|w|^2))^{\frac{1}{2}} - (E(|u|^2))^{\frac{1}{2}}\bar{w} \geq 0$, *then* $\sigma_p(M_w E M_u) = \sigma_{jp}(M_w E M_u)$.

*Proof.* If $u(E(|w|^2))^{\frac{1}{2}} - (E(|u|^2))^{\frac{1}{2}}\bar{w} \geq 0$, then by Theorem 3.7 $T = M_w E M_u$ is $p$-hyponormal for $p \in (0, \infty)$. Also, by [1, Theorem 1.3] we have that $T$ is weakly-hyponormal and then by Corollary 4.3 we get that $\sigma_p(M_w E M_u) = \sigma_{jp}(M_w E M_u)$. □

For a semi-hyponormal operator $S$ on a Hilbert space $\mathcal{H}$ we have $\sigma(S) = \{\lambda : \bar{\lambda} \in \sigma_a(S^*)\}$ (see [16]). So from Theorem 3.7 the following holds.

THEOREM 4.5. *If* $u(E(|w|^2))^{\frac{1}{2}} - (E(|u|^2))^{\frac{1}{2}}\bar{w} \geq 0$, *then* $\sigma(M_w E M_u) \setminus \{0\} = \{\lambda : \bar{\lambda} \in \sigma_a(M_{\bar{u}} E M_{\bar{w}})\} \setminus \{0\}$ *or equivalently* $\sigma_a(M_w E M_u) \setminus \{0\} =$ *ess range* $(E(uw)) \setminus \{0\}$.

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