

COMMON FIXED POINTS OF GENERALIZED CONTRACTIVE MAPPINGS IN UNIFORM SPACES

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Abstract. In order to establish some common fixed point theorems on Hausdorff uniform spaces endowed with a graph we will define a new kind of generalized contraction for self-mappings. A few related examples are also provided to support our main results. Finally an application of our results in b -metric spaces is exhibited.

1. Introduction

Following [6], a pair (X, ν) is called a uniform space, if X is a nonempty set and ν is a special kind of filter on $X \times X$ satisfying the following conditions:

(ν_1) for each $U \in \nu$, $\Delta = \{(x, x) : x \in X\} \subseteq U$,

(ν_2) $U \in \nu$ and $U \subseteq W \subseteq X \times X$ implies $W \in \nu$,

(ν_3) $U \in \nu$ and $W \in \nu$ implies $U \cap W \in \nu$,

(ν_4) $U \in \nu$ implies $U^{-1} \in \nu$,

(ν_5) if $U \in \nu$, then there exists $V \in \nu$ with $V \circ V \subseteq U$. (The composition of two subsets V and U of $X \times X$ is defined by $V \circ U = \{(x, z) : \exists y \in X : (x, y) \in V, (y, z) \in U\}$). A uniform space (X, ν) is said to be Hausdorff if the intersection of all members of ν reduces to the diagonal Δ of X . This guarantees the uniqueness of limits of sequences.

Knill [10] was the first who extended the notion of contractive mapping in uniform spaces. Later, a few mathematicians studied various types of fixed point theorems in non-metrizable spaces (e.g. [1–4, 7, 12, 14, 16, 17]). Aamri and El Moutawakil [1] introduced the concept of an A -distance and an E -distance to prove some common fixed point theorems for contractive and expansive maps in uniform spaces. In 2004, Ran and Reurings [13] obtained a generalization of Banach's fixed point theorem for continuous self-mappings on a complete metric space endowed with a partial ordering. Jachymski [9] noted that every partially ordered metric space (X, d, \preceq) can be

2010 Mathematics Subject Classification: 47H10, 54E15, 05C40, 54H25

Keywords and phrases: Fixed points; uniform spaces; graph; connectivity.

considered as a special case of a metric space (X, d) endowed with a directed graph G , where $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \preceq y\}$. This observation, motivated a few mathematicians to extend and unify some fixed point theorems in metric spaces endowed with a graph (e.g. [5, 8, 11, 15]).

The aim of this paper is to obtain common fixed point theorems for two self-mappings on a Hausdorff uniform space endowed with a graph when the space is equipped with an A -distance. More precisely, we obtain a general result for existence and uniqueness of common fixed points for two generalized contractive self-mappings. Our main results generalize [1, Theorem 3.1] and lead to some applications in b -metric spaces.

2. Preliminaries

In this section we introduce the concepts that we will use in the rest of the paper. We start with the following definition.

DEFINITION 2.1 ([1]). Let (X, ν) be a uniform space. A function $\rho : X \times X \rightarrow \mathbb{R}^{\geq 0}$ is called an A -distance, if for any $U \in \nu$ there exists $\delta > 0$ such that if $\rho(z, x) \leq \delta$ and $\rho(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in U$. If ρ also satisfies $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for each $x, y, z \in X$, then ρ is called an E -distance.

EXAMPLE 2.2. Let (X, d) be a metric space, then the metric d is an E -distance for the uniformity generated by the metric.

EXAMPLE 2.3. Consider $X = [0, +\infty)$ with the uniformity generated by the Euclidean metric. Then $\rho(x, y) = \max\{x, y\}$ is an E -distance defined on X .

The following examples show that there are A -distances which are not E -distances.

EXAMPLE 2.4. Let X be a nonempty set and $d : X \times X \rightarrow \mathbb{R}^{\geq 0}$ be such that (i) $d(x, y) = d(y, x)$, (ii) $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$ implies that $d(x, z) < 2\varepsilon$. Define $\nu = \{V_\varepsilon : \varepsilon > 0\}$ in which $V_\varepsilon = \{(x, y) \in X^2 : d(x, y) < \varepsilon\}$. Then ν defines a uniformity on X and d is an A -distance on (X, ν) . For example if $X = \{a, b, c\}$ and $d : X \times X \rightarrow \mathbb{R}^{\geq 0}$ is a symmetric function which is defined by $d(a, b) = 3$, $d(b, c) = 2$, $d(a, c) = 6$, $d(a, a) = d(b, b) = d(c, c) = 0$, then it is easy to verify that conditions (i) and (ii) hold, (X, ν_d) is a uniformity and d is an A -distance on X . Note that $d(a, c) \not\leq d(a, b) + d(b, c)$. Therefore d is not an E -distance.

EXAMPLE 2.5. Let X be a nonempty set and $d : X \times X \rightarrow [0, \infty)$ for some $s > 1$ satisfies the following properties:

(i) $d(x, y) = 0$ iff $x = y$, (ii) $d(x, y) = d(y, x)$, (iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$. Then (X, d) is called a b -metric space.

We may consider (X, d) as a Hausdorff uniform space with the uniformity ν_d generated by $U_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\}$ for $\varepsilon > 0$. Let $U \in \nu_d$, then there is $\varepsilon > 0$ such that $U_\varepsilon \subseteq U$. Let $\delta = \frac{\varepsilon}{2s}$, then $d(z, x) < \delta$ and $d(z, y) < \delta$ imply that $d(x, y) \leq$

$s(d(x, z) + d(z, y)) < 2s\delta = \varepsilon$. Hence $(x, y) \in U_\varepsilon$ if $d(z, x) < \delta$ and $d(z, y) < \delta$. This means that d is an A -distance. However, the triangle inequality is not true. Therefore d is not an E -distance.

We also need the following notions.

DEFINITION 2.6 ([1]). Let (X, ν) be a uniform space endowed with an A -distance ρ .
 (i) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is called ρ -Cauchy if $\lim_{n, m \rightarrow \infty} \rho(x_n, x_m) = 0$. Two sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ are said to be ρ -Cauchy equivalent if each of them is ρ -Cauchy and $\lim_{n \rightarrow \infty} \rho(x_n, y_n) = 0$.

(ii) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to be ρ -convergent to a point $x \in X$, if $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$.

(iii) X is called S -complete if every ρ -Cauchy sequence in X is ρ -convergent.

(iv) $f : X \rightarrow X$ is called ρ -continuous if $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$ implies $\lim_{n \rightarrow \infty} \rho(fx_n, fx) = 0$.

(v) For $A \subseteq X$ define $\text{diam}(A) = \sup\{\rho(x, y) : x, y \in A\}$. A is said to be ρ -bounded if $\text{diam}(A) < \infty$.

The following lemma implies uniqueness of limit of ρ -convergent sequences in Hausdorff uniform spaces.

LEMMA 2.7 ([14]). Let (X, ν) be a Hausdorff uniform space and ρ be an A -distance on X . Let $\{x_n\}$ be an arbitrary sequence in X . Then for each $x, y, z \in X$, the following conditions hold.

(a) If $\lim_{n \rightarrow \infty} \rho(x_n, y) = 0$ and $\lim_{n \rightarrow \infty} \rho(x_n, z) = 0$ then $y = z$. Especially if $\rho(x, y) = 0$ and $\rho(x, z) = 0$, then $y = z$.

(b) If $\lim_{n, m \rightarrow \infty} \rho(x_n, x_m) = 0$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in (X, ν) .

Let (X, ν) be a uniform space and G be a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G by the pair $(V(G), E(G))$. If G is such a graph, we say that X is endowed with the graph G .

By G^{-1} we denote the conversion of a graph G . That is $V(G^{-1}) = V(G)$ and $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. The letter \tilde{G} denotes the undirected graph obtained from G by ignoring the direction of edges. Under this convention $E(\tilde{G}) = E(G) \cup E(G^{-1})$.

A graph G is called connected if there is a path between any two vertices of it. G is weakly connected if \tilde{G} is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G , then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x . In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule: yRx if and only if there is a path in G from x to y . Clearly G_x is connected.

3. Results

We denote by Ψ the set of all functions $\psi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$, which are non-decreasing, $\psi(0) = 0$, $\psi(r) > 0$ for each $r > 0$ and $\lim_{n \rightarrow \infty} \psi^n(r) = 0$. It follows from the definition that $\psi(r) < r$ for all $\psi \in \Psi$ and $r > 0$.

In this section, we obtain some results on existence of common fixed points for two generalized contractive mappings in uniform spaces endowed with an A -distance ρ , which may not satisfy the triangle's inequality. In order to achieve this goal, we need to the following definition.

DEFINITION 3.1. Let (X, v) be a Hausdorff uniform space endowed with a graph G and A -distance ρ , $\psi \in \Psi$ and $f, g : X \rightarrow X$. We say that f is a (ρ, ψ, G) -contraction with respect to g if the following statements hold:

- (i) For each $x \in X$ there exists $y \in [x]_{\bar{G}}$ such that $fx = gy$.
- (ii) f and g are G -invariant, i.e., $(x, y) \in E(G)$ implies that $(fx, fy), (gx, gy) \in E(G)$.
- (iii) If $x \in X$ and $y \in [x]_{\bar{G}}$, then $\rho(fx, fy) \leq \psi(\rho(gx, gy))$.

EXAMPLE 3.2. Let (X, d) be a b -metric space and let $f : X \rightarrow X$ be a mapping such that for some $0 \leq \alpha < 1$ satisfies $d(fx, fy) \leq \alpha d(x, y)$ for all $x, y \in X$. Define graph G_0 with $V(G_0) = X$ and $E(G_0) = X \times X$ and define function $\psi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ by $\psi(r) = \alpha r$ for each $r \in \mathbb{R}^{\geq 0}$. Then f is a (d, ψ, G_0) -contraction with respect to $g = I$, where I is a identity mapping on X .

EXAMPLE 3.3. Let $X = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{\frac{-1}{2^n} : n \in \mathbb{N}\} \cup \mathbb{Z} \setminus \{0\}$. For each $x, y \in X$ define $\rho(x, y) = |x - y|^2$. Then (X, ρ) satisfies conditions (i)–(iii) in Example 2.5 for $s = 2$, so ρ is a b -metric on X . Thus ρ defines a Hausdorff uniformity v_ρ on X . By Example 2.5, ρ is an A -distance on (X, v_ρ) . Define graph G by $V(G) = X$ and

$$E(G) = \Delta(X) \cup \left\{ (n+1, n) : n \in \mathbb{N} \right\} \cup \left\{ (-n-1, -n) : n \in \mathbb{N} \right\} \cup \left\{ \left(\frac{1}{2^n}, \frac{1}{2^{n+1}} \right) : n \in \mathbb{N} \right\} \\ \cup \left\{ \left(\frac{-1}{2^n}, \frac{-1}{2^{n+1}} \right) : n \in \mathbb{N} \right\} \cup \left\{ (x, -x) : x \in X \right\} \cup \left\{ \left(-1, -\frac{1}{2} \right), \left(1, \frac{1}{2} \right) \right\}.$$

Then G is weakly connected. Let $\psi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ be defined by $\psi(r) = \frac{r}{3}$ which belongs to Ψ and let $f, g : X \rightarrow X$ be defined by

$$f x = \begin{cases} \frac{1}{2^{n+1}} & \text{if } x = n \text{ for some } n \in \mathbb{N} \\ \frac{-1}{2^{n+1}} & \text{if } x = -n \text{ for some } n \in \mathbb{N} \\ \frac{1}{2^{n+2}} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ \frac{-1}{2^{n+2}} & \text{if } x = \frac{-1}{2^n} \text{ for some } n \in \mathbb{N} \end{cases} \quad \text{and} \quad g x = \begin{cases} \frac{1}{2^n} & \text{if } x = n \text{ for some } n \in \mathbb{N} \\ \frac{-1}{2^n} & \text{if } x = -n \text{ for some } n \in \mathbb{N} \\ \frac{1}{2^{n+1}} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ \frac{-1}{2^{n+1}} & \text{if } x = \frac{-1}{2^n} \text{ for some } n \in \mathbb{N} \end{cases}.$$

We show that f is a (ρ, ψ, G) -contraction with respect to g .

- (i) G is weakly connected and

$$f(X) = \left\{ \pm \frac{1}{4}, \pm \frac{1}{8}, \pm \frac{1}{16}, \dots, \pm \frac{1}{2^n}, \dots \right\} \subseteq g(X) = \left\{ \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{8}, \pm \frac{1}{16}, \dots, \pm \frac{1}{2^n}, \dots \right\}.$$

Thus for each $x \in X$ there exists $y \in [x]_{\bar{G}} = X$ such that $fx = gy$.

- (ii) For each $(x, y) \in E(G)$ we have $(fx, fy), (gx, gy) \in E(G)$.

(iii) For each $x \in X$ and $y \in [x]_{\tilde{G}} = X$, $\rho(fx fy) \leq \psi(\rho(gx, gy))$.

Note that (X, v_ρ) is not S -complete. Since $\{\frac{1}{2^n}\}_{n \in \mathbb{N}}$ is a ρ -Cauchy sequence in X and there is no element in X to which $\{\frac{1}{2^n}\}_{n \in \mathbb{N}}$ converges.

The following lemma is direct consequence of Definition 3.1.

LEMMA 3.4. *Let (X, v) be a Hausdorff uniform space endowed with a graph G and A -distance ρ . Assume that $\psi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ belongs to Ψ and $f, g : X \rightarrow X$. Suppose that f is a (ρ, ψ, G) -contraction with respect to g . Then f is also (ρ, ψ, G^{-1}) and (ρ, ψ, \tilde{G}) -contraction with respect to g .*

REMARK 3.5. Let (X, v) be a Hausdorff uniform space endowed with a graph G and an A -distance ρ and let $\psi \in \Psi$. Assume that $f, g : X \rightarrow X$ be such that f is a (ρ, ψ, G) -contraction with respect to g . Let $x_0 \in X$. Definition 3.1(i) implies that there exists $x_1 \in [x_0]_{\tilde{G}}$ such that $fx_0 = gx_1$. Again there exists $x_2 \in [x_1]_{\tilde{G}} = [x_0]_{\tilde{G}}$ such that $fx_1 = gx_2$. By continuing this procedure, we can obtain a sequence $\{fx_n\}$ such that for each $n \in \mathbb{N}$, $x_n \in [x_0]_{\tilde{G}}$ and $fx_{n-1} = gx_n$.

In what follows, whenever $x_0 \in X$, $\{fx_n\}$ will be the sequence described above.

DEFINITION 3.6. Let $f, g : X \rightarrow X$. The mapping f is called orbitally bounded with respect to g at $x_0 \in X$ if for every choice $x_n \in [x_0]_{\tilde{G}}$ with $fx_{n-1} = gx_n$, the set $\text{orb}(x_0, f, g) = \{x_0, fx_0, fx_1, \dots\}$ is ρ -bounded. f is called orbitally bounded with respect to g if it is orbitally bounded with respect to g at each point of X .

EXAMPLE 3.7. Let X, ρ, G, f and g be as was described in Example 3.3. Trivially X is not ρ -bounded. For each arbitrary element $x_0 \in X$ we have

$$\text{diam}(\text{orb}(x_0, f, g)) = \sup\{\rho(fx_i, fx_j), \rho(x_0, fx_i) : i, j \in \mathbb{N}\} \leq (x_0)^2.$$

Thus f is orbitally bounded with respect to g .

In order to state the main results of this section, we need some auxiliary results.

LEMMA 3.8. *Let (X, v) be a Hausdorff uniform space endowed with a graph G and A -distance ρ . Assume that $\psi \in \Psi$ and $f, g : X \rightarrow X$. Let f be a (ρ, ψ, G) -contraction with respect to g and let f be orbitally bounded with respect to g at $x_0, y_0 \in X$. If $[x_0]_{\tilde{G}} = [y_0]_{\tilde{G}}$, then the corresponding sequences $\{fx_n\}$ and $\{fy_n\}$, where $fx_{n-1} = gx_n$ and $fy_{n-1} = gy_n$ for all $n \in \mathbb{N}$, are ρ -Cauchy equivalent.*

Proof. Since for each $n \in \mathbb{N}$ we have $x_n \in [x_{n-1}]_{\tilde{G}} = [x_0]_{\tilde{G}}$, it follows that

$$\begin{aligned} \rho(fx_n, fx_{n+m}) &\leq \psi(\rho(gx_n, gx_{n+m})) = \psi(\rho(fx_{n-1}, fx_{n+m-1})) \leq \psi^2(\rho(gx_{n-1}, gx_{n+m-1})) \\ &= \psi^2(\rho(fx_{n-2}, fx_{n+m-2})) \leq \dots \leq \psi^n(\rho(fx_0, fx_m)) \leq \psi^n(\text{diam}(\text{orb}(x, f, g))), \end{aligned}$$

for all $n, m \in \mathbb{N}$. Hence $\lim_{n, m \rightarrow \infty} \rho(fx_n, fx_{n+m}) = 0$. By Lemma 2.7(b), $\{fx_n\}$ is a ρ -Cauchy sequence. Similarly, one can see that $\{fy_n\}$ is also ρ -Cauchy. Moreover, since for each $n \in \mathbb{N}$, $[y_n]_{\tilde{G}} = [x_n]_{\tilde{G}} = [x]_{\tilde{G}} = [y]_{\tilde{G}}$, we have

$$\begin{aligned} \rho(fx_n, fy_n) &\leq \psi(\rho(gx_n, gy_n)) = \psi(\rho(fx_{n-1}, fy_{n-1})) \leq \psi^2(\rho(gx_{n-1}, gy_{n-1})) \\ &= \psi^2(\rho(fx_{n-2}, fy_{n-2})) \leq \dots \leq \psi^n(\rho(fx, fy)) \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Therefore $\{fx_n\}$ and $\{fy_n\}$ are ρ -Cauchy equivalent. \square

The next result states that in a Hausdorff uniform space (X, v) , endowed with a graph G and an A -distance ρ with $\rho(x, x) = 0$ for all $x \in X$, under certain circumstances, the condition of weak connectedness of G is equivalent to two other conditions.

LEMMA 3.9. *Let (X, v) be a Hausdorff uniform space endowed with a graph G and A -distance ρ . Assume that $\psi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ belongs to Ψ and $f, g : X \rightarrow X$. Let f be a (ρ, ψ, G) -contraction with respect to g . If $\rho(x, x) = 0$, for each $x \in X$, then the following conditions are equivalent.*

(a) G is weakly connected.

(b) If f is orbitally bounded with respect to g at $x, y \in X$, then the sequences $\{fx_n\}$ and $\{fy_n\}$ are ρ -Cauchy equivalent, where $x_0 = x$, $y_0 = y$, $fx_{n-1} = gx_n$, $fy_{n-1} = gy_n$, $x_n \in [x_{n-1}]_{\tilde{G}}$ and $y_n \in [y_{n-1}]_{\tilde{G}}$ for each $n \in \mathbb{N}$.

(c) f and g have at most one common fixed point.

Proof. (a) \Rightarrow (b) follows immediately from Lemma 3.8.

Let (b) hold. If a_0 and b_0 are distinct common fixed points of f and g , by Definition 3.1(i), there exists $a_1 \in [a_0]_{\tilde{G}}$ such that $fa_0 = ga_1$. If $\rho(a_0, fa_1) \neq 0$, we have $\rho(a_0, fa_1) = \rho(fa_0, fa_1) \leq \psi(\rho(ga_0, ga_1)) = \psi(\rho(a_0, a_0)) = 0$, which is a contradiction. Therefore $\rho(a_0, fa_1) = 0$. By Lemma 2.7(a), $fa_1 = a_0$. Fix some $n \in \mathbb{N}$ and let $fa_i = a_0$ for $i \leq n$. There is $a_{n+1} \in [a_n]_{\tilde{G}}$ such that $fa_n = ga_{n+1}$. If $fa_{n+1} \neq a_0$, then by Lemma 2.7(a), $\rho(a_0, fa_{n+1}) \neq 0$. Therefore we have $\rho(a_0, fa_{n+1}) = \rho(fa_0, fa_{n+1}) \leq \psi(\rho(ga_0, ga_{n+1})) = \psi(\rho(ga_0, fa_n)) = 0$, which is a contradiction. Therefore $fa_n = a_0$ for each n . Similarly, one can show that there is a sequence $\{b_n\}$ such that $b_{n+1} \in [b_n]_{\tilde{G}} = [b_0]_{\tilde{G}}$, $fb_n = gb_{n+1}$ and $fb_n = b_0$, for each $n \in \mathbb{N}$. By our assumption, $\{fa_n\}$ and $\{fb_n\}$ are ρ -Cauchy equivalent. Since for each $n \in \mathbb{N}$, $fa_n = a_0$ and $fb_n = b_0$, by Lemma 2.7(a), $a_0 = b_0$. Thus (c) holds.

If (c) is true but G is not weakly connected, i.e., \tilde{G} is disconnected, then for some $a_0 \in X$, both sets $[a_0]_{\tilde{G}}$ and $X \setminus [a_0]_{\tilde{G}}$ are nonempty. Fix $b_0 \in X \setminus [a_0]_{\tilde{G}}$ and define $f, g : X \rightarrow X$ by

$$fx = \begin{cases} a_0 & \text{if } x \in [a_0]_{\tilde{G}} \\ b_0 & \text{if } x \in X \setminus [a_0]_{\tilde{G}} \end{cases}$$

and $gx = x$ for all $x \in X$. Trivially $\text{fix}\{f, g\} = \{a_0, b_0\}$. It is enough to show that f is a (ρ, ψ, G) -contraction with respect to g .

(i) Let $x \in X$. Then either $x \in [a_0]_{\tilde{G}}$ or $x \in X \setminus [a_0]_{\tilde{G}}$. Hence either $fx = a_0$ or $fx = b_0$. If $fx = a_0$, then $a_0 \in [x]_{\tilde{G}}$ and $fx = ga_0 = a_0$. If $fx = b_0$ then $b_0, x \in X \setminus [a_0]_{\tilde{G}}$, so $[b_0]_{\tilde{G}} = [x]_{\tilde{G}}$. Thus $b_0 \in [x]_{\tilde{G}}$ and $fx = gb_0 = b_0$.

(ii) Let $(x, y) \in E(G)$, then either $x, y \in [a_0]_{\tilde{G}}$ or $x, y \in X \setminus [a_0]_{\tilde{G}}$. By the definition either $fx = fy = a_0$ or $fx = fy = b_0$ in both cases $(fx, fy) \in E(G)$, also $(gx, gy) = (x, y) \in E(G)$.

(iii) Fix $x \in X$ and $y \in [x]_{\tilde{G}}$. Then we have two following cases: 1) $x, y \in [a_0]_{\tilde{G}}$; 2) $x, y \in X \setminus [a_0]_{\tilde{G}}$. In the first case, we get $\rho(fx, fy) = \rho(a_0, a_0) = 0 \leq \psi(\rho(gx, gy))$,

and in the second case, we have $\rho(fx, fy) = \rho(b_0, b_0) = 0 \leq \psi(\rho(gx, gy))$ for any arbitrary $\psi \in \Psi$. \square

We also need the following result.

LEMMA 3.10. *Let (X, ν) be a Hausdorff uniform space endowed with a graph G and an A -distance ρ . Let f and g be self-mappings on X and $\psi \in \Psi$ be such that f is a (ρ, ψ, G) -contraction with respect to g . Assume that $fx_0, gx_0 \in [x_0]_{\tilde{G}}$, for some $x_0 \in X$. Let \tilde{G}_{x_0} be the component of \tilde{G} containing x_0 . Then $[x_0]_{\tilde{G}}$ is both f and g -invariant and $f|_{[x_0]_{\tilde{G}}}$ is a $(\rho, \psi, \tilde{G}_{x_0})$ -contraction with respect to $g|_{[x_0]_{\tilde{G}}}$.*

Moreover, for arbitrary $y_0, z_0 \in [x_0]_{\tilde{G}}$, if f is orbitally bounded with respect to g at y_0 and z_0 , the sequences $\{fy_n\}$ and $\{fz_n\}$ are ρ -Cauchy equivalent, where $fy_n = gy_{n-1}$ and $fz_n = gz_{n-1}$ for each $n \geq 1$.

Proof. Let $x \in [x_0]_{\tilde{G}}$. We will show that $fx, gx \in [x_0]_{\tilde{G}}$. By our assumption, there exists a path $\{r_i\}_{i=0}^N$ in \tilde{G} from x_0 to x , i.e., $r_0 = x_0$, $r_N = x$ and $(r_{i-1}, r_i) \in E(\tilde{G})$ for all $1 \leq i \leq N$.

By Definition 3.1(ii), we get $(fr_{i-1}, fr_i) \in E(\tilde{G})$ for all $1 \leq i \leq N$. It means that $\{fr_i\}_{i=0}^N$ is a path in \tilde{G} from $fr_0 = fx_0$ to $fr_N = fx$. It follows that $fr_N = fx \in [fx_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$. Similarly one can see that $gx \in [x_0]_{\tilde{G}}$. Thus $[x_0]_{\tilde{G}}$ is both f and g -invariant.

Now, we will show that $f|_{[x_0]_{\tilde{G}}}$ is a $(\rho, \psi, \tilde{G}_{x_0})$ -contraction with respect to $g|_{[x_0]_{\tilde{G}}}$.

(i) Let $y_0 \in [x_0]_{\tilde{G}}$. Since f is a (ρ, ψ, G) -contraction with respect to g , by Definition 3.1(i), there exists $y_1 \in [y_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$ such that $fy_0 = gy_1$.

(ii) $(x, y) \in E(\tilde{G}_{x_0})$ implies $(x, y) \in E(\tilde{G})$. Thus $(fx, fy), (gx, gy) \in E(\tilde{G})$. In order to show that $(fx, fy), (gx, gy) \in E(\tilde{G}_{x_0})$, we note that if $(x, y) \in E(\tilde{G}_{x_0})$, then $x, y \in [x_0]_{\tilde{G}}$. By the above argument $fx, fy, gx, gy \in [x_0]_{\tilde{G}}$. Therefore (fx, fy) and (gx, gy) are in $E(\tilde{G}_{x_0})$.

(iii) Since $E(\tilde{G}_{x_0}) \subseteq E(\tilde{G})$ and f is a (ρ, ψ, \tilde{G}) -contraction with respect to g , we get $\rho(fx_0, fy_0) \leq \psi(\rho(gx_0, gy_0))$, for all $y_0 \in [x_0]_{\tilde{G}}$.

Now, let $y_0, z_0 \in [x_0]_{\tilde{G}}$ be such that f is orbitally bounded with respect to g at y_0 and z_0 . Since $[y_0]_{\tilde{G}} = [z_0]_{\tilde{G}}$, by Lemma 3.8, the sequences $\{fy_n\}$ and $\{fz_n\}$ are ρ -Cauchy equivalent, where $fy_n = gy_{n-1}$ and $fz_n = gz_{n-1}$ for each $n \geq 1$. \square

Now, we are ready to state of the main result of this section which gives some sufficient conditions for the existence and uniqueness of a common fixed point for self-mappings f and g where f is a (ρ, ψ, G) -contraction with respect to g on a Hausdorff uniform space (X, ν) .

THEOREM 3.11. *Let (X, ν) be a Hausdorff uniform space endowed with a graph G and an A -distance ρ , such that $\rho(x, x) = 0$ for all $x \in X$. Let $\psi \in \Psi$, X be S -complete and the triple (X, ρ, G) have the following property.*

() For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in X with $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$ and $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, there exists a subsequence $\{x_{k_n}\}_{n \in \mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$ for each $n \in \mathbb{N}$.*

Assume that $f, g : X \rightarrow X$ are commuting ρ -continuous mappings on X such that f is a (ρ, ψ, G) -contraction with respect to g and f is orbitally bounded with respect to g . Define $X_{f,g} = \{x_0 \in X : fx_0, gx_0 \in [x_0]_{\tilde{G}} \text{ and } (gx_n, fx_n) \in E(G) \text{ for all } n \in \mathbb{N}\}$, where $fx_{n-1} = gx_n$, $x_n \in [x_{n-1}]_{\tilde{G}}$ for each $n \in \mathbb{N}$. Then for each $x \in X_{(f,g)}$, the mappings $f|_{[x]_{\tilde{G}}}$ and $g|_{[x]_{\tilde{G}}}$ have a unique common fixed point. In particular, if $X_{(f,g)} \neq \emptyset$ and G is weakly connected, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X_{(f,g)}$, then $fx_0, gx_0 \in [x_0]_{\tilde{G}}$, $(gx_n, fx_n) = (fx_{n-1}, fx_n) \in E(G)$ for each $n \in \mathbb{N}$. Since f is orbitally bounded with respect to g at each point of X , Lemma 3.8 implies that for all $y_0 \in [x_0]_{\tilde{G}}$, the sequences $\{fx_n\}_{n \in \mathbb{N}}$ and $\{fy_n\}_{n \in \mathbb{N}}$ are ρ -Cauchy equivalent where $fx_{n-1} = gx_n$ and $fy_{n-1} = gy_n$, for each $n \in \mathbb{N}$. Since X is S -complete, there is $u \in X$ such that $\lim_{n \rightarrow \infty} \rho(fx_n, u) = 0$. Since for each $n \in \mathbb{N}$, $fx_{n-1} = gx_n$, we get $\lim_{n \rightarrow \infty} \rho(fx_n, u) = \lim_{n \rightarrow \infty} \rho(gx_n, u)$. Therefore $\lim_{n \rightarrow \infty} \rho(gx_n, u) = 0$. By our assumption f and g are ρ -continuous, hence $\lim_{n \rightarrow \infty} \rho(gfx_n, gu) = \lim_{n \rightarrow \infty} \rho(fgx_n, fu) = 0$. Since $fg = gf$, we have $\lim_{n \rightarrow \infty} \rho(fgx_n, fu) = \lim_{n \rightarrow \infty} \rho(fgx_n, gu) = 0$, and by Lemma 2.7(a), $gu = fu$. We will show that fu is a common fixed point of f and g . Since $fx_0, gx_0 \in [x_0]_{\tilde{G}}$, by Lemma 3.10, $[x_0]_{\tilde{G}}$ is both f and g -invariant. Moreover, for each $n \in \mathbb{N}$, $x_n \in [x_0]_{\tilde{G}}$, therefore $fx_n \in [x_0]_{\tilde{G}}$, for all n .

On the other hand $\lim_{n \rightarrow \infty} \rho(fx_n, u) = 0$ and $(fx_{n-1}, fx_n) \in E(G)$, for all $n \in \mathbb{N}$. Therefore by (*) there exists a subsequence $\{fx_{k_n}\}_{n \in \mathbb{N}}$ such that $(fx_{k_n}, u) \in E(G)$ for all $n \in \mathbb{N}$. Hence $(ffx_{k_n}, fu) \in E(G)$ for all $n \in \mathbb{N}$. Since for each n , $ffx_{k_n} \in [x_0]_{\tilde{G}}$, there is a finite sequence $r_0 = x_0, r_1, r_2, \dots, r_{M-1} = ffx_{k_1}, r_M = fu$ such that $(r_{i-1}, r_i) \in E(\tilde{G})$. It means $fu \in [x_0]_{\tilde{G}}$. By applying a similar argument, we see that $u \in [x_0]_{\tilde{G}}$. Thus $[fu]_{\tilde{G}} = [u]_{\tilde{G}}$. If $\rho(fu, ffu) \neq 0$, we have $\rho(fu, ffu) \leq \psi(\rho(gu, gfu)) = \psi(\rho(fu, ffu)) < \rho(fu, ffu)$ which is a contradiction. On the other hand $\rho(fu, fu) = 0$, by Lemma 2.7(a). Hence $ffu = fu$ and $gfu = fgfu = ffu = fu$. Therefore fu is a common fixed point of f and g . Since \tilde{G}_{x_0} is weakly connected, by Lemma 3.9, fu is a unique common fixed point of f and g .

If G is weakly connected then $[x]_{\tilde{G}} = X$. Therefore $f = f|_{[x]_{\tilde{G}}}$ and $g = g|_{[x]_{\tilde{G}}}$ have a unique common fixed point. \square

In 2004, Aamri and El Moutawakil [1] investigated the existence and uniqueness of common fixed point for two self-mappings on a Hausdorff uniform space as follows.

THEOREM 3.12 ([1, Theorems 3.1 and 3.2]). *Let (X, ν) be a Hausdorff uniform spaces and ρ be an A -distance on X . Suppose X is ρ -bounded and S -complete. Suppose that $\psi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ satisfies $\psi(t) > 0$ and $\lim_{n \rightarrow \infty} \psi^n(t) = 0$ for each $t > 0$. Let f and g be commuting ρ -continuous or $\tau(\nu)$ -continuous self mappings of X such that (i) $f(X) \subseteq g(X)$, (ii) $\rho(f(x), f(y)) \leq \psi(\rho(g(x), g(y)))$, for all $x, y \in X$. Then f and g have a common fixed point. Moreover if ρ is an E -distance, then f and g have a unique common fixed point.*

Let X be ρ -bounded and f be a (ρ, ψ, G) -contraction with respect to g . Then trivially f is orbitally bounded with respect to g . Thus Theorem 3.11 is a refinement of Theorem 3.12.

The following result shows that one can replace ρ -continuity of f by continuity of the A -distance ρ in Theorem 3.11.

THEOREM 3.13. *Let (X, ν) be a Hausdorff uniform space endowed with a graph G and a continuous A -distance ρ such that $\rho(x, x) = 0$ for all $x \in X$ and $\psi \in \Psi$. Let X be S -complete and the triple (X, ρ, G) have the property (*).*

Assume that f and g are commuting mappings on X such that f is a (ρ, ψ, G) -contraction with respect to g . Let g be ρ -continuous and let f be orbitally bounded with respect to g . Define $X_{f,g} = \{x_0 \in X : fx_0, gx_0 \in [x_0]_{\tilde{G}} \text{ and } (gx_n, fx_n) \in E(G) \text{ for all } n \in \mathbb{N}\}$, where $fx_{n-1} = gx_n$, $x_n \in [x_{n-1}]_{\tilde{G}}$ for each $n \in \mathbb{N}$.

Then for each $x \in X_{(f,g)}$, the mappings $f|_{[x]_{\tilde{G}}}$ and $g|_{[x]_{\tilde{G}}}$ have a unique common fixed point. In particular, if $X_{(f,g)} \neq \emptyset$ and G is weakly connected, then f and g have a unique common fixed point.

Proof. By applying the same argument as in the beginning of the proof of Theorem 3.11, we can find a sequence $\{x_n\}_{n \geq 0}$ and $u \in X$ such that $fx_0, gx_0 \in [x_0]_{\tilde{G}}$, $(gx_n, fx_n) \in E(G)$, $fx_{n-1} = gx_n$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} \rho(fx_n, u) = \lim_{n \rightarrow \infty} \rho(gx_n, u) = 0$.

Since $gx_0 \in [x_0]_{\tilde{G}}$ and for each $n \geq 0$, $(gx_n, gx_{n+1}) \in E(G)$, by (*) there exists a subsequence $\{gx_{k_n}\}$ of $\{gx_n\}$ such that $(gx_{k_n}, u) \in E(G)$ for each $n \in \mathbb{N}$. Hence $[gx_{k_n}]_{\tilde{G}} = [u]_{\tilde{G}}$, for each $n \in \mathbb{N}$. Thus

$$\rho(fgx_{k_n}, fu) \leq \psi(\rho(ggx_{k_n}, gu)) \leq \rho(ggx_{k_n}, gu), \quad (1)$$

Since $\lim_{n \rightarrow \infty} \rho(gx_{k_n}, u) = 0$ and g is ρ -continuous, Definition 2.6(iv) implies that $\lim_{n \rightarrow \infty} \rho(ggx_{k_n}, gu) = 0$. By (1) we get $\lim_{n \rightarrow \infty} \rho(fgx_{k_n}, fu) = 0$.

On the other hand, ρ -continuity of g implies that $\lim_{n \rightarrow \infty} \rho(gfx_n, gu) = 0$. Since f and g are commuting, $\lim_{n \rightarrow \infty} \rho(fgx_n, gu) = 0$. By Lemma 2.7(a), $fu = gu$.

The equality $[fx_n]_{\tilde{G}} = [u]_{\tilde{G}}$, for each $n \in \mathbb{N}$ together with continuity of ρ and ρ -continuity of g implies that

$$\lim_{n \rightarrow \infty} \rho(ffx_n, fu) \leq \lim_{n \rightarrow \infty} \psi(\rho(gfx_n, gu)) \leq \lim_{n \rightarrow \infty} \rho(gfx_n, gu) = \rho(gu, gu) = 0.$$

It means that the sequence $\{ffx_n\}$ is ρ -convergent to fu . The rest of the proof is similar to the end part of the proof of Theorem 3.11. \square

COROLLARY 3.14. *Let (X, d) be a complete b -metric space endowed with a graph G with the following property.*

*(**) For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in X with $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ and $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, there exists a subsequence $\{x_{k_n}\}_{n \in \mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$ for each $n \in \mathbb{N}$.*

Let d be continuous, $\psi \in \Psi$ and $f, g : X \rightarrow X$ be commuting mappings such that g is continuous and f is orbitally bounded with respect to g and the following conditions holds.

(i) For each $x \in X$ there exists $y \in [x]_{\tilde{G}}$ such that $fx = gy$.

(ii) For each $x, y \in X$, if $(x, y) \in E(G)$ then $(fx, fy), (gx, gy) \in E(G)$.

(iii) For each $x \in X$ and each $y \in [x]_{\tilde{G}}$, we have $d(fx, fy) \leq \psi(d(gx, gy))$.

Define $X_{f,g} = \{x_0 \in X : fx_0, gx_0 \in [x_0]_{\bar{G}} \text{ and } (gx_n, fx_n) \in E(G) \text{ for all } n \in \mathbb{N}\}$, where $fx_{n-1} = gx_n$, $x_n \in [x_{n-1}]_{\bar{G}}$ for each $n \in \mathbb{N}$. The mappings $f|_{[x]_{\bar{G}}}$ and $g|_{[x]_{\bar{G}}}$ have a unique common fixed point for each $x \in X_{(f,g)}$. In particular, if $X_{(f,g)} \neq \emptyset$ and G is weakly connected, then f and g have a unique common fixed point.

Proof. By Example 2.5, d generates a Hausdorff uniformity on X and, with respect to it, d is an A -distance for X . Conditions (i)-(iii) imply that f is a (d, ψ, G) -contraction with respect to g . Thus the result follows from Theorem 3.13. \square

EXAMPLE 3.15. Let $X = \left\{\frac{1}{n} : n \geq 1\right\} \cup \left\{\frac{-1}{n} : n \geq 1\right\} \cup \{0\}$. For $x, y \in X$ define $d(x, y) = |x - y|^2$. Then d is a b -metric on X . Indeed (X, d) satisfies conditions (1)-(3) in Example 2.5 for $s = 2$. Thus d defines a Hausdorff uniformity v_d on X . By Example 2.5, d is an A -distance on (X, v_d) .

Let $\{x_n\}$ be a Cauchy sequence in X . It means that for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $m, n > N_0$ implies that $d(x_m, x_n) < \varepsilon$. Therefore either $x_n = x$, for some $x \in X$ and for large enough n , or $x_n \rightarrow 0$ as $n \rightarrow \infty$. Thus X is complete.

Define graph G , by $V(G) = X$ and

$$E(G) = \Delta(X) \cup \left\{\left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{-1}{2}, \frac{-1}{3}\right)\right\} \cup \left\{\left(\frac{1}{n}, \frac{1}{n+1}\right) : n \geq 4\right\} \\ \cup \left\{\left(\frac{-1}{n}, \frac{-1}{n+1}\right) : n \geq 4\right\} \cup \left\{\left(\frac{1}{n}, 0\right) : n \geq 1\right\} \cup \left\{\left(\frac{-1}{n}, 0\right) : n \geq 1\right\}.$$

Then G is weakly connected. Assume that $\psi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ is defined by $\psi(r) = \frac{r}{2}$ which belongs to Ψ and let $f, g : X \rightarrow X$ be defined by

$$fx = \begin{cases} \frac{1}{3} & \text{if } x = 1 \\ \frac{-1}{3} & \text{if } x = -1 \\ 0 & \text{if } x \neq 1, -1 \end{cases} \quad \text{and} \quad gx = \begin{cases} x & \text{if } x = 0, 1, -1, \frac{1}{3}, \frac{-1}{3} \\ \frac{1}{1+n} & \text{if } x = \frac{1}{n}, n > 1, n \neq 3 \\ \frac{-1}{1+n} & \text{if } x = \frac{-1}{n}, n > 1, n \neq 3 \end{cases}.$$

Then $fgx = gfx$ for all $x \in X$, and $f(X) = \left\{0, \frac{1}{3}, \frac{-1}{3}\right\} \subseteq g(X) = \left\{0, \pm 1, \pm \frac{1}{3}, \pm \frac{1}{5}, \pm \frac{1}{6}, \dots\right\}$. Moreover, f is orbitally bounded with respect to g at each point of X .

Assume that $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ for some $x \in X$. By the definition of d , we get $\lim_{n \rightarrow \infty} |x_n - x|^2 = 0$.

Hence $\lim_{n \rightarrow \infty} |x_n - x| = 0$. It means for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $n \geq N_\varepsilon$ implies that $|x_n - x| < \varepsilon$. Hence either $x_n = x$ for large enough n or $x = 0$. In both cases we get $\lim_{n \rightarrow \infty} d(gx_n, gx) = 0$, thus g is continuous.

Also, the triple (X, d, G) satisfies the property (***) of Corollary 3.14. One can easily check that the following conditions hold.

(i) G is weakly connected and $f(X) \subseteq g(X)$. Thus for each $x \in X$ there exists $y \in [x]_{\bar{G}} = X$ such that $fx = gy$.

(ii) For each $(x, y) \in E(G)$ we have $(fx, fy), (gx, gy) \in E(G)$.

(iii) For each $x \in X$ and $y \in [x]_{\bar{G}} = X$, $d(fx, fy) \leq \psi(d(gx, gy))$.

Therefore f is (ρ, ψ, G) -contraction with respect to g . Moreover $0 \in X_{f,g} \neq \emptyset$. Since G is weakly connected $[0]_{\bar{G}} = X$. By Corollary 3.14, f and g have a unique common fixed point on $[0]_{\bar{G}} = X$, i.e. $x = 0$.

ACKNOWLEDGEMENT. The authors would like to thank the anonymous reviewers for their careful reading of the manuscript and useful suggestions.

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(received 22.12.2018; in revised form 10.09.2019; available online 23.12.2019)

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