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A NOTE ON THE MINIMAL DISPLACEMENT FUNCTION

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Abstract. Let (X,d) be a metric space and $\mathrm{Iso}(X,d)$ the associated isometry group. We study the subadditivity of the minimal displacement function $f:\mathrm{Iso}(X,d)\to\mathbb{R}$ for different metric spaces. When (X,d) is ultrametric, we prove that the minimal displacement function is subadditive. We show, by a simple algebraic argument, that subadditivity does not hold for the direct isometry group of the hyperbolic plane. The same argument can be used for other metric spaces.

1. Introduction

Fundamental in the form we sense the world is the concept of symmetry. Mathematically, symmetry is realized by means of isometries in appropriate spaces. Following [6], given a metric space (X,d), we define an isometry as a function $\phi: X\to X$ that preserves distances, $d(\phi(x),\phi(y))=d(x,y)$ and is onto. Associated to isometries there is a well defined number, the so called minimal displacement or minimal translation length, which is defined purely analytically as follows. For any isometry ϕ,

$$f(\phi)=\inf_{x\in X} d(\phi x,x).$$

The study of the minimal displacement function can be traced back to the following problem in functional analysis [3]: given a subset K of a Banach space and a mapping $T:K\to K$, estimate the quantity $\inf\{\|x-Tx\|: x\in K\}$. As pointed out by Goebel and Kirk [4, p. 210], this problem is meaningful only in situations where it is not already known that the mapping T has a fixed point. An example is the case of certain continuous mappings $T:B\to B$ on the unit ball B of an infinitely dimensional Banach space, where Brouwer's fixed point theorem does not hold.

In geometry and metric group theory, the minimal displacement function is used to classify isometries into the three types (elliptic, hyperbolic and parabolic), generalizing the work of Felix Klein in the 19th century on the classification of isometries of the hyperbolic plane $\mathbb{H}$, see [6, p. 258].

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A generalization of the minimal displacement function is now in use, with many applications to metric group theory, see for instance the recent preprint by Breuillard and Fujiwara [1] and the bibliography therein.

We recall that, given a set S with some algebraic structure, (e.g. a group or a semigroup) a subadditive function to $\mathbb{R}$ is any function $f : S \to \mathbb{R}$ satisfying

$$f(g \cdot h) \leqslant f(g) + f(h). \tag{1}$$

Subadditive functions are particularly interesting due to their remarkable properties (e.g. every numerical semigroup can be obtained from a subadditive, periodic function $f : \mathbb{N} \to \mathbb{Q}_0^+$, see [8, Theorem 5.5]).

Let (X, d) be a metric space and $\mathrm{Iso}(X, d)$ the associated isometry group. A natural question is to understand if the minimal displacement function $f : \mathrm{Iso}(X, d) \to \mathbb{R}$ is subadditive. In other words, given any isometries ϕ, ψ we want to know if the following inequality holds:

$$\inf_{x \in X} d(\phi\psi x, x) \leqslant \inf_{x \in X} d(\phi x, x) + \inf_{x \in X} d(\psi x, x). \tag{2}$$

In this note we provide some examples showing that the inequality (2) does hold for some metric spaces but not always. As expected, the inequality (2) depends on the metric d over the space X. When (X, d) is ultrametric, we show that the minimal displacement function is always subadditive. In spite of the analytic nature of the minimal displacement function we give a purely algebraic argument which proves that the minimal displacement function of the hyperbolic plane, when restricted to the direct isometry group, is not subadditive. The same argument can easily be applied to other isometry groups under certain conditions.

2. Properties of minimal displacement function

Let f be the minimal displacement function defined on the set $\mathrm{Iso}(X, d)$ of isometries of X. It is well known that $\mathrm{Iso}(X, d)$ has a group structure with product given by composition: $(g \cdot h)(x) = g(h(x))$, for every $g, h \in \mathrm{Iso}(X, d)$. We have already noted that the minimal displacement function is used to characterize isometries. Following several authors (see [6] and the references therein), we say that an isometry is *elliptic* if the infimum is attained and is zero, *hyperbolic* if the infimum is attained and is greater than zero and *parabolic* if the infimum is not attained. Define $I_{e\ell\ell}$ as the subset of elliptic isometries.

For any $\phi \in \mathrm{Iso}(X, d)$ we have

$$d(\phi x, x) = d(\phi^{-1}\phi x, \phi^{-1} x) = d(x, \phi^{-1} x) = d(\phi^{-1} x, x),$$

where we have used the definition of isometry and the symmetry of the metric d. Taking the infimum, we have

$$f(g^{-1}) = f(g), \tag{3}$$

that is, f is inverse invariant. We also have $d(\phi\psi\phi^{-1} x, x) = d(\psi\phi^{-1} x, \phi^{-1} x)$. Again, taking the infimum we get

$$f(\phi\psi\phi^{-1}) = \inf_{x\in X} d(\psi\phi^{-1}x, \phi^{-1}x) = \inf_{y\in X} d(\psi y, y) = f(\psi), \tag{4}$$

that is, f is conjugacy invariant. Hence, we have the following well known facts.

Proposition 2.1. *Let (X, d) be a metric space and f the minimal displacement function. Then, for all isometries ϕ, ψ, $f(\phi) = f(\phi^{-1})$ and $f(\phi\psi\phi^{-1}) = f(\psi)$.*

For a given metric space (X, d), there is always a subgroup of $\mathrm{Iso}(X, d)$ for which the restriction of the minimal displacement function is trivially subadditive. In fact, given $p \in X$, define the set $\mathrm{Iso}_p(X, d) = \{\phi \in \mathrm{Iso}(X, d) : \phi(p) = p\}$. It can easily be proved that $\mathrm{Iso}_p(X, d)$ is a subgroup of $\mathrm{Iso}(X, d)$, called the *isotropy group* of p (or stabilizer of p). Since every element of $\mathrm{Iso}_p(X, d)$ is elliptic we conclude the following.

Proposition 2.2. *The restriction $f|_{\mathrm{Iso}_p(X,d)} : \mathrm{Iso}_p(X, d) \to \mathbb{R}$ of the minimal displacement function is subadditive.*

3. Examples

In this section we give examples of metric spaces and verify if the inequality (2) holds.

3.1 Euclidean line

The first example is the usual Euclidean metric space $(\mathbb{R}, d)$, with $d(x, y) = |x - y|$. The group of isometries of $(\mathbb{R}, d)$ is the semidirect product: $\mathrm{Iso}(\mathbb{R}, d) \cong \mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z}$, where $(\mathbb{R}, +)$ is the additive group of $\mathbb{R}$, identified as the group of translations of $\mathbb{R}$, and $\mathbb{Z}/2\mathbb{Z} = \{-1, 1\}$ is the group with two elements. The latter is the orthogonal group $O(1)$. It follows that $\mathrm{Iso}(\mathbb{R}, d)$ is not simple, having $\mathbb{R}$ as a normal subgroup. The group $\mathrm{Iso}(\mathbb{R}, d)$ may be realized as follows: $\mathrm{Iso}(\mathbb{R}, d) = \langle \phi_k, \psi_k : k \in \mathbb{R}\rangle$, where $\phi_k(x) = x + k$ and $\psi_k(x) = -x + k$. Since ψ_k has a fixed point, it is an elliptic isometry, for every $k \in \mathbb{R}$, whereas ϕ_k is hyperbolic for $k \neq 0$. Take ψ_m and ψ_n with $m \neq n$. Then, $\psi_n\psi_m = \phi_{n-m}$. It follows that the minimal displacement function for the space $(\mathbb{R}, d)$ is not subadditive. Notice that the argument works for every Euclidean space $(\mathbb{E}, d)$.

3.2 Hyperbolic plane

The upper plane model of hyperbolic geometry is the set $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$, endowed with the metric $d_{\mathbb{H}}$ given by

$$\cosh d_{\mathbb{H}}(z, w) = 1 + \frac{|z - w|^2}{2\mathrm{Im}\, z \cdot \mathrm{Im}\, w}$$

The group of order preserving isometries $\mathrm{Iso}^+(\mathbb{H}, d)$, also known as the direct isometry group, is given by the Möbius transformations

$$z \mapsto \frac{az + b}{cz + d},$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

Now, take

$$\phi_d(z) = \frac{1}{-z + 2d}.$$

This is an elliptic isometry precisely when $d \in]-1, 1[$ since $z_0 = d + i\sqrt{1 - d^2}$ is a fixed point of ϕ_d and $z_0 \in \mathbb{H}$. Pick two isometries $\phi_d, \phi_{d'}$. The composition is

$$\phi_d \phi_{d'}(z) = \frac{z - 2d'}{2dz + (1 - 4dd')}.$$

Choosing d, d' such that $dd' < 0$ we can see that $\phi_d \phi_{d'}$ fixes two real points,

$$\frac{dd' \pm \sqrt{(dd')^2 - dd'}}{d},$$

in which case the isometry is hyperbolic. Again, the minimal displacement function for the space $(\mathbb{H}, d_{\mathbb{H}})$ is not subadditive.

3.3 French railway metric space

Let (X, d) be a metric space and fix $p \in X$. Define a new metric $d_{F,p} = d_F$ on X by

$$d_F(x, y) = \begin{cases} d(x, p) + d(p, y) & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Note that, since d is a metric on X, $d_F(x, y) = 0$ if, and only if, $x = y$. The above metric is called the *French Railway metric*.

In [2, Theorem 2.2], the author proved that, for every isometry $\phi \in \mathrm{Iso}(X, d_F)$, p is a fixed point of ϕ. Since $\mathrm{Iso}_p(X, d_F) = \mathrm{Iso}(X, d_F)$, from Proposition 2.2 we obtain:

PROPOSITION 3.1. *The minimal displacement function for the space (X, d_F) is subadditive.*

3.4 Ultrametric spaces

The last example includes a large class of metric spaces. A metric space (X, d) is called ultrametric if it satisfies the strong triangle inequality

$$d(x, y) \leq \max \{d(x, z), d(z, y)\}, \tag{5}$$

for every $x, y, z \in X$.

Given any two isometries ϕ, ψ of (X, d) and $x \in X$, consider the three points x, $\phi^{-1}x$ and ψx. Now, we use the fact that in an ultrametric space, every triangle is an acute isosceles. This means that either

$$d(\psi x, \phi^{-1}x) = d(\phi^{-1}x, x) \quad \text{or} \quad d(\psi x, \phi^{-1}x) = d(\psi x, x) \quad \text{or} \quad d(\phi^{-1}x, x) = d(\psi x, x).$$

If $d(\psi x, \phi^{-1}x) = d(\phi^{-1}x, x) = d(\phi x, x)$ then taking the infimum on both sides we have $\inf_{x \in X} d(\psi x, \phi^{-1}x) = \inf_{x \in X} d(\phi x, x)$ and so

$$\inf_{x \in X} d(\phi\psi x, x) = \inf_{x \in X} d(\phi x, x) \leq \inf_{x \in X} d(\phi x, x) + \inf_{x \in X} d(\psi x, x).$$

If $d(\psi x, \phi^{-1}x) = d(\psi x, x)$, again we conclude by the same reasoning that

$$\inf_{x\in X} d(\phi\psi x, x) = \inf_{x\in X} d(\psi x, x) \leq \inf_{x\in X} d(\psi x, x) + \inf_{x\in X} d(\phi x, x).$$

Finally, suppose that $d(\phi^{-1}x, x) = d(\psi x, x)$. By the strong triangle inequality (5) we have $d(\psi x, \phi^{-1}x) \leq \max\{d(\psi x, x), d(x, \phi^{-1}x)\} = d(\phi^{-1}x, x)$. So, $d(\phi\psi x, x) \leq d(\phi x, x)$. Taking the infimum we obtain

$$\inf_{x\in X} d(\phi\psi x, x) \leq \inf_{x\in X} d(\phi x, x) \leq \inf_{x\in X} d(\phi x, x) + \inf_{x\in X} d(\psi x, x).$$

We have concluded the following result for ultrametric spaces.

Theorem 3.2. *Let (X, d) be an ultrametric space. Then the minimal displacement function is subadditive.*

4. A class of subadditive functions

Let G be a group. We focus on functions $f : G \to \mathbb{R}$ that are subadditive (1) and inverse invariant (3). Denoting by e the identity of G, we have

$$f(e) = f(gg^{-1}) \leqslant f(g) + f(g^{-1}) = 2f(g).$$

Taking $g = e$ in the above inequality we obtain $f(e) \geqslant 0$. Hence, $f(g) \geqslant 0$ for every $g \in G$. Suppose there exists an element $g_o \in G$ such that $f(g_o) = 0$. Define the subset of G, $K = \{g \in G : f(g) = 0\}$. Since $0 \leqslant f(e) \leqslant 2f(g_o)$ then $f(e) = 0$ and $e \in K$. Also if $g \in K$ then $g^{-1} \in K$ by (3). On the other hand, if $g, h \in K$ it follows that $0 \leqslant f(gh) \leqslant f(g) + f(h) = 0$ and $gh \in K$. So we conclude the following

Proposition 4.1. *Let f be subadditive and inverse invariant and suppose K is not empty. Then K is a subgroup of G.*

Recall that a subgroup N of G is called *normal* if $gng^{-1} \in N$ for all $g \in G$ and $n \in N$. A group G is said to be *simple* if the only normal subgroups of G are $\{e\}$ and G itself.

Suppose that we have a subadditive, inverse invariant function f on G. Let K be defined as above. If it happens that f is also conjugacy invariant (4) on G we have, for all $g \in G$, $h \in K$, $f(ghg^{-1}) = f(h) = 0$ and so K is a normal subgroup of G.

5. The hyperbolic plane

In Examples 3.1 and 3.2 we saw that the inequality (2) fails, by constructing two elliptic isometries whose composition is a hyperbolic isometry. In this section we will use a simple algebraic criteria to achieve the same conclusion for the hyperbolic plane.

Let f be the minimal displacement function and assume that f is also subadditive. Clearly K defined as above is a subgroup of $\mathrm{Iso}^+(\mathbb{H}, d_{\mathbb{H}})$. By conjugacy invariance K is even a normal subgroup of $\mathrm{Iso}^+(\mathbb{H}, d_{\mathbb{H}})$. By definition of elliptic isometries we clearly have $I_{e\ell\ell} \subset K$. So $K \neq \{e\}$. On the other hand, $\mathrm{Iso}^+(\mathbb{H}, d_{\mathbb{H}})$ has plenty hyperbolic isometries (see Example 3.2), which implies that $K \neq \mathrm{Iso}^+(\mathbb{H}, d_{\mathbb{H}})$. Now, we note that in this setting $\mathrm{Iso}^+(\mathbb{H}, d_{\mathbb{H}})$ is a simple group, see [5, 7]. And this is a contradiction. We conclude the following.

THEOREM 5.1. *The minimal displacement function* $f : \mathrm{Iso}^+(\mathbb{H}, d_{\mathbb{H}}) \to \mathbb{R}$ *is not subadditive.*

REMARK 5.2. Let (X, d) be a metric space with $G = \mathrm{Iso}(X, d)$ simple and $I_{e\ell\ell} \backslash \{id\}$ nonempty. Then the above result applies and the minimal displacement function for (X, d) is not subadditive.

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