

SEMILATTICE DECOMPOSITION OF LOCALLY ASSOCIATIVE Γ -AG**-GROUPOIDS

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Abstract. In this paper, we have shown that a locally associative Γ -AG**-groupoid S has associative powers and S/ρ_Γ is a maximal separative homomorphic image of S , where $a\rho_\Gamma b$ implies that $a\Gamma b_\Gamma^n = b_\Gamma^{n+1}$, $b\Gamma a_\Gamma^n = a_\Gamma^{n+1}$, $\forall a, b \in S$. The relation η_Γ is the least left zero semilattice congruence on S , where η_Γ is defined on S as $a\eta_\Gamma b$ if and only if there exist some positive integers m, n such that $b_\Gamma^m \subseteq a\Gamma S$ and $a_\Gamma^n \subseteq b\Gamma S$.

1. Introduction

An Abel-Grassmann's groupoid [11] (abbreviated as an AG-groupoid), is a groupoid S whose elements satisfy the invertive law $(ab)c = (cb)a$, for all $a, b, c \in S$. It is also called a left almost semigroup [3, 7, 8]. In [2], the same structure is called a left invertive groupoid. It is a useful non-associative algebraic structure, midway between a groupoid and a commutative semigroup.

An AG-groupoid S is medial [3], that is, $(ab)(cd) = (ac)(bd)$, for all $a, b, c, d \in S$. If an AG-groupoid satisfies the following property:

$$a(bc) = b(ac), \text{ for all } a, b, c \in S, \quad (1)$$

then it is called an AG**-groupoid (cf. [5, 10]). In an AG**-groupoid S the law $(ab)(cd) = (db)(ca)$ holds for all $a, b, c, d \in S$ (cf. [10]).

An AG-groupoid S is called a locally associative AG-groupoid if $(aa)a = a(aa)$ holds for all $a \in S$. If S is a locally associative AG-groupoid, then it is easy to see that $(Sa)S = S(aS)$ or $(SS)S = S(SS)$. If a locally associative AG-groupoid S satisfies the identity (1), then S is known as a locally associative AG**-groupoid.

An element a of S is called left zero if $ax = a$, for all $x \in S$.

Locally associative LA-semigroups have been studied by Mushtaq et al. [6, 7]. Other notions and results on AG-groupoids and AG**-groupoids, one can find in [2–5, 8–11, 13].

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M.K. Sen [12] introduced the concept of Γ -semigroup in 1981. The non-associative Γ -AG-groupoid is the generalization of an associative Γ -semigroup.

Let S and Γ be two non-empty sets. Denote by the letters of English alphabet the elements of S and by the letters of Greek alphabet the elements of Γ . Any map from $S \times \Gamma \times S$ to S will be called a Γ -multiplication in S and denoted by $(\cdot)_{\Gamma}$. The result of this multiplication for $a, b \in S$ and $\alpha \in \Gamma$ is denoted by $a\alpha b$. A Γ -AG-groupoid [1] S is an ordered pair $(S, (\cdot)_{\Gamma})$ where S and Γ are non-empty sets and $(\cdot)_{\Gamma}$ is a Γ -multiplication on S which satisfies the following Γ -left invertive law: $\forall(a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2$,

$$(x\alpha y)\beta z = (z\alpha y)\beta x. \quad (2)$$

A Γ -AG-groupoid also satisfies the Γ -medial law $\forall(w, x, y, z, \alpha, \beta, \gamma) \in S^4 \times \Gamma^3$,

$$(w\alpha x)\beta(y\gamma z) = (w\alpha y)\beta(x\gamma z). \quad (3)$$

Note that if a Γ -AG-groupoid contains a left identity, then it becomes an AG-groupoid with left identity. A Γ -AG-groupoid is called a Γ -AG** -groupoid [1] if it satisfies the following law $\forall(x, y, z, \alpha, \beta) \in S^3 \times \Gamma^2$,

$$x\alpha(y\beta z) = y\alpha(x\beta z). \quad (4)$$

A Γ -AG** -groupoid also satisfies the following Γ -paramedial law

$$\forall(w, x, y, z, \alpha, \beta, \gamma) \in S^4 \times \Gamma^3, (w\alpha x)\beta(y\gamma z) = (z\alpha y)\beta(x\gamma w).$$

Other concepts and results on Γ -AG** -groupoids one can find in [1].

In this paper, we introduce a new non-associative algebraic structure namely locally associative Γ -AG** -groupoids and decompose it using Γ -congruences. An AG-groupoid S is called a locally associative Γ -AG-groupoid if $(aaa)\beta a = a\alpha(a\beta a)$ holds for all $a \in S$ and $\alpha, \beta \in \Gamma$. If S is a locally associative AG-groupoid, then it is easy to see that $(S\Gamma a)\Gamma S = S\Gamma(a\Gamma S)$ or $(S\Gamma S)\Gamma S = S\Gamma(S\Gamma S)$. For particular $\alpha \in \Gamma$, let us denote $aaa = a_{\alpha}^2$ for some $\alpha \in \Gamma$ and $aaa = a_{\Gamma}^2$, for all $\alpha \in \Gamma$, i.e., $a\Gamma a = a_{\Gamma}^2$ and generally $a\Gamma a\Gamma a \dots a\Gamma a = a_{\Gamma}^n$ (n times).

2. Main results

Let S be an Γ -AG** -groupoid and a relation ρ_{Γ} be defined on S as follows : $a\rho_{\Gamma}b$ if and only if there exists a positive integer n such that $a\Gamma b_{\Gamma}^n = b_{\Gamma}^{n+1}$ and $b\Gamma a_{\Gamma}^n = a_{\Gamma}^{n+1}$, for all a and b in S .

PROPOSITION 2.1. *If S is a locally associative Γ -AG** -groupoid, then $a\Gamma a_{\Gamma}^{n+1} = (a_{\Gamma}^{n+1})\Gamma a$, for all a in S and positive integer n .*

Proof. $a\Gamma a_{\Gamma}^{n+1} = a\Gamma(a_{\Gamma}^n\Gamma a) = a_{\Gamma}^n\Gamma(a\Gamma a) = (a_{\Gamma}^{n-1}\Gamma a)\Gamma(a\Gamma a)$
 $= (a\Gamma a)\Gamma(a\Gamma a_{\Gamma}^{n-1}) = (a\Gamma a)\Gamma a_{\Gamma}^n = (a_{\Gamma}^n\Gamma a)\Gamma a = (a_{\Gamma}^{n+1})\Gamma a. \quad \square$

PROPOSITION 2.2. *In a locally associative Γ -AG** -groupoid S , $a_{\Gamma}^m\Gamma a_{\Gamma}^n = a_{\Gamma}^{m+n}$, for all $a \in S$ and positive integers m, n .*

$$\begin{aligned}
\text{Proof. } a_{\Gamma}^{m+1}\Gamma a_{\Gamma}^n &= (a_{\Gamma}^m\Gamma a)\Gamma a_{\Gamma}^n = (a_{\Gamma}^n\Gamma a)\Gamma a_{\Gamma}^m = (a\Gamma a_{\Gamma}^n)\Gamma a_{\Gamma}^m = (a_{\Gamma}^m\Gamma a_{\Gamma}^n)\Gamma a \\
&= a_{\Gamma}^{m+n}\Gamma a = a_{\Gamma}^{m+n+1}. \quad \square
\end{aligned}$$

PROPOSITION 2.3. *If S is a locally associative Γ -AG^{**}-groupoid, then for all $a, b \in S$, $(a\Gamma b)_{\Gamma}^n = a_{\Gamma}^n\Gamma b_{\Gamma}^n$ for a positive integer $n \geq 1$ and $(a\Gamma b)_{\Gamma}^n = b_{\Gamma}^n\Gamma a_{\Gamma}^n$, for $n \geq 2$.*

Proof. We have

$$\begin{aligned}
(a\Gamma b)_{\Gamma}^2 &= (a\Gamma b)\Gamma(a\Gamma b) = (a\Gamma a)\Gamma(b\Gamma b) = a_{\Gamma}^2\Gamma b_{\Gamma}^2 \\
(a\Gamma b)_{\Gamma}^{k+1} &= (a\Gamma b)_{\Gamma}^k\Gamma(a\Gamma b) = (a_{\Gamma}^k\Gamma b_{\Gamma}^k)\Gamma(a\Gamma b) = (a_{\Gamma}^k\Gamma a)\Gamma(b_{\Gamma}^k\Gamma b) = a_{\Gamma}^{k+1}\Gamma b_{\Gamma}^{k+1}.
\end{aligned}$$

Let $n \geq 2$. Then by (4) and (2), we get

$$\begin{aligned}
(a\Gamma b)_{\Gamma}^n &= a_{\Gamma}^n\Gamma b_{\Gamma}^n = (a\Gamma a_{\Gamma}^{n-1})\Gamma(b\Gamma b_{\Gamma}^{n-1}) = b\Gamma((a\Gamma a_{\Gamma}^{n-1})\Gamma b_{\Gamma}^{n-1}) = b\Gamma((b_{\Gamma}^{n-1}\Gamma a_{\Gamma}^{n-1})\Gamma a) \\
&= b\Gamma((b\Gamma a)_{\Gamma}^{n-1}\Gamma a) = (b\Gamma a)_{\Gamma}^{n-1}\Gamma(b\Gamma a) = (b\Gamma a)_{\Gamma}^n = b_{\Gamma}^n\Gamma a_{\Gamma}^n. \quad \square
\end{aligned}$$

PROPOSITION 2.4. *In a locally associative Γ -AG^{**}-groupoid S , $(a_{\Gamma}^m)_{\Gamma}^n = a_{\Gamma}^{mn}$ for all $a \in S$ and positive integers m, n .*

$$\text{Proof. } (a_{\Gamma}^{m+1})_{\Gamma}^n = (a_{\Gamma}^m\Gamma a)_{\Gamma}^n = (a_{\Gamma}^m)_{\Gamma}^n\Gamma a_{\Gamma}^n = a_{\Gamma}^{mn}\Gamma a_{\Gamma}^n = a_{\Gamma}^{mn+n} = a_{\Gamma}^{n(m+1)}. \quad \square$$

THEOREM 2.5. *Let S be a locally associative Γ -AG^{**}-groupoid. If $a\Gamma b_{\Gamma}^m = b_{\Gamma}^{m+1}$ and $b\Gamma a_{\Gamma}^n = a_{\Gamma}^{n+1}$, for $a, b \in S$ and positive integers m, n , then $a\rho_{\Gamma}b$.*

$$\text{Proof. If } n > m, \text{ then } b_{\Gamma}^{n-m}\Gamma(a\Gamma b_{\Gamma}^m) = b_{\Gamma}^{n-m}\Gamma b_{\Gamma}^{m+1}, \quad a\Gamma(b_{\Gamma}^{n-m}\Gamma b_{\Gamma}^m) = b_{\Gamma}^{n-m+m+1}, \\
a\Gamma b_{\Gamma}^{n-m+m} = b_{\Gamma}^{n+1}, \quad a\Gamma b_{\Gamma}^n = b_{\Gamma}^{n+1}. \quad \square$$

THEOREM 2.6. *The relation ρ_{Γ} on a locally associative Γ -AG^{**}-groupoid is a congruence relation.*

Proof. Evidently ρ_{Γ} is reflexive and symmetric. For transitivity we may proceed as follows.

Let $a\rho_{\Gamma}b$ and $b\rho_{\Gamma}c$ so that there exist positive integers n, m such that $a\Gamma b_{\Gamma}^n = b_{\Gamma}^{n+1}$, $b\Gamma a_{\Gamma}^n = a_{\Gamma}^{n+1}$, and $b\Gamma c_{\Gamma}^m = c_{\Gamma}^{m+1}$, $c\Gamma b_{\Gamma}^m = b_{\Gamma}^{m+1}$.

Let $k = (n+1)(m+1) - 1$, that is, $k = n(m+1) + m$. Using (2), (4) and Proposition 2.2, 2.3 and 2.4, we get

$$\begin{aligned}
a\Gamma c_{\Gamma}^k &= a\Gamma c_{\Gamma}^{n(m+1)+m} = a\Gamma(c_{\Gamma}^{n(m+1)}\Gamma c_{\Gamma}^m) = a\Gamma\{(c_{\Gamma}^{m+1})_{\Gamma}^n\Gamma c_{\Gamma}^m\} = a\Gamma\{(b\Gamma c_{\Gamma}^m)_{\Gamma}^n\Gamma c_{\Gamma}^m\} \\
&= a\Gamma\{(b_{\Gamma}^n\Gamma c_{\Gamma}^{mn})\Gamma c_{\Gamma}^m\} = a\Gamma(c_{\Gamma}^{m(n+1)}\Gamma b_{\Gamma}^n) = c_{\Gamma}^{m(n+1)}\Gamma(a\Gamma b_{\Gamma}^n) \\
&= c_{\Gamma}^{m(n+1)}\Gamma b_{\Gamma}^{n+1} = (c_{\Gamma}^m\Gamma b)_{\Gamma}^{n+1} = b_{\Gamma}^{n+1}\Gamma c_{\Gamma}^{m(n+1)} = (b\Gamma c_{\Gamma}^m)_{\Gamma}^{n+1} = c_{\Gamma}^{k+1}.
\end{aligned}$$

Similarly, $c\Gamma a^k = a_{\Gamma}^{k+1}$. Thus ρ_{Γ} is an equivalence relation. To show that ρ_{Γ} is compatible, assume that $a\rho_{\Gamma}b$ such that for some positive integer n , $a\Gamma b_{\Gamma}^n = b_{\Gamma}^{n+1}$ and $b\Gamma a_{\Gamma}^n = a_{\Gamma}^{n+1}$.

Let $c \in S$, then by identity (3) and Propositions 2.4 and 2.1, we get

$$(a\Gamma c)\Gamma(b\Gamma c)_{\Gamma}^n = (a\Gamma c)\Gamma(b_{\Gamma}^n\Gamma c_{\Gamma}^n) = (a\Gamma b_{\Gamma}^n)\Gamma(c\Gamma c_{\Gamma}^n) = b_{\Gamma}^{n+1}\Gamma c_{\Gamma}^{n+1} = (b\Gamma c)_{\Gamma}^{n+1}.$$

Similarly, $(b\Gamma c)\Gamma(a\Gamma c)_{\Gamma}^n = (a\Gamma c)_{\Gamma}^{n+1}$. Hence ρ_{Γ} is a congruence relation on S . \square

LEMMA 2.7. *Let S be a locally associative Γ -AG^{**}-groupoid; then $a\Gamma b\rho_\Gamma b\Gamma a$, for all $a, b \in S$.*

$$\begin{aligned} \text{Proof.} \quad (a\Gamma b)\Gamma(b\Gamma a)_\Gamma^{n+1} &= (a\Gamma b)\Gamma(a_\Gamma^{n+1}\Gamma b_\Gamma^{n+1}) = (a\Gamma a_\Gamma^{n+1})\Gamma(b\Gamma b_\Gamma^{n+1}) \\ &= a_\Gamma^{n+2}\Gamma b_\Gamma^{n+2} = (b\Gamma a)_\Gamma^{n+2}. \end{aligned}$$

Similarly, $(b\Gamma a)\Gamma(a\Gamma b)_\Gamma^{n+1} = (a\Gamma b)_\Gamma^{n+2}$. Hence $a\Gamma b\rho_\Gamma b\Gamma a$, for all $a, b \in S$. \square

A relation ρ_Γ on an AG-groupoid S is called separative if $a\Gamma b\rho_\Gamma a_\Gamma^2$ and $a\Gamma b\rho_\Gamma b_\Gamma^2$ imply that $a\rho_\Gamma b$.

THEOREM 2.8. *The relation ρ_Γ is separative.*

Proof. Let $a, b \in S$, $a\Gamma b\rho_\Gamma a_\Gamma^2$ and $a\Gamma b\rho_\Gamma b_\Gamma^2$. Then by the definition of ρ_Γ , there exist positive integers m and n such that,

$$(a\Gamma b)\Gamma(a_\Gamma^2)_\Gamma^m = (a_\Gamma^2)_\Gamma^{m+1}, a_\Gamma^2\Gamma(a\Gamma b)_\Gamma^m = (a\Gamma b)_\Gamma^{m+1}$$

$$\text{and} \quad (a\Gamma b)\Gamma(b_\Gamma^2)_\Gamma^n = (b_\Gamma^2)_\Gamma^{n+1}, b_\Gamma^2\Gamma(a\Gamma b)_\Gamma^n = (a\Gamma b)_\Gamma^{n+1}.$$

$$\begin{aligned} \text{Then} \quad (a\Gamma b)\Gamma a_\Gamma^{2m} &= (a\Gamma b)\Gamma(a_\Gamma^m\Gamma a_\Gamma^m) = (a\Gamma a_\Gamma^m)\Gamma(b\Gamma a_\Gamma^m) \\ &= (a_\Gamma^{m+1})\Gamma(b\Gamma a_\Gamma^m) = b\Gamma(a_\Gamma^{m+1}\Gamma a_\Gamma^m) = b\Gamma a_\Gamma^{2m+1}, \end{aligned}$$

but $(a\Gamma b)\Gamma a_\Gamma^{2m} = (a_\Gamma^2)_\Gamma^{m+1} = a_\Gamma^{2m+2}$, which implies that $b\Gamma a_\Gamma^{2m+1} = a_\Gamma^{2m+2}$. Also, $(a\Gamma b)\Gamma(b_\Gamma^2)_\Gamma^n = (b_\Gamma^2)_\Gamma^{n+1}$ implies that $b_\Gamma^{2n+1}\Gamma a = b_\Gamma^{2n+2}$. Also, we get $b_\Gamma^{2n+2}\Gamma b_\Gamma^2 = (b_\Gamma^{2n+1}\Gamma a)\Gamma b_\Gamma^2$, which implies that $b_\Gamma^{2n+4} = b_\Gamma^2\Gamma(a\Gamma b_\Gamma^{2n+1}) = a\Gamma(b_\Gamma^2\Gamma b_\Gamma^{2n+1}) = a\Gamma b_\Gamma^{2n+3}$. Hence by Theorem 2.5, $a\rho_\Gamma b$. \square

THEOREM 2.9. *Let S be a locally associative Γ -AG^{**}-groupoid. Then S/ρ_Γ is a maximal separative commutative image of S .*

Proof. By Theorem 2.8, ρ_Γ is separative, and hence S/ρ_Γ is separative. We now show that ρ_Γ is contained in every separative congruence relation σ_Γ on S . Let $a\rho_\Gamma b$, so that there exists a positive integer n such that $a\Gamma b_\Gamma^n = b_\Gamma^{n+1}$ and $b\Gamma a_\Gamma^n = a_\Gamma^{n+1}$.

We need to show that $a\sigma_\Gamma b$, where σ_Γ is a separative congruence on S . Let k be any positive integer such that

$$a\Gamma b_\Gamma^k\Gamma\sigma_\Gamma b_\Gamma^{k+1} \quad \text{and} \quad b\Gamma a_\Gamma^k\Gamma\sigma_\Gamma a_\Gamma^{k+1} \quad (5)$$

Suppose that $k \geq 3$.

$$\begin{aligned} (a\Gamma b_\Gamma^{k-1})_\Gamma^2 &= (a\Gamma b_\Gamma^{k-1})\Gamma(a\Gamma b_\Gamma^{k-1}) = a_\Gamma^2\Gamma b_\Gamma^{2k-2} = (a\Gamma a)\Gamma(b_\Gamma^{k-2}\Gamma b_\Gamma^k) \\ &= (a\Gamma b_\Gamma^{k-2})\Gamma(a\Gamma b_\Gamma^k) = (a\Gamma b_\Gamma^{k-2})\Gamma b_\Gamma^{k+1}. \end{aligned}$$

Therefore $(a\Gamma b_\Gamma^{k-2})\Gamma(a\Gamma b_\Gamma^k)\sigma_\Gamma(a\Gamma b_\Gamma^{k-2})\Gamma b_\Gamma^{k+1}$.

Using the identity (2) and Proposition 2.2, we get

$$(a\Gamma b_\Gamma^{k-2})\Gamma b_\Gamma^{k+1} = (b_\Gamma^{k+1}\Gamma b_\Gamma^{k-2})\Gamma a = b_\Gamma^{2k-1}\Gamma a = (b_\Gamma^k\Gamma b_\Gamma^{k-1})\Gamma a = (a\Gamma b_\Gamma^{k-1})\Gamma b_\Gamma^k$$

$$\text{Also} \quad (a\Gamma b_\Gamma^{k-1})\Gamma b_\Gamma^k = (b_\Gamma^k\Gamma b_\Gamma^{k-1})\Gamma a = b_\Gamma^{2k-1}\Gamma a = (b_\Gamma^{k-1}\Gamma b_\Gamma^k)\Gamma a = (a\Gamma b_\Gamma^k)\Gamma b_\Gamma^{k-1},$$

implying that $(a\Gamma b_\Gamma^{k-1})_\Gamma^2\sigma_\Gamma(a\Gamma b_\Gamma^k)\Gamma b_\Gamma^{k-1}$.

Since $a\Gamma b_\Gamma^k\sigma_\Gamma b_\Gamma^{k+1}$ and $(a\Gamma b_\Gamma^k)\Gamma b_\Gamma^{k-1}\sigma_\Gamma b_\Gamma^{k+1}\Gamma b_\Gamma^{k-1}$, hence $(a\Gamma b_\Gamma^{k-1})_\Gamma^2\sigma_\Gamma(b_\Gamma^k)_\Gamma^2$. It further implies that $(a\Gamma b_\Gamma^{k-1})_\Gamma^2\sigma_\Gamma(a\Gamma b_\Gamma^{k-1})\Gamma b_\Gamma^k\sigma_\Gamma(b_\Gamma^k)_\Gamma^2$. Thus $a\Gamma b_\Gamma^{k-1}\sigma_\Gamma b_\Gamma^k$. Similarly, $b\Gamma a_\Gamma^{k-1}\sigma_\Gamma a_\Gamma^k$. Thus if (5) holds for k , it holds for $k-1$.

Now obviously (5) yields $a\Gamma b_\Gamma^3 \sigma'_\Gamma b_\Gamma^4$ and $b\Gamma a_\Gamma^3 \sigma'_\Gamma a_\Gamma^4$. Also, we get

$$\begin{array}{lll} (a\Gamma b_\Gamma^3)\Gamma a_\Gamma^2 \sigma'_\Gamma b_\Gamma^4 \Gamma a_\Gamma^2 & \text{and} & (b\Gamma a_\Gamma^3)\Gamma b_\Gamma^2 \sigma'_\Gamma a_\Gamma^4 \Gamma b_\Gamma^2, \\ (a_\Gamma^2 \Gamma b_\Gamma^3)\Gamma a \sigma'_\Gamma \Gamma b_\Gamma^4 \Gamma a_\Gamma^2 & \text{and} & (b_\Gamma^2 \Gamma a_\Gamma^3)\Gamma b \sigma'_\Gamma \Gamma a_\Gamma^4 \Gamma b_\Gamma^2, \\ (b_\Gamma^3 \Gamma a_\Gamma^2)\Gamma a \sigma'_\Gamma \Gamma a_\Gamma^2 \Gamma b_\Gamma^4 & \text{and} & (a_\Gamma^3 \Gamma b_\Gamma^2)\Gamma b \sigma'_\Gamma \Gamma b_\Gamma^2 \Gamma a_\Gamma^4, \\ a_\Gamma^3 \Gamma b_\Gamma^3 \sigma'_\Gamma a_\Gamma^2 \Gamma b_\Gamma^4 & \text{and} & b_\Gamma^3 \Gamma a_\Gamma^3 \sigma'_\Gamma b_\Gamma^2 \Gamma a_\Gamma^4, \\ a_\Gamma^3 \Gamma b_\Gamma^3 \sigma'_\Gamma a_\Gamma^2 \Gamma b_\Gamma^4 & \text{and} & a_\Gamma^3 \Gamma b_\Gamma^3 \sigma'_\Gamma b_\Gamma^2 \Gamma a_\Gamma^4, \end{array}$$

which imply that $(b_\Gamma^2 \Gamma a)_\Gamma^2 \sigma'_\Gamma a_\Gamma^3 \Gamma b_\Gamma^3 \sigma'_\Gamma (a_\Gamma^2 \Gamma b)_\Gamma^2$, and as σ'_Γ is separative and $(b_\Gamma^2 \Gamma a)\Gamma(a_\Gamma^2 \Gamma b) = (b_\Gamma^2 \Gamma a_\Gamma^2)\Gamma(a\Gamma b) = (a_\Gamma^2 \Gamma b_\Gamma^2)\Gamma(a\Gamma b) = a_\Gamma^3 \Gamma b_\Gamma^3$, so $a_\Gamma^2 \Gamma b \sigma'_\Gamma b_\Gamma^2 \Gamma a$. Now we get: $(a_\Gamma^2 \Gamma b)\Gamma a \sigma'_\Gamma \Gamma (b_\Gamma^2 \Gamma a)\Gamma a$, $(a\Gamma b)\Gamma a_\Gamma^2 \sigma'_\Gamma a_\Gamma^2 \Gamma b_\Gamma^2$, $a_\Gamma^2 \Gamma (b\Gamma a)\sigma'_\Gamma a_\Gamma^2 \Gamma b_\Gamma^2$, $b\Gamma a_\Gamma^3 \sigma'_\Gamma a_\Gamma^2 \Gamma b_\Gamma^2$, but $b\Gamma a_\Gamma^3 \sigma'_\Gamma a_\Gamma^4$.

Thus $(b\Gamma a)_\Gamma^2 \sigma'_\Gamma b\Gamma a_\Gamma^3 \sigma'_\Gamma (a_\Gamma^2)_\Gamma^2$. Now since σ'_Γ is separative and $a_\Gamma^2 \Gamma (b\Gamma a) = b\Gamma a_\Gamma^3$, so we get $b\Gamma a \sigma'_\Gamma a_\Gamma^2$.

Similarly we can obtain $a\Gamma b \sigma'_\Gamma b_\Gamma^2$.

Also it is easy to show that (5) holds for $k = 2$. Thus if (5) holds for k , it holds for $k = 1$. By induction down from k , it follows that (5) holds for $k = 1$, $a\Gamma b \sigma_\Gamma b_\Gamma^2$ and $b\Gamma a \sigma_\Gamma a_\Gamma^2$. Now using (2) and Proposition 2.4 on $a\Gamma b \sigma_\Gamma b_\Gamma^2$, we get $(b\Gamma a)_\Gamma^2 \sigma_\Gamma b_\Gamma^3 \Gamma a$, and again using (4) and (2) on $a\Gamma b \sigma_\Gamma b_\Gamma^2$ we get $b_\Gamma^3 \Gamma a \sigma_\Gamma b_\Gamma^4$. So $(b\Gamma a)_\Gamma^2 \sigma_\Gamma b_\Gamma^3 \Gamma a \sigma_\Gamma b_\Gamma^4$ implies that $b\Gamma a \sigma_\Gamma b_\Gamma^2$ which further implies that $a\Gamma b \sigma_\Gamma b\Gamma a$. Thus we obtain $a \sigma_\Gamma b$. Hence $\rho_\Gamma \subseteq \sigma_\Gamma$ and so S/ρ_Γ is the maximal separative commutative image of S . \square

LEMMA 2.10. *If $x\Gamma a = x$ ($a = a_\Gamma^2$) for some x in a locally associative Γ -AG^{**}-groupoid S , then $x_\Gamma^n \Gamma a = x_\Gamma^n$ for some positive integer n .*

Proof. Let $n = 2$. By using (3), we get

$$x_\Gamma^2 \Gamma a = (x\Gamma x)\Gamma(a\Gamma a) = (x\Gamma a)\Gamma(x\Gamma a) = x\Gamma x = x_\Gamma^2.$$

Let the result be true for k , that is, $x_\Gamma^k \Gamma a = x_\Gamma^k$. Then by (3) and Proposition 2.1, we get $x_\Gamma^{k+1} \Gamma a = (x\Gamma x_\Gamma^k)\Gamma(a\Gamma a) = (x\Gamma a)\Gamma(x_\Gamma^k \Gamma a) = x\Gamma x_\Gamma^k = x_\Gamma^{k+1}$. Hence $x_\Gamma^n \Gamma a = x_\Gamma^n$ for all positive integers n . \square

LEMMA 2.11. *If S is a Γ -AG-groupoid, then $Q_\Gamma = \{x \in S \mid x\Gamma a = x \text{ and } a = a_\Gamma^2\}$ is a commutative subsemigroup.*

Proof. As $a\Gamma a = a$, we have $a \in Q_\Gamma$. Now if $x, y \in Q_\Gamma$, then by identity (3), $x\Gamma y = (x\Gamma a)\Gamma(y\Gamma a) = (x\Gamma y)\Gamma(a\Gamma a) = (x\Gamma y)\Gamma a$.

To prove that Q_Γ is commutative and associative, assume that x, y and z belong to Q_Γ . Then by using (2), we get $x\Gamma y = (x\Gamma a)\Gamma y = (y\Gamma a)\Gamma x = y\Gamma x$. Also, $(x\Gamma y)\Gamma z = (z\Gamma y)\Gamma x = x\Gamma(y\Gamma z)$. Hence Q_Γ is a commutative subsemigroup of S . \square

THEOREM 2.12. *Let ρ_Γ and σ_Γ be separative congruences on locally associative Γ -AG^{**}-groupoid S and $x_\Gamma^2 \Gamma a = x_\Gamma^2$ ($a = a_\Gamma^2$) for all $x \in S$. If $\rho_\Gamma \cap (Q_\Gamma \times Q_\Gamma) \subseteq \sigma_\Gamma \cap (Q_\Gamma \times Q_\Gamma)$, then $\rho_\Gamma \subseteq \sigma_\Gamma$.*

Proof. If $x\rho_\Gamma y$, then $(x_\Gamma^2 \Gamma (x\Gamma y))_\Gamma^2 \rho_\Gamma (x_\Gamma^2 \Gamma (x\Gamma y)\Gamma(x_\Gamma^2 \Gamma y_\Gamma^2))_\Gamma^2 \rho_\Gamma (x_\Gamma^2 \Gamma y_\Gamma^2)_\Gamma^2$. It follows that $(x_\Gamma^2 \Gamma (x\Gamma y))_\Gamma^2, (x_\Gamma^2 \Gamma y_\Gamma^2)_\Gamma^2 \subseteq Q_\Gamma$. Now by (3), (2), (4), respectively, we get

$$(x_\Gamma^2 \Gamma (x\Gamma y))_\Gamma^2 \Gamma (x_\Gamma^2 \Gamma y_\Gamma^2)_\Gamma^2 = (x_\Gamma^2 \Gamma x_\Gamma^2)_\Gamma^2 \Gamma (x\Gamma y)\Gamma y_\Gamma^2 = (x_\Gamma^2 \Gamma x_\Gamma^2)_\Gamma^2 \Gamma (y_\Gamma^3 \Gamma x)$$

Also since $b\rho_\Gamma b_\Gamma^2$ and ρ_Γ is compatible, so we get $b\Gamma y\rho_\Gamma b_\Gamma^2\Gamma y$. We can easily see that $b\Gamma a\rho_\Gamma a\Gamma b\rho_\Gamma a\rho_\Gamma b\Gamma y\rho_\Gamma b_\Gamma^2\Gamma y$ which implies that $b\Gamma a\rho_\Gamma b_\Gamma^2\Gamma y$. Similarly, we can show that $a\Gamma b\rho_\Gamma a_\Gamma^2\Gamma x$. So $a\rho_\Gamma b\Gamma y\rho_\Gamma b_\Gamma^2\Gamma y\rho_\Gamma b\Gamma a\rho_\Gamma a\Gamma b\rho_\Gamma a_\Gamma^2\Gamma x\rho_\Gamma a\Gamma x\rho_\Gamma b$ implies that $a\rho_\Gamma b$. Thus η_Γ is a least semilattice congruence on S . \square

THEOREM 2.15. η_Γ is separative.

Proof. Let $a_\Gamma^2\eta_\Gamma a\Gamma b$ and $a\Gamma b\eta_\Gamma b_\Gamma^2$, then there exist positive integers m, m', n, n' such that $(a_\Gamma^2)_\Gamma^m = (a\Gamma b)_\Gamma^2\Gamma x$, $(a\Gamma b)_\Gamma^{m'} = (a_\Gamma^2)_\Gamma^m\Gamma x'$ and $(a\Gamma b)_\Gamma^{n'} = (b_\Gamma^2)_\Gamma^2\Gamma y'$, $(b_\Gamma^2)_\Gamma^n = (a\Gamma b)_\Gamma^2\Gamma y$.

$$\begin{aligned} \text{Now we get, } a_\Gamma^{2m+2} &= a_\Gamma^{2m}\Gamma a_\Gamma^2 = (a_\Gamma^2)_\Gamma^m\Gamma a_\Gamma^2 = ((a\Gamma b)_\Gamma^2\Gamma x)\Gamma a_\Gamma^2 \\ &= (a_\Gamma^2\Gamma x)\Gamma (a\Gamma b)_\Gamma^2 = (a_\Gamma^2\Gamma x)\Gamma (a_\Gamma^2\Gamma b_\Gamma^2) = (a_\Gamma^2\Gamma x)\Gamma (b_\Gamma^2\Gamma a_\Gamma^2) \\ &= b_\Gamma^2\Gamma ((a_\Gamma^2\Gamma x)\Gamma a_\Gamma^2) = b_\Gamma^2\Gamma t_6, \text{ where } t_6 = ((a_\Gamma^2\Gamma x)\Gamma a_\Gamma^2) \end{aligned}$$

$$\begin{aligned} \text{Similarly, } b_\Gamma^{2n+2} &= b_\Gamma^{2n}\Gamma b_\Gamma^2 = ((a\Gamma b)_\Gamma^2\Gamma y)\Gamma b_\Gamma^2 = (b_\Gamma^2\Gamma y)\Gamma (a_\Gamma^2\Gamma b_\Gamma^2) \\ &= a_\Gamma^2\Gamma ((b_\Gamma^2\Gamma y)\Gamma b_\Gamma^2) = a_\Gamma^2\Gamma t_7, \text{ where } t_7 = ((b_\Gamma^2\Gamma y)\Gamma b_\Gamma^2). \end{aligned}$$

Hence η_Γ is separative. \square

THEOREM 2.16. Let S be a locally associative Γ -AG^{**}-groupoid. Then S/η_Γ is a maximal semilattice separative image of S .

Proof. By Theorem 2.14, η_Γ is the least semilattice congruence on S and S/η_Γ is a semilattice. Hence S/η_Γ is a maximal semilattice separative image of S . \square

THEOREM 2.17. Every locally associative Γ -AG^{**}-groupoid S is uniquely expressible as a semilattice Y of Archimedean locally associative Γ -AG^{**}-groupoids $(S_\pi)_\Gamma$ ($\pi \in Y$). The semilattice Y is isomorphic with the maximal semilattice separative image S/η_Γ of S and $(S_\pi)_\Gamma$ ($\pi \in Y$) are the equivalence classes of $S \text{ mod } \eta_\Gamma$.

Proof. By Theorem 2.14, η_Γ is the least semilattice congruence on S . Next we will prove that the equivalence classes $\text{mod } \eta_\Gamma$ are Archimedean locally associative Γ -AG^{**}-groupoids and the semilattice Y is isomorphic to S/η_Γ . Let $a, b \in (S_\pi)_\Gamma$, where $\pi \in Y$; then $a\eta_\Gamma b$ implies that $a_\Gamma^m \subseteq b\Gamma S, b_\Gamma^n \subseteq a\Gamma S$, so $a_\Gamma^m = b\Gamma x$ and $b_\Gamma^n = a\Gamma y$, where $x, y \in S$. If $x \in S_\vartheta, \vartheta \neq \pi$, then $\pi = \pi\vartheta$, using (4), and we get $a_\Gamma^{m+1} = a\Gamma a_\Gamma^m = a\Gamma (b\Gamma x) = b\Gamma (a\Gamma x) \subseteq b\Gamma (S_{\pi\vartheta})_\Gamma = b\Gamma (S_\pi)_\Gamma$. Similarly, one can show that $b_\Gamma^{n+1} \subseteq a\Gamma (S_\pi)_\Gamma$. This shows that $(S_\pi)_\Gamma$ is right Archimedean and so it is locally associative Archimedean Γ -AG^{**}-groupoid S .

Next we show the uniqueness. Let S be a semilattice Y of Archimedean AG^{**}-groupoid $(S_\pi)_\Gamma, \pi \in Y$. We need to show that $(S_\pi)_\Gamma$ are equivalent classes of $S \text{ mod } \eta_\Gamma$. Let $a, b \in S$. Then we show that $a\eta_\Gamma b$ if and only if a and b belong to the same $(S_\pi)_\Gamma$. If a and b both belong to the same $(S_\pi)_\Gamma$, then each divides the power of the other. Since $(S_\pi)_\Gamma$ is Archimedean, $a\eta_\Gamma b$ by the definition. Conversely, if $a\eta_\Gamma b$, then $a\Gamma x = b_\Gamma^m$ and $b\Gamma y = a_\Gamma^n$ for some $x, y \in S$ and some $m, n \in \mathbb{Z}^+$. If $x \in (S_\vartheta)_\Gamma$, then $a\Gamma x \subseteq (S_{\pi\vartheta})_\Gamma$ and $b_\Gamma^m \subseteq (S_\vartheta)_\Gamma$, so that $\pi\vartheta = \vartheta$. Hence $\vartheta \leq \pi$, in the semilattice Y . By symmetry, it follows that $\pi \leq \vartheta$, that is, $\pi = \vartheta$. \square

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