MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 01.06.2020

research paper
оригинални научни рад

SOME COMMON FIXED POINT THEOREMS FOR CONTRACTIVE MAPPINGS IN CONE 2-METRIC SPACES EQUIPPED WITH A TERNARY RELATION

V. H. Badshah, Prakash Bhagat and Satish Shukla

Abstract. In this paper, we study some common fixed point results in cone 2-metric spaces equipped with a ternary relation $\mathcal{T}$. A weaker version of weakly compatible mappings, the notions of g-contractions with respect to $\mathcal{T}$ and g-φ-contractions with respect to $\mathcal{T}$ are introduced. Some common fixed point results for g-contractions and g-φ-contractions with respect to an arbitrary ternary relation $\mathcal{T}$ and a transitive ternary relation respectively, are proved. To justify the newly introduced notions and results several examples are provided.

1. Introduction

In a series of papers, Gähler [11–13] introduced the concept of 2-metric spaces. He considered the metric function as the area of a triangle formed by three points in the space and extended the triangular inequality of an ordinary metric. On the other hand, Huang and Zhang [14] introduced the notion of cone metric spaces by replacing the codomain of metric function by an ordered Banach space and obtained some basic versions of fixed point theorems in this new setting. Recently, in the papers [7, 8, 16, 17] it was shown that several notions and fixed point results in cone metric spaces are the consequences of their metric counterparts. Inspired by this fact, Liu and Xu [18] defined the cone metric spaces over Banach algebras and proved the fixed point results for the contractive mappings with vector contractive constants. Singh et al. [25] combined the concepts of cone metric and 2-metric, and introduced a new type of spaces called cone 2-metric spaces and proved a fixed point result in this new setting. Following the idea of Liu and Xu [18], Wang et al. [26] improved the result of Singh et al. [25] by introducing the cone 2-metric spaces over Banach algebras.

2010 Mathematics Subject Classification: 47H10, 54H25
Keywords and phrases: Fixed point; cone 2-metric; J-weakly compatible pair; α-φ-contractive mapping; ternary relation.

Abbas and Junck [1] first introduced common fixed point results in the setting of cone metric spaces. They used weak compatibility of mappings defined on cone metric spaces. Bari and Vetro [5] introduced the notion of ψ-pairs and proved common fixed point theorems in cone metric spaces.

Ran and Reurings [22] initiated the study of fixed point theorems in metric spaces endowed with a partial order relation. Recently, Alam and Imdad [24] proved fixed point results in metric spaces endowed with an arbitrary binary relation and generalized and extended several fixed point theorems.

In this paper, we introduce a weaker version of weakly compatible mappings in cone 2-metric spaces, which generalizes the notion of weakly compatible mappings in cone 2-metric spaces. Our results extend the results of Abbas and Junck [1] and Alam and Imdad [24] in cone 2-metric spaces over Banach algebras equipped with a ternary relation $\mathcal{T}$. The notions of g-contraction with respect to $\mathcal{T}$ and g-φ-contraction with respect to $\mathcal{T}$ in cone 2-metric spaces are introduced. Some common fixed point results for g-contractions and g-φ-contraction with respect to an arbitrary ternary relation $\mathcal{T}$ and a transitive ternary relation, respectively, are proved.

2. Preliminaries

We first state some definitions and properties which will be used in the sequel. Throughout the paper, we assume that A is a real Banach algebra with a multiplicative unit e.

It is well known that if the spectral radius $\rho(x)$ of $x \in A$ is less than 1, that is, $\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} < 1$, then $e - x$ is invertible, and $(e - x)^{-1} = \sum_{i=0}^{\infty} x^i$.

A subset P of A is called a cone if:

(i) P is nonempty, closed and $\{\theta, e\} \subset P$, where θ is the zero vector of A;

(ii) $\alpha P + \beta P \subset P$ for all nonnegative real numbers α, β;

(iii) $P^2 = PP \subset P$; (iv) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset A$, we define a partial ordering $\preceq$ in A with respect to P by $x \preceq y$ (or equivalently $y \succeq x$) if and only if $y - x \in P$. We shall write $x \prec y$ (or equivalently $y \succ x$) to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ (or equivalently $y \gg x$) will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P.

The cone P is called normal if there exists a number $K > 0$ such that for all $x, y \in P$ with $x \preceq y$ we have $\|x\| \leq K\|y\|$. The least number K satisfying this inequality is called the normal constant of P. The cone P is called solid if $\text{int}P \neq \emptyset$.

In what follows, we always assume that P is a solid cone in A and $\preceq$ is the partial ordering with respect to P.

Lemma 2.1 ([17,20]). *(a) If $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

(b) If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.

(c) If $\|x_n\| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n > n_0$.

(d) If $a, b, c \in P$ such that $a \preceq b$ and $b \ll c$, then $a \ll c$.

(e) If $a, b, c \in P$ such that $a \ll b$ and $b \preceq c$, then $a \ll c$.

(f) If $a, b, c \in P$ such that $a \ll b$ and $b \ll c$, then $a \ll c$.

REMARK 2.2. Clearly, if $\rho(x) < 1$ then $\|x^n\| \to 0$ as $n \to \infty$.

DEFINITION 2.3 ([6]). A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n > n_0$.

PROPOSITION 2.4 ([27]). *If $\{u_n\}$ ia a c-sequence in P and $k \in P$ then $\{ku_n\}$ is a c-sequence.*

PROPOSITION 2.5 ([27]). *For any $a, b \in A$, $c \in P$ with $a \preceq b$ we have $ac \preceq bc$.*

DEFINITION 2.6 ([2, 25, 26]). Let X be a nonempty set. Suppose that a mapping $d \colon X \times X \times X \to P$ satisfies:
(i) for every $x, y \in X$ with $x \neq y$ there exists $z \in X$ such that $d(x, y, z) \neq \theta$;

(ii) if at least two of $x, y, z \in X$ are equal, then $d(x, y, z) = \theta$;

(iii) $d(x, y, z) = d(p(x, y, z))$ for all $x, y, z \in X$, where $p(x, y, z)$ denotes all the permutations of x, y, z;

(iv) $d(x, y, z) \preceq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.
Then d is called a cone 2-metric on X and (X, d) is called a cone 2-metric space over the Banach algebra A. Cone 2-metric space will be called normal if its cone P is normal.

DEFINITION 2.7 ([25]). Let (X, d) be a cone 2-metric space with a solid cone P in a Banach algebra A. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in A$ with $\theta \ll c$ (that is, $c \in \text{int}P$) there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x, a) \ll c$ for all $a \in X$ and for all $n > n_0$, then $\{x_n\}$ is said to converge to x. We denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. Obviously, the limit of a convergent sequence is unique.

If for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m, a) \ll c$ for all $a \in X$ and for all $n, m > n_0$, then $\{x_n\}$ is said to be a Cauchy sequence. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be a complete cone 2-metric space.

Let X be a nonempty set and $f, g \colon X \to X$ be two mappings. Recall (see, e.g., [1]) that a point $x \in X$ is called a coincidence point and a point $y \in X$ is called the corresponding point of coincidence of the pair (f, g) if $y = fx = gx$. By $CO(f, g)$, we denote the set of all coincidence points of the pair (f, g). The pair (f, g) is called weakly compatible if f and g commute at every coincidence point of the pair (f, g). Further, a sequence $\{x_n\}$ in X is called a Picard sequence of f with initial value $x_0 \in X$ if $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. One can always construct a Picard sequence with an arbitrary initial point in X. A sequence $\{y_n\}$ in X is called a Jungck sequence of the pair (f, g) with initial point $x_0 \in X$, if $y_n = gx_n = fx_{n-1}$ for all $n \in \mathbb{N}$. Obviously, it is not always possible to construct a Jungck sequence for a given initial point $x_0 \in X$. In particular, if $x_0 \in X$ is such that $fx_0 \notin g(X)$, then it is not possible to construct a Jungck sequence of the pair (f, g) with initial point $x_0 \in X$.

3. Main results

We first introduce a weaker version of weakly compatible pairs of mappings.

Definition 3.1. Let (X, d) be a 2-metric space and $f, g\colon X \to X$ be two mappings. Then, the pair (f, g) is called J-weakly compatible if whenever a Jungck sequence of the pair (f, g) converges to some gx^* with $x^* \in CO(f, g)$, then f and g commute at x^*.

Remark 3.2. The concept of J-weakly compatible pair is more general than that of weakly compatible. Indeed, every weakly compatible pair (f, g) is a J-weakly compatible pair, but the converse of this fact is not necessarily true.

Example 3.3. Let $X = [0, \infty)$, $A = C^1([0, 1], \mathbb{R})$ be the Banach algebra with pointwise multiplication and the norm defined by $\|x(t)\| = \|x(t)\|_\infty + \|x'(t)\|_\infty$. Let $P = \{x(t) \in C^1([0, 1], \mathbb{R})\colon x(t) \geq 0 \text{ for all } t \in [0, 1]\}$; then P is a solid cone in A. Define a function $d\colon X \times X \times X \to P$ by

$$d(x, y, z) = \begin{cases} (xy + yz + zx)e^t, & \text{if } x \neq y \neq z; \\ 0, & \text{otherwise.} \end{cases}$$

Then (X, d) is a cone 2-metric space over Banach algebras A. Let $f, g\colon X \to X$ be defined by $fx = x^2$ for all $x \in X$ and $gx = \dfrac{x}{2}$ for all $x \in X$. Then, it is obvious that the pair (f, g) is J-weakly compatible, but it is not a weakly compatible pair. Indeed, $x = \dfrac{1}{2}$ is a coincidence point of the pair (f, g), but $fg\frac{1}{2} = \frac{1}{16} \neq gf\frac{1}{2} = \frac{1}{8}$.

Let X be a nonempty set and by $P(x, y, z)$ denote the collection of all permutations of $x, y, z \in X$. A nonempty subset $\mathcal{T} \subseteq X \times X \times X$ is called a ternary relation on X and $(X, \mathcal{T})$ is called a ternary structure. Three elements $x, y, z \in X$ are called related by $\mathcal{T}$ if at least one permutation of (x, y, z) is an element of $\mathcal{T}$. A ternary relation $\mathcal{T}$ on X is called transitive if the following implication holds: $(x, y, z), (y, w, z) \in \mathcal{T} \implies (x, w, z) \in \mathcal{T}$.

A sequence $\{x_n\}$ in X is called a $\mathcal{T}$-sequence if $(x_{n-1}, x_n, a) \in \mathcal{T}$ for all $a \in X$.

Definition 3.4. Let $(X, \mathcal{T})$ be a ternary structure and $f\colon X \to X$ be a mapping. Then, f is called $\mathcal{T}$ preserving if: $(x, y, z) \in \mathcal{T} \implies (fx, fy, z) \in \mathcal{T}$ for all $x, y, z \in X$.

Example 3.5. Let $(X, \mathcal{T})$ be an arbitrary ternary structure and $f\colon X \to X$ be the mapping defined by $fx = x$ for all $x \in X$. Then, f is $\mathcal{T}$ preserving.

Example 3.6. Let $X = \mathbb{R}^3$ and $\|\cdot\|$ be the Euclidean norm on X. Define a ternary relation $\mathcal{T}$ on X by: $\mathcal{T} = \{(x, y, z)\colon \|x\| \leq \|y\| \leq 1, \|z\| = 1\}$. Let $f\colon X \to X$ be the rotation about the axis of x through an angle θ in anti-clockwise direction, i.e., $f(x_1, x_2, x_3) = (x_1, x_2 \cos\theta - x_3 \sin\theta, x_2 \sin\theta + x_3 \cos\theta)$. Then f is $\mathcal{T}$ preserving.

Definition 3.7. Let $(X, \mathcal{T})$ be a ternary structure and $f, g\colon X \to X$ be two mappings. Then, f is called $\mathcal{T}$ preserving with respect to g if: $(gx, gy, z) \in \mathcal{T} \implies (fx, fy, z) \in \mathcal{T}$ for all $x, y, z \in X$.

It is obvious that if we choose g as the identity mapping, then a $\mathcal{T}$ preserving mapping with respect to g is reduced into $\mathcal{T}$ preserving mapping. The next example shows that a $\mathcal{T}$ preserving mapping with respect to g may not be a $\mathcal{T}$ preserving mapping.

Example 3.8. Let $X = \mathbb{R}^3$ and $\mathcal{T}$ be the ternary relation which relates all the points of xy-plane inside and on the unit circle $C\colon x^2 + y^2 = 1$, i.e.,

$$\mathcal{T} = \{((x_1, x_2, 0), (y_1, y_2, 0), (z_1, z_2, 0))\colon (x_1, x_2), (y_1, y_2), (z_1, z_2) \in C\}.$$

Let $f, g\colon X \to X$ be mappings defined by $f(x_1, x_2, x_3) = (2x_1, 2x_2, \sin x_3)$; $g(x_1, x_2, x_3) = (4x_1, 4x_2, x_3 - \pi)$ for all $x_1, x_2, x_3 \in \mathbb{R}$. Then f is not $\mathcal{T}$ preserving, but $\mathcal{T}$ preserving with respect to g.

Definition 3.9. Let $(X, \mathcal{T})$ be a ternary structure, (X, d) be a cone 2-metric space and $f\colon X \to X$ be a mapping. Then, f is called a contraction with respect to $\mathcal{T}$ if there exists $k \in P$ such that $\rho(k) < 1$ and the following condition is satisfied: $d(fx, fy, a) \preceq kd(x, y, a)$ for all $x, y, a \in X$ with $(x, y, a) \in \mathcal{T}$. The vector k is called the contractive vector of T.

Example 3.10. Let (X, d) be an arbitrary cone 2-metric space over Banach algebra and $\mathcal{T} = \{(x, x, a)\colon x, a \in X\}$ be a ternary relation on X. Then, the mapping $f\colon X \to X$ defined by $fx = x$ for all $x \in X$ is a contraction with respect to $\mathcal{T}$ with an arbitrary contractive vector $k \in P$ such that $\rho(k) < 1$.

Example 3.11. Let $A = \mathbb{R}^2$ be equipped with vector multiplication $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$, $P = \{(x_1, x_2)\colon x_1, x_2 \geq 0\}$. Let $X = \mathbb{R}^3$ and define $d\colon X \times X \times X \to A$ by $d(x, y, z) = (\rho^n, \alpha\rho)$, where $\rho = \min\{\|x - y\|, \|y - z\|, \|z - x\|\}$, $\|\cdot\|$ is the norm on $\mathbb{R}^3$ defined by $\|x\| = |x_1| + |x_2| + |x_3|$ for all $x = (x_1, x_2, x_3) \in X$; finally, let α, n be some fixed positive integers. Then (X, d) is a cone 2-metric space. Define a ternary relation $\mathcal{T}$ on X by $\mathcal{T} = \{((0, x_1, x_2), (0, y_1, y_2), (z_1, z_2, z_3))\colon x_1, x_2, y_1, y_2 \leq 1, z_1, z_2 \geq 2\}$. If $f\colon X \to X$ is defined by $f(x_1, x_2, x_3) = \left(x_1 + 1, \frac{x_2}{2}, \frac{x_3}{2}\right)$, then f is a contraction with respect to $\mathcal{T}$ with a contractive vector $k = \left(\frac{1}{2}, \frac{1}{2}\right) \in P$.

A more general class of contractions is defined as follows:

Definition 3.12. Let $(X, \mathcal{T})$ be a ternary structure, (X, d) be a cone 2-metric space and $f, g\colon X \to X$ be two mappings. Then, f is called a g-contraction with respect to $\mathcal{T}$ if there exists $k \in P$ such that $\rho(k) < 1$ and the following condition is satisfied:

$$d(fx, fy, a) \preceq kd(gx, gy, a) \tag{1}$$

for all $x, y, a \in X$ with $(gx, gy, a) \in \mathcal{T}$. The vector k is called the contractive vector of T.

Note that, for $g = I_X$, the identity mapping of X, a g-contraction with respect to $\mathcal{T}$ reduces to a contraction with respect to $\mathcal{T}$.

Theorem 3.13. *Let $(X, \mathcal{T})$ be a ternary structure and (X, d) be a cone 2-metric space. Suppose $f, g\colon X \to X$ are two mappings such that $f(X) \subset g(X)$. Suppose f is a g-contraction with respect to $\mathcal{T}$ with contractive vector k and the following conditions hold:*

(i) $g(X)$ is complete;

(ii) there exists $x_0 \in X$ such that $(gx_0, fx_0, a) \in \mathcal{T}$ for all $a \in X$;

(iii) f is $\mathcal{T}$ preserving with respect to g;

(iv) if $\{gx_n\}$ is a $\mathcal{T}$-sequence such that $gx_n \to gz$, then $(gx_n, gz, a) \in \mathcal{T}$ and $(gz, ggz, a) \in \mathcal{T}$ for all $a \in X$.
Then f and g have a coincidence point $x^* \in X$. In addition, if the pair (f, g) is J-weakly compatible, then f and g have a common fixed point.

Proof. Since $f(X) \subset g(X)$, we can construct a Jungck sequence of the pair (f, g) with initial point x_0, i.e., $fx_{n-1} = gx_n = y_n$, $n \in \mathbb{N}$. We shall show that the sequence $\{y_n\}$ is a $\mathcal{T}$-sequence. Then, since $(gx_0, fx_0, a) = (gx_0, gx_1, a) = (y_0, y_1, a) \in \mathcal{T}$ for all $a \in X$ and f is $\mathcal{T}$ preserving with respect to g we have $(fx_0, fx_1, a) = (y_1, y_2, a) \in \mathcal{T}$ for all $a \in X$. Repeating this process we obtain

$$(y_{n-1}, y_n, a) = (gx_{n-1}, gx_n, a) \in \mathcal{T} \text{ for all } a \in X, n \in \mathbb{N}. \tag{2}$$

Thus, $\{y_n\}$ is a $\mathcal{T}$-sequence. We next show that $\{y_n\}$ is a Cauchy sequence.

Using (2) and the fact that f is g-contraction with respect to $\mathcal{T}$ with contractive vector k we obtain $d(y_n, y_{n+1}, a) = d(fx_{n-1}, fx_n, a) \preceq kd(gx_{n-1}, gx_n, a) = kd(y_{n-1}, y_n, a)$ for all $a \in X$. Repeating the above process we obtain

$$d(y_n, y_{n+1}, a) \preceq k^n d(y_0, y_1, a) \text{ for all } a \in X, n \in \mathbb{N}. \tag{3}$$

If $r, s \in \mathbb{N}$ and $s > r$, then using (2) and the fact that f is g-contraction with respect to $\mathcal{T}$ with contractive vector k we obtain $d(y_{s+1}, y_s, y_r) = d(fx_s, fx_{s-1}, y_r) \preceq kd(gx_s, gx_{s-1}, y_r) \preceq kd(y_s, y_{s-1}, y_r)$. Repeating this process $s - r$ times we get $d(y_{s+1}, y_s, y_r) \preceq k^{s-r} d(y_{r+1}, y_r, y_r) = \theta$.

Suppose $n, m \in \mathbb{N}$ and $n > m$, then for all $a \in X$ using (3) we have

$$\begin{aligned}
d(y_n, y_m, a) &\preceq d(y_n, y_m, y_{n-1}) + d(y_n, y_{n-1}, a) + d(y_{n-1}, y_m, a) \\
&\preceq \theta + k^{n-1} d(y_0, y_1, a) + d(y_{n-1}, y_m, a) \\
&\preceq k^{n-1} d(y_0, y_1, a) + d(y_{n-1}, y_m, y_{n-2}) + d(y_{n-1}, y_{n-2}, a) + d(y_{n-2}, y_m, a) \\
&\preceq k^{n-1} d(y_0, y_1, a) + \theta + k^{n-2} d(y_0, y_1, a) + d(y_{n-2}, y_m, a) \\
&\preceq \left[k^{n-1} + k^{n-2}\right] d(y_0, y_1, a) + d(y_{n-2}, y_m, a).
\end{aligned}$$

Repetition of this process yields

$$\begin{aligned}
d(y_n, y_m, a) &\preceq \left[k^{n-1} + k^{n-2} + \cdots + k^m\right] d(y_0, y_1, a) \\
&\preceq k^m \left[e + k + k^2 + \cdots + k^{n-m-1} + \cdots\right] d(y_0, y_1, a).
\end{aligned}$$

Since $\rho(k) < 1$, the vector $e - k$ is invertible and $(e - k)^{-1} = \sum_{i=0}^{\infty} k^i$. By using this result in the above inequality we obtain $d(y_n, y_m, a) \preceq k^m (e - k)^{-1} d(y_0, y_1, a)$. Also, by Remark 2.2 we have $\|k^n\| \to 0$ as $n \to \infty$, therefore, by Lemma 2.1, for every $c \in P^\circ$ there exists $n_0 \in \mathbb{N}$ such that $k^m \ll c$ for all $n > n_0$, and so, k^n is a c-sequence. Using Proposition 2.4 we obtain $k^m (e - k)^{-1} d(y_0, y_1, a)$ is a c-sequence, and so, by Lemma 2.1, the sequence $d(y_n, y_m, a)$ is a c-sequence. This shows that the sequence $\{y_n\} = \{gx_n\}$ is a Cauchy sequence.

By completeness of $g(X)$, there exists $x^* \in X$ such that $y_n = gx_n = fx_{n-1} \to gx^*$ as $n \to \infty$. We shall show that x^* is a coincidence point of the pair (f,g), i.e., $fx^* = gx^*$.

By condition (iv) we have $(gx_n, gz, a) \in \mathcal{T}$. Then since f is a g-contraction with respect to $\mathcal{T}$ with contractive vector k, we obtain $d(y_{n+1}, fx^*, a) = d(fx_n, fx^*, a) \preceq kd(gx_n, gx^*, a) = kd(y_n, gx^*, a)$ for all $a \in X$.

Since $y_n \to gx^*$ as $n \to \infty$, for every $c \in P^\circ$ there exists $n_1 \in \mathbb{N}$ such that $d(y_n, gx^*, a) \ll c$ for all $n > n_1$. Therefore, $\{d(y_n, gx^*, a)\}$ is a c-sequence. By Proposition 2.4 we obtain $\{kd(y_n, gx^*, a)\}$ is a c-sequence. Therefore, by Lemma 2.1 $\{d(y_{n+1}, fx^*, a)\}$ is a c-sequence, and so, $y_n \to fx^*$ as $n \to \infty$. By the uniqueness of limit in cone 2-metric spaces we have $fx^* = gx^* = z^*$ (say). Thus, x^* is a coincidence point of the pair (f,g) and z^* is the corresponding point of coincidence of the pair (f,g).

Suppose f and g are J-weakly compatible. Then, since the Jungck sequence $\{y_n\}$ of the pair (f,g) converges to gx^*, we have $fgx^* = gfx^* = ffx^* = ggx^*$. By the assumption (iv), we have $(gx^*, ggx^*, a) \in \mathcal{T}$ for all $a \in X$. We claim that $fz^* = z^*$. On contrary, suppose that $fz^* \neq z^*$. Therefore, there exists $z \in X$ such that $d(z^*, fz^*, z) \neq \theta$. Using the condition (1) we obtain that, for all $a \in X$,

$$d(z^*, fz^*, a) = d(fx^*, fgx^*, a) \preceq kd(gx^*, gfx^*, a) = kd(z^*, fgx^*, a) = kd(z^*, fz^*, a).$$

Thus, $d(z^*, fz^*, a) \preceq kd(z^*, fz^*, a)$, and as $\rho(k) < 1$, we must have $d(z^*, fz^*, a) = \theta$ for all $a \in X$. This contradiction shows that $fz^* = z^*$. Again, $gz^* = gfx^* = fgx^* = fz^* = z^*$. Thus, z^* is a common fixed point of f and g. □

Example 3.14. Let X, A, P and d be as in Example 3.3. Define $f, g \colon X \to X$ by

$$fx = \begin{cases} x^2, & \text{if } x \in [0,1]; \\ 1/2, & \text{otherwise} \end{cases} \quad \text{and} \quad gx = \begin{cases} 2x^2, & \text{if } x \in [0,1]; \\ 1/x, & \text{otherwise.} \end{cases}$$

Define a ternary relation $\mathcal{T}$ on X by $\mathcal{T} = \{(x,y,z) \in X \times X \times X \colon x, y, \in [0,1], z \in X\}$. Then, by routine calculations, one can see that f is a g-contraction with respect to $\mathcal{T}$ with contractive vector $k(t) = 1/2$. All the conditions of Theorem 3.13 are satisfied, and so the pair (f,g) has a common fixed point, namely, $(0,0)$ is the desired common fixed point.

Corollary 3.15. *Let $(X, \mathcal{T})$ be a ternary structure and (X,d) be a cone 2-metric space. Suppose $f \colon X \to X$ is a contraction with respect to $\mathcal{T}$ with contractive vector k and the following conditions hold:*

(i) X is complete;

(ii) there exists $x_0 \in X$ such that $(x_0, fx_0, a) \in \mathcal{T}$ for all $a \in X$;

(iii) f is $\mathcal{T}$ preserving;

(iv) if $\{x_n\}$ is a $\mathcal{T}$-sequence such that $x_n \to z$, then $(x_n, z, a) \in \mathcal{T}$ for all $a \in X$.

Then f has a fixed point.

Proof. Assuming $g = I_X$ in Theorem 3.13 we obtain the desired result. Note that, when following the lines of proof of Theorem 3.13 with $g = I_X$, the condition

"$(gz, ggz, a) \in \mathcal{T}$ for all $a \in X$" in Theorem 3.13 (iv), is not required for the proof, so now it can be omitted. □

DEFINITION 3.16 ([5]). Let P be a cone in a Banach algebra A. A nondecreasing function $\varphi\colon P \to P$ (i.e., $x, y \in P, x \preceq y$ implies $\varphi(x) \preceq \varphi(y)$) is called a comparison function if it satisfies:
(i) $\varphi(\theta) = \theta$ and $\theta \prec \varphi(x) \prec x$ for all $x \in P \setminus \{\theta\}$.
(ii) If $x \in \mathrm{int}P$ then $x - \varphi(x) \in \mathrm{int}P$.
(iii) $\lim_{n\to\infty} \varphi^n(x) = \theta$ for all $x \in P \setminus \{\theta\}$.
We denote by Φ the set of all comparison functions on P.

DEFINITION 3.17. Let $(X, \mathcal{T})$ be a ternary structure and (X, d) be a cone 2-metric space. Suppose $f\colon X \to X$ is a mappings and $\varphi \in \Phi$. Then, f is called a φ-contraction with respect to $\mathcal{T}$ if: $d(fx, fy, a) \preceq \varphi(d(x, y, a))$ for all $x, y, a \in X$ with $(x, y, a) \in \mathcal{T}$.

DEFINITION 3.18. Let $(X, \mathcal{T})$ be a ternary structure and (X, d) be a cone 2-metric space. Suppose $f, g\colon X \to X$ are two mappings and $\varphi \in \Phi$. Then, f is called a g-φ-contraction with respect to $\mathcal{T}$ if

$$d(fx, fy, a) \preceq \varphi(d(gx, gy, a)) \tag{4}$$

for all $x, y, a \in X$ with $(gx, gy, a) \in \mathcal{T}$.

Again, it is easy to see that the φ-contractions with respect to $\mathcal{T}$ are a particular case of g-φ-contractions with respect to $\mathcal{T}$.

We next prove a common fixed point result for the pair (f, g) when f is a g-φ-contraction.

THEOREM 3.19. *Let $(X, \mathcal{T})$ be a ternary structure, (X, d) be a cone 2-metric space and $\mathcal{T}$ be transitive. Suppose $f, g\colon X \to X$ are two mappings such that $f(X) \subset g(X)$, f is a g-φ-contraction with respect to $\mathcal{T}$ and the following conditions hold:*
(i) $g(X)$ is complete;
(ii) there exists $x_0 \in X$ such that $(gx_0, fx_0, a) \in \mathcal{T}$ for all $a \in X$;
(iii) f is $\mathcal{T}$ preserving with respect to g;
(iv) if $\{gx_n\}$ is a $\mathcal{T}$-sequence such that $gx_n \to gz$, then $(gx_n, gz, a) \in \mathcal{T}$ and $(gz, ggz, a) \in \mathcal{T}$ for all $a \in X$.
Then f and g have a coincidence point. In addition, if the pair (f, g) is J-weakly compatible, then f and g have a common fixed point.

Proof. Since $f(X) \subset g(X)$, we can construct a Jungck sequence of the pair (f, g) with initial point x_0, i.e., $fx_{n-1} = gx_n = y_n$, $n \in \mathbb{N}$. Then, following the lines of proof of Theorem 3.13, we obtain

$$(y_{n-1}, y_n, a) = (gx_{n-1}, gx_n, a) \in \mathcal{T} \text{ for all } a \in X, n \in \mathbb{N}. \tag{5}$$

Thus, $\{y_n\}$ is a $\mathcal{T}$-sequence. We next show that $\{y_n\}$ is a Cauchy sequence. Then, using (5) and the fact that f is g-φ-contraction with respect to $\mathcal{T}$ we obtain

$$d(y_n, y_{n+1}, a) = d(fx_{n-1}, fx_n, a) \preceq \varphi(d(gx_{n-1}, gx_n, a)) = \varphi(d(y_{n-1}, y_n, a))$$

for all $a \in X$. Repeating the above process and using the properties of φ we obtain

$$d(y_n, y_{n+1}, a) \preceq \varphi^n(d(y_0, y_1, a)) \text{ for all } a \in X, n \in \mathbb{N}. \tag{6}$$

For $c \gg \theta$, we can choose $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $c - \varphi(c) + \{u \in E : \|u\| < \delta\} \subset \operatorname{int}P$, $\|\varphi^n(d(y_0, y_1, a))\| < \delta$ and $\varphi^n(d(y_0, y_1, a)) \ll c - \varphi(c)$ for all $n > n_0, a \in X$. Therefore, by (6) and the above inequality we obtain

$$d(y_n, y_{n+1}, a) \ll c - \varphi(c) \preceq c \text{ for all } n > n_0,\ a \in X. \tag{7}$$

Suppose $n, r \in \mathbb{N}$; then by (5) we have $(gx_n, gx_{n+1}, a), (gx_{n+1}, gx_{n+2}, a), \ldots, (gx_{n+r-1}, gx_{n+r}, a) \in \mathcal{T}$ for all $a \in X, n \in \mathbb{N}$. Since $\mathcal{T}$ is transitive we have $(gx_n, gx_{n+r}, a) \in \mathcal{T}$ for all $a \in X, n, r \in \mathbb{N}$. Then, for $n > n_0$, using (4), (7) and the above inclusion we have

$$\begin{aligned} d(y_n, y_{n+2}, a) &\preceq d(y_n, y_{n+2}, y_{n+1}) + d(y_n, y_{n+1}, a) + d(y_{n+1}, y_{n+2}, a) \\ &= d(y_n, fx_{n+1}, fx_n) + d(y_n, y_{n+1}, a) + d(fx_n, fx_{n+1}, a) \\ &\preceq \varphi(d(gx_n, gx_{n+1}, y_n)) + d(y_n, y_{n+1}, a) + \varphi(d(gx_n, gx_{n+1}, a)) \\ &= \varphi(d(y_n, y_{n+1}, y_n)) + d(y_n, y_{n+1}, a) + \varphi(d(y_n, y_{n+1}, a)) \\ &\ll \theta + c - \varphi(c) + \varphi(c) = c. \end{aligned}$$

Similarly, for $n > n_0$ we obtain

$$\begin{aligned} d(y_n, y_{n+3}, a) &\preceq d(y_n, y_{n+3}, y_{n+1}) + d(y_n, y_{n+1}, a) + d(y_{n+1}, y_{n+3}, a) \\ &= d(y_n, fx_{n+2}, fx_n) + d(y_n, y_{n+1}, a) + d(fx_n, fx_{n+2}, a) \\ &\preceq \varphi(d(y_n, gx_{n+2}, gx_n)) + d(y_n, y_{n+1}, a) + \varphi(d(gx_n, gx_{n+2}, a)) \\ &= \varphi(d(y_n, y_{n+2}, y_n)) + d(y_n, y_{n+1}, a) + \varphi(d(y_n, y_{n+2}, a)) \\ &\ll \theta + c - \varphi(c) + \varphi(c) = c. \end{aligned}$$

By induction, we obtain $d(y_n, y_{n+r}, a) \ll c$ for all $r \in \mathbb{N}$ and $n \geq n_0$. Thus, $\{y_n\} = \{gx_n\}$ is a Cauchy sequence. By the completeness of $g(X)$, there exists $x^* \in X$ such that $y_n = gx_n = fx_{n-1} \to gx^*$ as $n \to \infty$. In the same way as in the proof of Theorem 3.13 it can be shown that x^* is a coincidence point of the pair (f, g), with $z^* = fx^* = gx^*$ being the corresponding point of coincidence..

Suppose f and g are J-weakly compatible. Then, since the Jungck sequence $\{y_n\}$ of the pair (f, g) converges to gx^*, we have $fgx^* = gfx^* = ffx^* = ggx^*$. We shall show that $fz^* = z^*$. On contrary, suppose that $d(z^*, fz^*, z) \neq \theta$ for some $z \in X$. Then by the assumption (iv), we have $(gx^*, ggx^*, a) \in \mathcal{T}$ for all $a \in X$. Therefore, using the condition (4) we obtain that

$$d(z^*, fz^*, a) = d(fx^*, ffx^*, a) \preceq \varphi(d(gx^*, gfx^*, a)) = \varphi(d(z^*, fz^*, a) \prec d(z^*, fz^*, a).$$

This contradiction shows that $d(z^*, fz^*, a) = \theta$ for all $a \in X$. Therefore, $fz^* = z^*$. Again, $gz^* = gfx^* = fgx^* = fz^* = z^*$. Thus, z^* is a common fixed point of f and g. □

Corollary 3.20. *Let $(X, \mathcal{T})$ be a ternary structure, (X, d) be a cone 2-metric space and $\mathcal{T}$ be transitive. Suppose $f \colon X \to X$ is a φ-contraction with respect to $\mathcal{T}$ and the following conditions hold:*
(i) X is complete;

(ii) there exists $x_0 \in X$ such that $(x_0, fx_0, a) \in \mathcal{T}$ for all $a \in X$;

(iii) f is $\mathcal{T}$ preserving;

(iv) if $\{x_n\}$ is a $\mathcal{T}$-sequence such that $x_n \to gz$, then $(x_n, z, a) \in \mathcal{T}$ for all $a \in X$. Then f has a fixed point.

Proof. Assuming $g = I_X$ in Theorem 3.19 we obtain the desired result. Note that, when following the lines of proof of Theorem 3.19 with $g = I_X$, the condition "$(gz, ggz, a) \in \mathcal{T}$ for all $a \in X$" in Theorem 3.19 (iv), is not required for the proof, so now it can be omitted. □

Example 3.21. Let X, A and P be as in Example 3.3. Define a mapping $d\colon X \times X \times X \to P$ by $d(x, y, z) = \rho e^t$ for all $x, y, z \in X$, where $\rho = \min\{|x - y|, |y - z|, |z - x|\}$. Then (X, d) is a cone 2-metric space over Banach algebra A. Define a ternary relation $\mathcal{T}$ on X by $\mathcal{T} = \{(x, y, a)\colon x, y \in [0, 1] \cap \mathbb{Q}, a > 1\} \cup \{(0, 0, a)\colon a \in X\}$ and the mappings $f, g\colon X \to X$ by

$$fx = \begin{cases} \dfrac{x}{1+x}, & \text{if } x \in \mathbb{Q};\\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \quad g(x) = x \text{ for all } x \in X.$$

Then, it is easy to see that all the conditions of Theorem 3.19 are satisfied with $\varphi(\psi(t)) = \frac{\psi(t)}{1+\psi(t)}$ for all $\psi \in P$, and so, by Theorem 3.19 the pair (f, g) has a common fixed point, namely, $(0, 0)$ is the desired common fixed point.

Acknowledgement. The authors are thankful to the referee for critical remarks on this paper.

References

[1] M. Abbas, G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., **341** (2008), 416–420.

[2] V. Badshah, P. Bhagat, S. Shukla, *Some fixed point theorems for α-ϕ-contractive mappings in cone 2-metric spaces*, Appl. Math. Inform. Mech., **8(2)** (2016), 187–199.

[3] E. Breuillard, K. Fujiwara, *On the joint spectral radius for isometries of non-positively curved spaces and uniform growth*, arXiv:1804.00748

[4] V. Bulgărean, *The group of isometries of the French rail ways metric*, Stud. Univ. Babeş-Bolyai Math., **58(4)** (2013), 445–450.

[5] C. Di Bari, P. Vetro, *φ-Pairs and common fixed points in cone metric space*, Rend. Circ. Mat. Palermo, **57** (2008) 279–285.

[6] M. Djordević, D. Djorić, Z. Kadelburg, S. Radenović, D. Spasić, *Fixed point results under c-distance in tvs-cone metric spaces*, Fixed Point Theory Appl., **29** (2011). doi:10.1186/1687-1812-2011-29.

[7] W.S. Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Anal., **72(5)** (2010), 2259–2261.

[8] Y. Feng, W. Mao, The equivalence of cone metric spaces and metric spaces, Fixed Point Theory, ***11(2)*** (2010), 259–264.

[9] K. Goebel, *On the minimal displacement of points under Lipschitzian mappings*, Pacific J. Math., **45(1)** (1973), 151–163.

[10] K. Goebel, W. Kirk, *Topics in Metric Fixed Points Theor*, Cambridge Univ. Press, Cambridge, 1990.

[11] S.Gähler, *2-metricsche Räume und ihre topologische strukture*, Math. Nachr., **26** (1963), 115–148.

[12] S.Gähler, *Uber die Uniformisierbarkeit 2-metricsche Räume*, Math. Nachr., **28** (1965) 235–244.

[13] S.Gähler, *Zur geometric 2-metricsche Räume*, Revne Roumaine der Mathem. Pures et Appliques, **11** (1966), 665–667.

[14] L.G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468–1476.

[15] A. Katok, V. Climenhaga, *Lectures on Surfaces: (Almost) Everything you Wanted to Know about Them*, AMS Student Mathematical Library, 2008.

[16] Z. Kadelburg, S. Radenović, V. Rakočević, *A note on the equivalence of some metric and cone metric fixed point results*, Appl. Math. Lett., **24** (2011), 370–374.

[17] Z. Kadelburg, M. Pavlović, S. Radenović, *Common fixed point theorems for ordered contractions and quasi-contractions in ordered cone metric spaces*, Comput. Math. Appl., **59** (2010), 3148–3159.

[18] H. Liu, S.-Y. Xu, *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings*, Fixed Point Theory Appl., 2013, 2013:320.

[19] A. Papadopoulos, *Metric Spaces, Convexity and Nonpositive Curvature*, IRMA Lectures in Mathematics and Theoretical Physics Vol. 6, Strasbourg, 2005.

[20] S. Radenović, B.E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*, Comput. Math. Appl., **57** (2009) 1701–1707.

[21] A. Ramsay, R. Richtmyer, *Introduction to Hyperbolic Geometry*, Universitex, Springer, New York, 1995.

[22] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2003) 1435–1443.

[23] J. Rosales, P. García-Sánchez, *Numerical Semigroups*, Developments in Mathematics, Springer, New York, 2009.

[24] A. Alam, M. Imdad, Relation-theoretic contraction principle, J. Fixed Point Theory Appl., **17(4)** (2015), 693–702.

[25] B. Singh, S. Jain, P. Bhagat, *Cone 2-metric space and fixed point theorem of contractive mappings*, Comment. Math., **52(2)** (2012), 143–151 .

[26] T. Wang, J. Yin, Q. Yan, *Fixed point theorems on cone 2-metric spaces over Banach algebras and an application*, Fixed Point Theory Appl., (2015) 2015:204.

[27] S. Xu, S. Radenović, *Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality*, Fixed Point Theory Appl., (2014), 2014:102.

(received 21.02.2019; in revised form 31.05.2019; available online 01.06.2020)

School of Studies in Mathematics, Vikram University, Ujjain, (M.P.), India

E-mail: vhbadshah@yahoo.co.in

Department of Applied Mathematics, NMIMS, Shirpur, India

E-mail: prakash1175@yahoo.com

Department of Applied Mathematics, Shri Vaishnav Institute of Technology and Science, Gram Baroli, Sanwer Road, Indore, 453331, (M.P.) India

E-mail: satishmathematics@yahoo.co.in