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Corrected proof
Available online 31.05.2020

research paper
оригинални научни рад

EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS TO A FOURTH-ORDER MULTI-POINT BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we study the existence and multiplicity of positive solutions for a nonlinear fourth-order ODE with multi-point boundary conditions and an integral boundary condition. The main tool is Krasnosel'skii fixed point theorem on cones.

1. Introduction

Boundary value problems related to nonlocal conditions have many applications to problems in the theory of heat conduction, thermoelasticity, plasma physics, control theory, etc. The current analysis of these problems has a great interest and many methods are used to solve them. Recently, the study of existence of a positive solution to fourth-order boundary value problems has gained much attention and becomes a rapidly growing field, see [1, 2, 4, 6–9, 15]. However, the approaches used in the literature are usually by topological degree theory and fixed-point theorems in cones [5].

Multi-point boundary value problems have received considerable interest in the mathematical applications in different areas of science and engineering, see [3, 12–14].

In 2007, M. Zhang and Z. Wei [13] studied the existence of multiple positive solutions for fourth-order m-point boundary value problem

$$\begin{cases} u^{(4)}(t) + B(t)u'' - A(t)u = f(t,u),\ 0 < t < 1, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i),\ u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \sum_{i=1}^{m-2} a_i u''(\xi_i),\ u''(1) = \sum_{i=1}^{m-2} b_i u''(\xi_i). \end{cases}$$

In the same year, X. Zhang and L. Liu [14] considered the fourth-order multi-point boundary value problems with bending term

$$\begin{cases} x^{(4)}(t) = g(t)f(t,x(t),x''(t)),\ t \in (0,1), \\ x(0) = 0,\ x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ x''(0) = 0,\ x''(1) = \sum_{i=1}^{m-2} b_i x''(\xi_i). \end{cases}$$

2010 Mathematics Subject Classification: 34B15, 34B18
Keywords and phrases: Positive solutions; Krasnoselskii's fixed point theorem; fourth-order integral boundary value problems; cone.

In 2016, S. Benaicha and F. Haddouchi [2], considered the following fourth-order two-point boundary value problem

$$u''''(t) + f(u(t)) = 0, \ t \in (0,1),$$
$$u'(0) = u'(1) = u''(0) = 0, \ u(0) = \int_0^1 a(s)u(s)\, ds.$$

Bo Yang [12] studied the fourth-order differential equation $u''''(t) = g(t)f(u(t))$, $t \in (0,1)$, together with boundary conditions $u(0) = \alpha u'(0) - \beta u''(0) = \gamma u'(1) + \delta u''(1) = u'''(1) = 0$. Yan. D and R. Ma [11] investigated the global behavior of positive solutions of fourth-order boundary value problem $u'''' = \lambda f(x,u), \ x \in (0,1)$, together with boundary conditions $u(0) = u(1) = u''(0) = u''(1) = 0$, where $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function with $f(x,0) < 0$ in $(0,1)$, and $\lambda > 0$. The proof of main results are based upon bifurcation techniques. Recently, Wei. Y et al. [10] considered the following boundary value problem $u^{(4)}(t) = f(t, u(t), u'(t))$, $t \in (0,1)$, subject to the boundary conditions $u(0) = u'(0) = u'(1) = u''(1) = 0$. Under some conditions of f, the existence and uniqueness of this problem is obtained.

Motivated by these works, in this paper, we are concerned with the following fourth-order ODE with multi-point and integral boundary conditions:

$$u''''(t) + f(t, u(t)) = 0, \ t \in (0,1), \tag{1}$$

$$u'(0) = u'(1) = u''(0) = 0, \ u(0) = \alpha \int_0^1 u(s)\, ds + \sum_{i=1}^{n} \beta_i u(\eta_i), \tag{2}$$

where

(C1) $f \in C([0,1] \times [0,\infty), [0,\infty))$;

(C2) $\alpha \geq 0, \beta_i \geq 0, 1 \leq i \leq n$ and $0 < \eta_1 < \eta_2 < \ldots < \eta_n < 1$;

(C3) $\alpha + \sum_{i=1}^{n} \beta_i < 1$.

This paper is organized as follows. In Section 2, we present some theorems and lemmas that will be used to prove our main results. In Section 3, we discuss the existence of at least one positive solution for (1)-(2). In Section 4, we investigate the existence of multiple positive solutions for (1)-(2). Finally, we give some examples to illustrate our results in Section 5.

2. Preliminaries

At first, we consider the Banach space $C([0,1], \mathbb{R})$ equipped with the sup norm $\|u\| = \sup_{t \in [0,1]} |u(t)|$.

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $K \subset E$ is a cone if it satisfies the following two conditions:
(i) $x \in K$, $\lambda \geq 0$ imply $\lambda x \in K$; (ii) $x \in K$, $-x \in K$ imply $x = 0$.

Definition 2.2. An operator $T : E \to E$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.3. *A* function $u(t)$ is called a positive solution of (1)-(2) if $u \in C([0,1],\mathbb{R})$ and $u(t) > 0$ for all $t \in (0,1)$.

To prove our results, we need the following well-known fixed point theorem of cone expansion and compression of norm type due to Krasnosel'skii [5].

Theorem 2.4. *Let E be a Banach space, and let $K \subset E$ be a cone. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $A : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that*
(a) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$; or
(b) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.
Then A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Consider the multi-point boundary value problem

$$u''''(t) + y(t) = 0,\ t \in (0,1), \tag{3}$$

$$u'(0) = u'(1) = u''(0) = 0,\ u(0) = \alpha \int_0^1 u(s)\, ds + \sum_{i=1}^{n} \beta_i u(\eta_i). \tag{4}$$

For convenience, we denote $k = 1 - \left(\alpha + \sum_{i=1}^{n} \beta_i\right)$.

Lemma 2.5. *Let $k \neq 0$. Then for any $y \in C[0,1]$, the boundary value problem* (3)-(4) *has a unique solution which can be expressed by $u(t) = \int_0^1 H(t,s)y(s)\, ds$, where $H(t,s) : [0,1] \times [0,1] \to \mathbb{R}$ is the Green's function defined by*

$$H(t,s) = G(t,s) + \frac{\alpha}{k}\int_0^1 G(\tau,s)d\tau + \frac{1}{k}\sum_{i=1}^{n} \beta_i G(\eta_i, s), \tag{5}$$

and

$$G(t,s) = \frac{1}{6}\begin{cases} t^3(1-s)^2 - (t-s)^3, & 0 \leq s \leq t \leq 1; \\ t^3(1-s)^2, & 0 \leq t \leq s \leq 1. \end{cases} \tag{6}$$

Proof. Rewriting (3) as $u''''(t) = -y(t)$ and integrating four times over the interval $[0,t]$ for $t \in [0,1]$, we obtain

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 y(s)\, ds + \frac{1}{6}C_1 t^3 + \frac{1}{2}C_2 t^2 + C_3 t + C_4, \tag{7}$$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$ are constants. By (4), we get $C_1 = \int_0^1 (1-s)^2 y(s)\, ds$ and $C_2 = C_3 = 0$. Further,

$$\begin{aligned} C_4 =& u(0) = \alpha \int_0^1 \left(-\frac{1}{6}\int_0^\tau (\tau-s)^3 y(s)\, ds + \frac{\tau^3}{6}\int_0^1 (1-s)^2 y(s)\, ds + C_4\right) d\tau \\ &+ \sum_{i=1}^{n} \beta_i \left(-\frac{1}{6}\int_0^{\eta_i} (\eta_i - s)^3 y(s)\, ds + \frac{\eta_i^3}{6}\int_0^1 (1-s)^2 y(s)\, ds + C_4\right) \\ =& \alpha \int_0^1 \left(-\frac{1}{6}\int_0^\tau (\tau-s)^3 y(s)\, ds + \frac{\tau^3}{6}\int_0^1 (1-s)^2 y(s)\, ds\right) d\tau \\ &+ \sum_{i=1}^{n} \beta_i \left(-\frac{1}{6}\int_0^{\eta_i} (\eta_i - s)^3 y(s)\, ds + \frac{\eta_i^3}{6}\int_0^1 (1-s)^2 y(s)\, ds\right) + C_4\left(\alpha + \sum_{i=1}^{n} \beta_i\right), \end{aligned}$$

so $$C_4 = \frac{\alpha}{k}\int_0^1\left(-\frac{1}{6}\int_0^\tau(\tau-s)^3y(s)\,ds+\frac{\tau^3}{6}\int_0^1(1-s)^2y(s)\,ds\right)d\tau$$
$$+\frac{1}{k}\sum_{i=1}^n\beta_i\left(-\frac{1}{6}\int_0^{\eta_i}(\eta_i-s)^3y(s)\,ds+\frac{\eta_i^3}{6}\int_0^1(1-s)^2y(s)\,ds\right).$$

Replacing these expressions in (7), we get

$$\begin{aligned}
u(t) =& -\frac{1}{6}\int_0^t(t-s)^3y(s)\,ds+\frac{t^3}{6}\int_0^1(1-s)^2y(s)\,ds\\
&+\frac{\alpha}{k}\int_0^1\left(-\frac{1}{6}\int_0^\tau(\tau-s)^3y(s)\,ds+\frac{\tau^3}{6}\int_0^1(1-s)^2y(s)\,ds\right)d\tau\\
&+\frac{1}{k}\sum_{i=1}^n\beta_i\left(-\frac{1}{6}\int_0^{\eta_i}(\eta_i-s)^3y(s)\,ds+\frac{\eta_i^3}{6}\int_0^1(1-s)^2y(s)\,ds\right)\\
=&\frac{1}{6}\int_0^t[t^3(1-s)^2-(t-s)^3]y(s)\,ds+\frac{1}{6}\int_t^1 t^3(1-s)^2y(s)\,ds\\
&+\frac{\alpha}{6k}\int_0^1\left(\int_0^\tau[\tau^3(1-s)^2-(\tau-s)^3]y(s)\,ds+\int_\tau^1\tau^3(1-s)^2y(s)\,ds\right)d\tau\\
&+\frac{1}{6k}\sum_{i=1}^n\beta_i\left(\int_0^{\eta_i}[\eta_i^3(1-s)^2-(\eta_i-s)^3]y(s)\,ds+\eta_i^3\int_{\eta_i}^1(1-s)^2y(s)\,ds\right)\\
=&\int_0^1\left(G(t,s)+\frac{\alpha}{k}\int_0^1 G(\tau,s)d\tau+\frac{1}{k}\sum_{i=1}^n\beta_iG(\eta_i,s)\right)y(s)\,ds\\
=&\int_0^1 H(t,s)y(s)\,ds.
\end{aligned}$$
□

The proof of the following lemma can be found in [2, Lemma 2.3].

Lemma 2.6. *Let $\theta\in(0,\frac{1}{2})$ be fixed. Then $G(t,s)$ defined by (6) satisfies*
(i) $G(t,s)\geq 0$, for all $t,s\in[0,1]$,
(ii) $\rho(t)e(s)\leq G(t,s)\leq e(s)$, for all $(t,s)\in[0,1]\times[0,1]$, where $e(s)=\frac{1}{6}s(1-s)^2$, and

$$\rho(t)=\min\{t^3,t^2(1-t)\}=\begin{cases}t^3, & t\leq\frac{1}{2};\\ t^2(1-t), & t\geq\frac{1}{2}.\end{cases}$$

(iii) $\theta^3e(s)\leq G(t,s)\leq e(s)$, for all $(t,s)\in[\theta,1-\theta]\times[0,1]$.

In the remainder of this paper, we always assume that $k>0$.

Lemma 2.7. *Let $y(t)\in C([0,1],[0,\infty))$ and $\theta\in(0,\frac{1}{2})$. Then the unique solution of (3)-(4) is nonnegative and satisfies $\min_{t\in[\theta,1-\theta]}u(t)\geq\theta^3(1-2\theta)\|u\|$.*

Proof. The positiveness of $u(t)$ follows immediately from Lemma 2.5 and Lemma 2.6.

For all $t\in[0,1]$, we have

$$u(t)=\int_0^1 H(t,s)y(s)\,ds=\int_0^1\left(G(t,s)+\frac{\alpha}{k}\int_0^1G(\tau,s)d\tau+\frac{1}{k}\sum_{i=1}^n\beta_iG(\eta_i,s)\right)y(s)\,ds$$

$$\leq \int_0^1 \left(e(s) + \frac{\alpha}{k}\int_0^1 e(s)d\tau + \frac{1}{k}\sum_{i=1}^{n}\beta_i G(\eta_i, s)\right) y(s)\, ds$$
$$= \int_0^1 \left(\left(1+\frac{\alpha}{k}\right)e(s) + \frac{1}{k}\sum_{i=1}^{n}\beta_i G(\eta_i, s)\right) y(s)\, ds.$$

Then $\|u\| \leq \int_0^1 \left(\left(1+\frac{\alpha}{k}\right)e(s) + \frac{1}{k}\sum_{i=1}^{n}\beta_i G(\eta_i, s)\right) y(s)\, ds.$ (8)

For $t \in [\theta, 1-\theta]$, we have

$$u(t) = \int_0^1 H(t,s)y(s)\, ds$$
$$= \int_0^1 \left(G(t,s) + \frac{\alpha}{k}\int_0^1 G(\tau,s)d\tau + \frac{1}{k}\sum_{i=1}^{n}\beta_i G(\eta_i, s)\right) y(s)\, ds$$
$$\geq \int_0^1 \left(G(t,s) + \frac{\alpha}{k}\int_\theta^{1-\theta} G(\tau,s)d\tau + \frac{1}{k}\sum_{i=1}^{n}\beta_i G(\eta_i, s)\right) y(s)\, ds$$
$$\geq \int_0^1 \left(\theta^3 e(s) + \frac{\alpha}{k}\theta^3(1-2\theta)e(s) + \frac{1}{k}\sum_{i=1}^{n}\beta_i G(\eta_i, s)\right) y(s)\, ds$$
$$\geq \theta^3(1-2\theta)\int_0^1 \left(\left(1+\frac{\alpha}{k}\right)e(s) + \frac{1}{k}\sum_{i=1}^{n}\beta_i G(\eta_i, s)\right) y(s)\, ds. \tag{9}$$

From (8) and (9), we obtain $\min_{t\in[\theta,1-\theta]} u(t) \geq \theta^3(1-2\theta)\|u\|$. □

Let $\theta \in (0, \frac{1}{2})$. We define the cone

$$K = \left\{u \in C([0,1],\ \mathbb{R}) :\ u(t) \geq 0,\ t \in [0,1],\ \min_{t\in[\theta,1-\theta]} u(t) \geq \theta^3(1-2\theta)\|u\|\right\},$$

and the operator $T : K \to C[0,1]$ by

$$Tu(t) = \int_0^1 H(t,s)f(s,u(s))\, ds, \tag{10}$$

where $H(t,s)$ is defined by (5).

Remark 2.8. By Lemma 2.5, the fixed points of the operator T in K are the nonnegative solutions of the boundary value problem (1)-(2).

Lemma 2.9. *The operator T defined in* (10) *is completely continuous and satisfies $T(K) \subset K$.*

Proof. From Lemma 2.7, we obtain $T(K) \subset K$. By an applying Arzela-Ascoli theorem, T is completely continuous. □

For convenience, we introduce the following notations

$$f_0 = \lim_{u\to 0^+}\left\{\min_{0\leq t\leq 1}\frac{f(t,u)}{u}\right\}, \qquad f^0 = \lim_{u\to 0^+}\left\{\max_{0\leq t\leq 1}\frac{f(t,u)}{u}\right\},$$

$$f_\infty = \lim_{u\to+\infty}\left\{\min_{0\le t\le 1}\frac{f(t,u)}{u}\right\},\quad f^\infty = \lim_{u\to+\infty}\left\{\max_{0\le t\le 1}\frac{f(t,u)}{u}\right\},$$
$$\Psi = \theta^6(1-2\theta)^2\int_\theta^{1-\theta}\left(\left(1+\frac{\alpha}{k}\right)e(s)+\frac{1}{k}\sum_{i=1}^n\beta_i G(\eta_i,s)\right)ds,$$
$$\Phi = \frac{1}{6k},\qquad \Lambda_1 = \Phi^{-1},\qquad \Lambda_2=\Psi^{-1}.$$

3. Existence results

THEOREM 3.1. *Assume that one of the following hypotheses is satisfied.*
(H1) $f_0=\infty$ *and* $f^\infty=0$; *(H2)* $f^0=0$ *and* $f_\infty=\infty$.
Then the problem (1)-(2) *has at least one positive solution in* K.

Proof. Assume that (H1) holds. Since $f_0=\infty$, there exists $\rho_1>0$ such that $f(t,u)\ge\delta u$, for all $0<u\le\rho_1, t\in[0,1]$, where $\delta>0$ is chosen so that $\delta\Psi\ge 1$. Then, for $u\in K\cap\partial\Omega_1$ and $t\in[\theta,1-\theta]$ with $\Omega_1=\{u\in C[0,1]:\|u\|<\rho_1\}$, we obtain

$$\begin{aligned}
Tu(t) &= \int_0^1 H(t,s)y(s)\,ds\\
&= \int_0^1\left(G(t,s)+\frac{\alpha}{k}\int_0^1 G(\tau,s)d\tau+\frac{1}{k}\sum_{i=1}^n\beta_iG(\eta_i,s)\right)f(s,u(s))\,ds\\
&\ge \int_\theta^{1-\theta}\left(G(t,s)+\frac{\alpha}{k}\int_\theta^{1-\theta} G(\tau,s)d\tau+\frac{1}{k}\sum_{i=1}^n\beta_iG(\eta_i,s)\right)f(s,u(s))\,ds\\
&\ge \int_\theta^{1-\theta}\left(G(t,s)+\frac{\alpha}{k}\int_\theta^{1-\theta} G(\tau,s)d\tau+\frac{1}{k}\sum_{i=1}^n\beta_iG(\eta_i,s)\right)\delta u(s)\,ds\\
&\ge \delta\theta^3(1-2\theta)\|u\|\int_\theta^{1-\theta}\left(\theta^3e(s)+\frac{\alpha}{k}\theta^3(1-2\theta)e(s)+\frac{1}{k}\sum_{i=1}^n\beta_iG(\eta_i,s)\right)ds\\
&\ge \delta\theta^6(1-2\theta)^2\|u\|\int_\theta^{1-\theta}\left(\left(1+\frac{\alpha}{k}\right)e(s)+\frac{1}{k}\sum_{i=1}^n\beta_iG(\eta_i,s)\right)ds\\
&= \delta\Psi\|u\|\ge\|u\|. \qquad (11)
\end{aligned}$$

Hence, $\|Tu\|\ge\|u\|,\ u\in K\cap\partial\Omega_1$.

On the other hand, since $f^\infty=0$, there exists $\widehat{\rho}_2>0$ ($\widehat{\rho}_2>\rho_1$) such that $f(t,u)\le\eta u$ for all $t\in[0,1]$ with $u\ge\widehat{\rho}_2$ and $\eta>0$ satisfies $\eta\Phi\le1$. We consider two cases.

Case 1. Suppose that f is bounded, then there exists $L>0$ such that $f(t,u)\le L$.

Let $\Omega_2=\{u\in C[0,1]:\|u\|<\rho_2\}$ with $\rho_2=\max\{2\rho_1, L\Phi\}$. If $u\in K\cap\partial\Omega_2$, then by Lemma 2.6 we have

$$Tu(t)=\int_0^1 H(t,s)f(s,u(s))\,ds\le L\int_0^1\left(e(s)+\frac{\alpha}{k}\int_0^1 e(s)d\tau+\frac{1}{k}\sum_{i=1}^n\beta_ie(s)\right)ds$$

$$\leq L\int_0^1 e(s)\left(1+\frac{\alpha}{k}+\frac{1}{k}\sum_{i=1}^{n}\beta_i\right)ds=\frac{L}{k}\int_0^1 e(s)\,ds\leq L\Phi\leq\rho_2=\|u\|, \quad (12)$$

and consequently, $\|Tu\|\leq\|u\|,\ u\in K\cap\partial\Omega_2$.

Case 2. If f is unbounded, then from condition (C1), there exists $\sigma>0$ such that $f(t,u)\leq\eta\sigma$, with $0<u\leq\widehat{\rho}_2$ and $t\in[0,1]$.

Let $\Omega_2=\{u\in C[0,1]:\|u\|<\rho_2\}$, where $\rho_2=\max\{\sigma,\widehat{\rho}_2\}$. If $u\in K\cap\partial\Omega_2$, then we have $f(t,u)\leq\eta\rho_2$, and

$$Tu(t)=\int_0^1 H(t,s)f(s,u(s))\,ds\leq\int_0^1\left(e(s)+\frac{\alpha}{k}e(s)+\frac{1}{k}\sum_{i=1}^{n}\beta_i e(s)\right)\eta\rho_2\,ds$$
$$\leq\eta\rho_2\frac{1}{k}\int_0^1 e(s)\,ds\leq\eta\rho_2\Phi\leq\rho_2=\|u\|. \quad (13)$$

So, $\|Tu\|\leq\|u\|,\ u\in K\cap\partial\Omega_2$. Therefore by Theorem 2.4, T has at least one fixed point, which is a positive solution of (1)-(2) such that $\rho_1<\|u\|\leq\rho_2$.

Next, assume that (H2) holds. Since $f^0=0$, there exists $\rho_1>0$ such that $f(t,u)\leq\epsilon u$, for all $0<u\leq\rho_1, t\in[0,1]$, where $\epsilon>0$ satisfies $\epsilon\Phi\leq 1$. Then, for $u\in K\cap\partial\Omega_1$ with $\Omega_1=\{u\in C[0,1]:\|u\|<\rho_1\}$, we have

$$Tu(t)=\int_0^1 H(t,s)f(s,u(s))\,ds\leq\int_0^1\left(e(s)+\frac{\alpha}{k}e(s)+\frac{1}{k}\sum_{i=1}^{n}\beta_i e(s)\right)\epsilon u(s)\,ds$$
$$\leq\frac{1}{k}\epsilon\|u\|\int_0^1 e(s)\,ds\leq\epsilon\Phi\|u\|\leq\|u\|.$$

Therefore, $\|Tu\|\leq\|u\|,\ u\in K\cap\partial\Omega_1$.

By $f_\infty=\infty$, there exists $\widehat{\rho}_2>0$ such that $f(t,u)\geq\delta u$, for all $u>\widehat{\rho}_2$ and $t\in[\theta,1-\theta]$, where $\delta>0$ is chosen so that $\delta\Psi\geq 1$. Let $\rho_2=\max\{2\rho_1,\frac{\widehat{\rho}_2}{\theta^3(1-2\theta)}\}$ and $\Omega_2=\{u\in C[0,1],\|u\|<\rho_2\}$. So, for all $u\in K\cap\partial\Omega_2$, $u(t)\geq\widehat{\rho}_2$, $t\in[\theta,1-\theta]$ is satisfied. Similar to the estimates (11), we obtain

$$Tu(t)=\int_0^1 H(t,s)f(s,u(s))\,ds\geq\delta\Psi\|u\|\geq\|u\|.$$

The existence of a positive solution of (1)-(2) follows from Theorem 2.4. □

4. Multiplicity results

THEOREM 4.1. *Assume that the following assumptions are satisfied.*
(H3) $f_0=f_\infty=\infty$.

(H4) There exist constants $\rho_1>0$ *and* $M_1\in(0,\Lambda_1]$ *such that* $f(t,u)\leq M_1\rho_1$, *for* $u\in(0,\rho_1]$ *and* $t\in[0,1]$.
Then the problem (1)-(2) *has at least two positive solutions* u_1 *and* u_2 *such that* $0<\|u_1\|<\rho_1<\|u_2\|$.

Proof. First, assume that (H3) holds. Since $f_0 = \infty$, then for any $M_* \in [\Lambda_2, \infty)$, there exists $\rho_* \in (0, \rho_1)$ such that $f(t,u) \geq M_* u$, for all $t \in [\theta, 1-\theta]$ and $0 < u \leq \rho_*$. Set $\Omega_{\rho_*} = \{u \in C[0,1] : \|u\| < \rho_*\}$. By using Lemma 2.6, for $u \in K \cap \partial\Omega_{\rho_*}$ and $t \in [\theta, 1-\theta]$, we have

$$
\begin{aligned}
Tu(t) &= \int_0^1 H(t,s) f(s,u(s))\,ds \\
&\geq \int_\theta^{1-\theta} \Big(G(t,s) + \frac{\alpha}{k}\int_\theta^{1-\theta} G(\tau,s)d\tau + \frac{1}{k}\sum_{i=1}^n \beta_i G(\eta_i,s) \Big) f(s,u(s))\,ds \\
&\geq \int_\theta^{1-\theta} \Big(G(t,s) + \frac{\alpha}{k}\int_\theta^{1-\theta} G(\tau,s)d\tau + \frac{1}{k}\sum_{i=1}^n \beta_i G(\eta_i,s) \Big) M_* u(s)\,ds \\
&\geq M_*\theta^6(1-2\theta)^2 \Big[\int_\theta^{1-\theta} \Big(\Big(1+\frac{\alpha}{k}\Big) e(s) + \frac{1}{k}\sum_{i=1}^n \beta_i G(\eta_i,s) \Big)\,ds \Big] \rho_* \\
&= M_*\Lambda_2^{-1}\rho_* \geq \Lambda_2\Lambda_2^{-1}\rho_* = \|u\|,
\end{aligned}
$$

which means that

$$\|Tu\| \geq \|u\|, \; u \in K \cap \partial\Omega_{\rho_*}. \tag{14}$$

On the other hand, since $f_\infty = \infty$, then for any $M^* \in [\Lambda_2, \infty)$, there exists $\bar{\rho^*} > \rho_1$ such that $f(t,u) \geq M^* u$, for all $t \in [\theta, 1-\theta]$ and $u \geq \bar{\rho^*}$.

Let $\rho^* \geq \frac{\bar{\rho^*}}{\theta^3(1-2\theta)}$ and $\Omega_{\rho^*} = \{u \in C[0,1] : \|u\| < \rho^*\}$. For all $u \in K \cap \partial\Omega_{\rho^*}$, we have that $u(t) \geq \bar{\rho^*}$, $t \in [\theta, 1-\theta]$. Hence, for $t \in [\theta, 1-\theta]$, we get

$$
\begin{aligned}
Tu(t) &= \int_0^1 H(t,s) f(s,u(s))\,ds \\
&\geq \int_\theta^{1-\theta} \Big(G(t,s) + \frac{\alpha}{k}\int_0^1 G(\tau,s)d\tau + \frac{1}{k}\sum_{i=1}^n \beta_i G(\eta_i,s) \Big) f(s,u(s))\,ds \\
&\geq \int_\theta^{1-\theta} \Big(G(t,s) + \frac{\alpha}{k}\int_\theta^{1-\theta} G(\tau,s)d\tau + \frac{1}{k}\sum_{i=1}^n \beta_i G(\eta_i,s) \Big) M^* u(s)\,ds \\
&\geq \rho^* M^* \Lambda_2^{-1} \geq \rho^* \Lambda_2 \Lambda_2^{-1} = \|u\|.
\end{aligned} \tag{15}
$$

Therefore $\|Tu\| \geq \|u\|, \; u \in K \cap \partial\Omega_{\rho^*}$. (16)

Finally, set $\Omega_{\rho_1} = \{u \in C[0,1] : \|u\| < \rho_1\}$. Then for any $u \in K \cap \partial\Omega_{\rho_1}$, we get from (H4) that $f(t,u) \leq M_1\rho_1$ for all $t \in [0,1]$, and similarly to the estimates (12), we obtain

$$
\begin{aligned}
Tu(t) &= \int_0^1 \Big(G(t,s) + \frac{\alpha}{k}\int_0^1 G(\tau,s)d\tau + \frac{1}{k}\sum_{i=1}^n \beta_i G(\eta_i,s) \Big) f(s,u(s))\,ds \\
&\leq \int_0^1 e(s) \Big(1 + \frac{\alpha}{k} + \frac{1}{k}\sum_{i=1}^n \beta_i \Big) M_1\rho_1\,ds \\
&\leq \Lambda_1\rho_1 \frac{1}{k}\int_0^1 e(s)\,ds \leq \Lambda_1\Lambda_1^{-1}\rho_1 = \|u\|,
\end{aligned} \tag{17}
$$

yielding $\|Tu\| \leq \|u\|, \; u \in K \cap \partial\Omega_{\rho_1}$. (18)

Hence, since $\rho_* < \rho_1 < \rho^*$ and from (14), (16), (18), it follows from Theorem 2.4 that T has a fixed point u_1 in $K \cap (\bar{\Omega}_{\rho_1} \setminus \Omega_{\rho_*})$ and a fixed point u_2 in $K \cap (\bar{\Omega}_{\rho^*} \setminus \Omega_{\rho_1})$. Both are positive solutions of the problem (1)-(2) and $0 < \|u_1\| < \rho_1 < \|u_2\|$. □

THEOREM 4.2. *Assume that the following assumptions are satisfied:*
(H5) $f^0 = f^\infty = 0$.

(H6) There exist constants $\rho_2 > 0$ *and* $M_2 \in [\Lambda_2, \infty)$ *such that* $f(t,u) \geq M_2\rho_2$, *for* $u \in [\theta^3(1-2\theta)\rho_2, \rho_2]$ *and* $t \in [\theta, 1-\theta]$.
Then the problem (1)-(2) has at least two positive solutions u_1 *and* u_2 *such that* $0 < \|u_1\| < \rho_2 < \|u_2\|$.

Proof. Assume that (H5) holds. Firstly, since $f^0 = 0$, for any $\epsilon \in (0, \Lambda_1]$, there exists $\rho_* \in (0, \rho_2)$ such that $f(t,u) \leq \epsilon u$, for all $t \in [0,1]$ where $0 < u \leq \rho_*$. Then, for $u \in K \cap \partial\Omega_{\rho_*}$ with $\Omega_{\rho_*} = \{u \in C[0,1] : \|u\| < \rho_*\}$, we have

$$Tu(t) = \int_0^1 H(t,s) f(s,u(s))\,ds \leq \int_0^1 \left(e(s) + \frac{\alpha}{k}e(s) + \frac{1}{k}\sum_{i=1}^n \beta_i e(s)\right) f(s,u(s))\,ds$$
$$\leq \int_0^1 e(s)\left(1 + \frac{\alpha}{k} + \frac{1}{k}\sum_{i=1}^n \beta_i\right) \epsilon u(s)\,ds \leq \epsilon\rho_* \frac{1}{k}\int_0^1 e(s)\,ds \leq \epsilon\Lambda_1^{-1}\rho_* \leq \rho_* = \|u\|.$$

Therefore $$\|Tu\| \leq \|u\|, \; u \in K \cap \partial\Omega_{\rho_*}. \tag{19}$$

Secondly, in view of $f^\infty = 0$, for any $\epsilon_1 \in (0, \Lambda_1]$, there exists $\widetilde{\rho} > \rho_2$ such that $f(t,u) \leq \epsilon_1 u$, for all $t \in [0,1]$ with $u \geq \widetilde{\rho}$.

We consider two cases.

Case 1. Suppose that f is bounded. Let $L > 0$ be such that $f(t,u) \leq L$, for all $u \in [0,\infty)$ and $t \in [0,1]$. Taking $\rho^* \geq \max\{\widetilde{\rho}, \frac{L}{\epsilon_1}\}$, for $u \in K$ with $\|u\| = \rho^*$, we have

$$Tu(t) = \int_0^1 \left(G(t,s) + \frac{\alpha}{k}\int_0^1 G(\tau,s)d\tau + \frac{1}{k}\sum_{i=1}^n \beta_i G(\eta_i,s)\right) f(s,u(s))\,ds$$
$$\leq L\Phi \leq \rho^*\epsilon_1\Lambda_1^{-1} \leq \rho^* = \|u\|,$$

and consequently

$$\|Tu\| \leq \|u\|, \; u \in K \cap \partial\Omega_{\rho^*}. \tag{20}$$

Case 2. Suppose that f is unbounded; then from condition (C1), there exists $\sigma > 0$ such that $f(t,u) \leq \epsilon_1\sigma$, with $0 \leq u \leq \widetilde{\rho}$, and $t \in [0,1]$. For $u \in K$ with $\|u\| = \rho^*$, where $\rho^* \geq \max\{\sigma, \widetilde{\rho}\}$, we obtain

$$Tu(t) = \int_0^1 H(t,s) f(s,u(s))\,ds \leq \int_0^1 \left(e(s) + \frac{\alpha}{k}e(s) + \frac{1}{k}\sum_{i=1}^n \beta_i e(s)\right) f(s,u(s))\,ds$$
$$\leq \int_0^1 e(s)\left(1 + \frac{\alpha}{k} + \frac{1}{k}\sum_{i=1}^n \beta_i\right)\epsilon_1\rho^*\,ds \leq \epsilon_1\rho^*\Lambda_1^{-1} \leq \rho^* = \|u\|.$$

We conclude that

$$\|Tu\| \leq \|u\|, \; u \in K \cap \partial\Omega_{\rho^*}. \tag{21}$$

Hence, in either case, we may always set $\Omega_{\rho^*} = \{u \in C[0,1] : \|u\| < \rho^*\}$ such that $\|Tu\| \le \|u\|$, for $u \in K \cap \partial\Omega_{\rho^*}$.

Now, set $\Omega_{\rho_2} = \{u \in C[0,1] : \|u\| < \rho_2\}$. Then for any $u \in K \cap \partial\Omega_{\rho_2}$, we get from (H6) that there exists $M_2 \in [\Lambda_2, \infty)$ such that $f(t,u) \ge M_2\rho_2$ for all $t \in [\theta, 1-\theta]$, and $u \in [\theta^3(1-2\theta)\rho_2, \rho_2]$. Similarly to the estimates of (15), we get

$$\begin{aligned} Tu(t) &\ge M_2\theta^3(1-2\theta)\rho_2 \int_\theta^{1-\theta} \left(\left(1+\frac{\alpha}{k}\right)e(s) + \frac{1}{k}\sum_{i=1}^n \beta_i G(\eta_i, s)\right) ds \\ &\ge M_2\rho_2\theta^6(1-2\theta)^2 \int_\theta^{1-\theta} \left(\left(1+\frac{\alpha}{k}\right)e(s) + \frac{1}{k}\sum_{i=1}^n \beta_i G(\eta_i, s)\right) ds \\ &= M_2\Lambda_2^{-1}\rho_2 \ge \rho_2 = \|u\|. \end{aligned}$$

Then

$$\|Tu\| \ge \|u\|,\ u \in K \cap \partial\Omega_{\rho_2}. \tag{22}$$

Hence, from (19)–(22) and Theorem 2.4 it follows that there exist at least two positive solutions u_1 in $K \cap (\bar{\Omega}_{\rho_2} \setminus \Omega_{\rho_*})$ and u_2 in $K \cap (\bar{\Omega}_{\rho^*} \setminus \Omega_{\rho_2})$ of the problem (1)-(2) such that $0 < \|u_1\| < \rho_2 < \|u_2\|$. □

5. Examples

Example 5.1. Consider the boundary value problem

$$\begin{cases} u''''(t) + t + |\cos u| = 0,\ 0 < t < 1, \\ u'(0) = u'(1) = u''(0) = 0, \\ u(0) = \frac{1}{3}\int_0^1 u(s)\,ds + \frac{1}{7}u(\frac{7}{15}) + \frac{1}{4}u(\frac{2}{3}) + \frac{3}{84}u(\frac{11}{13}), \end{cases} \tag{23}$$

where $f(t,u) = t + |\cos u|$, $\alpha = \frac{1}{3}$, $\beta_1 = \frac{1}{7}$, $\beta_2 = \frac{1}{4}$, $\beta_3 = \frac{3}{84}$, $\eta_1 = \frac{7}{15}$, $\eta_2 = \frac{2}{3}$, and $\eta_3 = \frac{11}{13}$. We have $k = 1 - (\frac{1}{3} + \frac{1}{7} + \frac{1}{4} + \frac{3}{84}) = \frac{5}{21} > 0$, $f_0 = \infty$, $f^\infty = 0$. Then, by (H1) of Theorem 3.1, the problem (23) has at least one positive solution.

Example 5.2. As a second example we consider the following boundary value problem

$$\begin{cases} u''''(t) + u^2e^u\ln(1+t+u) = 0,\ 0 < t < 1, \\ u'(0) = u'(1) = u''(0) = 0, \\ u(0) = \frac{1}{4}\int_0^1 u(s)\,ds + \frac{1}{12}u(\frac{1}{8}) + \frac{1}{6}u(\frac{1}{4}), \end{cases} \tag{24}$$

where $f(t,u) = u^2e^u\ln(1+t+u) \ge 0$, $\alpha = \frac{1}{4}$, $\beta_1 = \frac{1}{12}$, $\beta_2 = \frac{1}{6}$, $\eta_1 = \frac{1}{8}$ and $\eta_2 = \frac{1}{4}$. We have $k = 1 - (\frac{1}{4} + \frac{1}{12} + \frac{1}{6}) = \frac{1}{2} > 0$, $f^0 = 0$, $f_\infty = \infty$. So, by (H2) of Theorem 3.1, the problem (24) has at least one positive solution.

Example 5.3. Consider the following boundary value problem

$$\begin{cases} u''''(t) + (1+t)e^u = 0,\ 0 < t < 1, \\ u'(0) = u'(1) = u''(0) = 0, \\ u(0) = \frac{1}{30}\int_0^1 u(s)\,ds + \frac{1}{60}u(\frac{1}{4}) + \frac{1}{120}u(\frac{1}{3}) + \frac{1}{240}u(\frac{1}{2}), \end{cases} \tag{25}$$

where $f(t,u) = (1+t)e^u$, $\alpha = \frac{1}{30}$, $\beta_1 = \frac{1}{60}$, $\beta_2 = \frac{1}{120}$, $\beta_3 = \frac{1}{240}$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{3}$ and $\eta_3 = \frac{1}{2}$.

Then $f_0 = f_\infty = \infty, k = 1 - (\frac{1}{30} + \frac{1}{60} + \frac{1}{120} + \frac{1}{240}) = \frac{15}{16}$. On the other hand, choose $\rho_1 = 1$ and $M_1 = \Lambda_1$. Then $f(t,u) \leq 2e$, for $(t,u) \in [0,1] \times (0,\rho_1]$ and $\Lambda_1 = 6k = \frac{45}{8} = 5,625$. So $f(t,u) \leq 2e \leq 5,625 = M_1\rho_1$.

By Theorem 4.1, the problem (25) has at least two positive solutions.

Example 5.4. Consider the following boundary value problem

$$\begin{cases} u''''(t) + 6528 \times 10^9 u^2 e^{1-u} = 0, \ 0 < t < 1, \\ u'(0) = u'(1) = u''(0) = 0, \\ u(0) = \frac{1}{10}\int_0^1 u(s)\,ds + \frac{1}{20}u(\frac{1}{2}), \end{cases} \tag{26}$$

where $f(t,u) = f(u) = 6528 \times 10^9 u^2 e^{1-u}, \alpha = \frac{1}{10}, \beta_1 = \beta = \frac{1}{20}$ and $\eta_1 = \eta = \frac{1}{2}$. Then $f^0 = f^\infty = 0$, $k = \frac{17}{20} > 0$, $(1 + \frac{\alpha}{k}) = \frac{19}{17}$, $\frac{\beta}{k} = \frac{1}{17}$, and $\Psi = \theta^6(1-2\theta)^2 \int_\theta^{1-\theta}((1 + \frac{\alpha}{k})e(s) + \frac{\beta}{k}G(\frac{1}{2}, s))\,ds$.

So,

$$\begin{aligned} \Psi =& \theta^6(1-2\theta)^2\Big[\int_\theta^{1-\theta} \frac{19}{17}\frac{1}{6}s(1-s)^2\,ds + \frac{1}{17}\frac{1}{6}\int_\theta^{\frac{1}{2}} \frac{1}{8}(1-s)^2\,ds \\ & - \frac{1}{17}\frac{1}{6}\int_\theta^{\frac{1}{2}}\Big(\frac{1}{2} - s\Big)^3 ds + \frac{1}{17}\frac{1}{6}\int_{\frac{1}{2}}^{1-\theta} \frac{1}{8}(1-s)^2\,ds\Big] \\ =& \frac{\theta^6(1-2\theta)^2}{102}\Big[19\Psi_1 + \frac{1}{8}\Psi_2^1 - \Psi_2^2 + \frac{1}{8}\Psi_3\Big], \end{aligned}$$

with

$$\begin{aligned} \Psi_1 =& \int_\theta^{1-\theta} s(1-s)^2\,ds = \frac{1}{6}(1-2\theta)\Big(\frac{1}{2} + \theta - \theta^2\Big), \\ \Psi_2^1 =& \int_\theta^{\frac{1}{2}} (1-s)^2\,ds = \frac{1}{6}(1-2\theta)\Big(\frac{7}{4} - \frac{5}{2}\theta + \theta^2\Big), \\ \Psi_2^2 =& \int_\theta^{\frac{1}{2}} \Big(\frac{1}{2} - s\Big)^3 ds = \frac{1}{64}(1-2\theta)^4, \\ \Psi_3 =& \int_{\frac{1}{2}}^{1-\theta} (1-s)^2\,ds = \frac{1}{6}(1-2\theta)\Big(\frac{1}{4} + \frac{1}{2}\theta + \theta^2\Big), \\ \Psi =& \frac{1}{6528}\theta^6(1-2\theta)^3(103 + 206\theta - 212\theta^2 + 8\theta^3). \end{aligned}$$

So, $\Lambda_2 = 6528 \times \theta^{-6}(1-2\theta)^{-3}(103 + 206\theta - 212\theta^2 + 8\theta^3)^{-1}$. On the other hand, let us choose $\rho_2 = 1$ and $M_2 = \Lambda_2$. Then $f(t,u) = f(u) \geq 6528 \times 10^9\theta^6(1-2\theta)^2$, for $(t,u) \in [\theta, 1-\theta] \times [\theta^3(1-2\theta)\rho_2, \rho_2]$. So, $f(t,u) \geq 10^9\theta^{12}(1-2\theta)^5(103 + 206\theta - 212\theta^2 + 8\theta^3)\Lambda_2$. Using the Mathematica software, we easily check that

$$10^9\theta^{12}(1-2\theta)^5(103 + 206\theta - 212\theta^2 + 8\theta^3) \geq 1, \text{ for all } \theta \in \Big[\frac{17}{125}, \frac{12}{25}\Big],$$

and consequently $f(t,u) \geq \Lambda_2 = M_2$.

By Theorem 4.2, the problem (26) has at least two positive solutions.

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(received 24.03.2019; in revised form 05.06.2019; available online 31.05.2020)

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