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EXISTENCE OF INFINITELY MANY EIGENGRAPH SEQUENCES OF THE $p(\cdot)$-BIHARMONIC EQUATION

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Abstract. The aim of this paper is to study the nonlinear eigenvalue problem

$$(P)\quad \begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) - \lambda\zeta(x)|u|^{\alpha(x)-2}u = \mu\xi(x)|u|^{\beta(x)-2}u, \quad x \in \Omega, \\ u \in W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $N \geq 1$, λ, μ are real parameters, ζ and ξ are nonnegative functions, p, α, and β are continuous functions on $\overline{\Omega}$ such that $1 < \alpha(x) < \beta(x) < p(x) < \frac{N}{2}$.

We show that the $p(\cdot)$-biharmonic operator possesses infinitely many eigengraph sequences and also prove that the principal eigengraph exists. Our analysis mainly relies on variational method and we prove Ljusternik-Schnirelemann theory on C^1-manifold.

1. Introduction

Nonlinear elliptic equations and variational problems involving variable exponents growth conditions has received a lot of attention in the last decades. This is a consequence of the fact that such equations can be used to model phenomena which arise in mathematical physics, such as in the electrorheological fluids, nonlinear porous medium, and image processing see e.g. [15, 16, 19].

A typical model of an elliptic equation with $p(\cdot)$-biharmonic operator is

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) = V(x)f(\lambda, x, u) \quad \text{in} \quad \Omega, \tag{1}$$

where V is a weight function, and $f : [0, +\infty] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a suitable function.

Problems like (1) have been largely considered in the literature in the recent years. For instance, Ayoujil and El Amrouss in [1] considered the problem (1), assuming that the nonlinearity has the form $f(\lambda, x, u) = \lambda|u|^{p(x)-2}u$, $V(x) = 1$ and subject to Navier boundary conditions. In this setting, the existence of a sequence of eigenvalues by

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using the Ljusternik-Schnirelmann critical point theory is established. Successively, in [2] the same authors considered the case $f(\lambda, x, u) = \lambda|u|^{q(x)-2}u$, $V(x) = 1$. Also in [13], the authors are interested in the existence of a continuous family of eigenvalues of problem (1) with $f(\lambda, x, u) = \lambda|u|^{q(x)-2}u$ and V is an indefinite weight function. The reader is referred to [5, 9–12, 20, 21] for some recent works on this subject.

We investigate in the present work the following two-parameters eigenvalue problem

$$(P) \qquad \begin{cases} \Delta^2_{p(x)}u - \lambda\zeta(x)|u|^{\alpha(x)-2}u = \mu\xi(x)|u|^{\beta(x)-2}u, & x \in \Omega, \\ u \in W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega), \end{cases}$$

where

- Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $N \geq 1$,
- λ is a positive real number,
- μ stands for a function depending on λ generating the corresponding eigengraphs. More precisely, we mean by eigengraphs the sets in $\mathbb{R}^2$ defined by $\{(\lambda, \mu(\lambda)) \text{ such that } \lambda \in \mathbb{R}^+\}$,
- p, α and β are continuous functions on $\overline{\Omega}$,
- ζ and ξ are nonnegative functions.

$\Delta^2_{p(\cdot)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the so-called $p(\cdot)$-biharmonic operator. It is reduced to the p-biharmonic (for a constant exponent $p > 1$). For variable exponent the $p(\cdot)$-biharmonic presents a more complicated nonlinearity than the p-biharmonic, we loose homogenity of any order.

The main goal of this paper is to show that for any parameter $\lambda \in \mathbb{R}^+$, the problem (P) has infinitely many eigengraph sequences $(\mu_k(\lambda))_{k\geq 1}$, by using the Ljusternik-Schnirelmann theory on C^1-manifolds [17]. We also give a direct characterization of the principal eigengraph.

This article is divided into five sections. In Section 2 we recall some basic facts about the variable exponent Lebesgue and Sobolev spaces. In Section 3, we present some important basic lemmas which allow us to prove our main results. In Section 4, we prove our first main result related to the existence of infinitely many eigengraph sequences for problem (P). The existence and a direct characterization of the principal eigengraph of problem (P) is derived in the last section.

2. Terminology and abstract framework

To study the problem (P), we need to recall some results on the spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$, respectively, which will be used later. For a deeper treatment, we refer the reader to [3, 14] and the references therein. Suppose that Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and let us denote by

$$C_1^+(\overline{\Omega}) := \{h \mid h \in C(\overline{\Omega}) \text{ and } h(x) > 1 \text{ for all } x \in \overline{\Omega}\},$$

$$M := \{u : \Omega \to \mathbb{R} \text{ and } u \text{ is a measurable real-valued function}\},$$
$$h^+ := \max_{x\in\overline{\Omega}} h(x), \quad h^- := \min_{x\in\overline{\Omega}} h(x), \text{ for any } h \in C_1^+(\overline{\Omega}).$$

For each fixed $p \in C_1^+(\overline{\Omega})$, we define the generalized Lebesgue space by

$$L^{p(\cdot)}(\Omega) := \{u \in M \mid \rho_{p(\cdot)}(u) < \infty\},$$

endowed with the so-called Luxemburg norm

$$|u|_{L^{p(\cdot)}(\Omega)} := |u|_{p(\cdot)} := \inf\Big\{\alpha > 0 \;\Big|\; \rho_{p(\cdot)}(\frac{u}{\alpha}) \leq 1\Big\},$$

where $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ is the convex modular mapping $\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx$. For each $m \in \mathbb{N}^*$, we define the variable exponent Sobolev space $W^{m,p(\cdot)}(\Omega)$,

$$W^{m,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) \mid D^\alpha u \in L^{p(\cdot)}(\Omega), \ |\alpha| \leq m\},$$

with the norm $\|u\|_{W^{m,p(\cdot)}(\Omega)} := \|u\|_{m,p(\cdot)} := \sum_{|\alpha|\leq m} |D^\alpha u|_{p(\cdot)}$.

Both $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$ are Banach, separable and reflexive spaces. We denote by $W_0^{m,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$.

Denote by $L^{q(\cdot)}(\Omega)$ the dual space of $L^{p(\cdot)}(\Omega)$ where q is the conjugate function of p, i.e., $q(x) = \frac{p(x)}{p(x)-1}$ for all $x \in \Omega$.

For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, we have the following Hölder-type inequality,

$$\Big|\int_\Omega uv\,dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(\cdot)}|v|_{q(\cdot)} \leq 2|u|_{p(\cdot)}|v|_{q(\cdot)}. \tag{2}$$

Moreover, if p_1, p_2 and $p_3 \in C_1^+(\overline{\Omega})$ are Lipschitz continuous functions such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, and for any functions $u \in L^{p_1(\cdot)}(\Omega)$, $v \in L^{p_2(\cdot)}(\Omega)$ and $w \in L^{p_3(\cdot)}(\Omega)$, the generalized Hölder-type inequality is given by

$$\Big|\int_\Omega uvw\,dx\Big| \leq \Big(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-}\Big)|u|_{p_1(\cdot)}|v|_{p_2(\cdot)}|w|_{p_3(\cdot)} \leq 3|u|_{p_1(\cdot)}|v|_{p_2(\cdot)}|w|_{p_3(\cdot)}, \tag{3}$$

see [7, Proposition 2.5].

Proposition 2.1 ([8, Theorem 1.3]). *Let u_n, $u \in L^{p(\cdot)}$. The following relations hold*

1. $|u|_{p(\cdot)} = a \Leftrightarrow \rho_{p(\cdot)}(\frac{u}{a}) = 1$ *for* $u \neq 0$ *and* $a > 0$.
2. $|u|_{p(\cdot)} <$ *(respectively* $=; > 1$) $\Leftrightarrow \rho_{p(\cdot)}(u) <$ *(respectively* $=; > 1$).
3. $|u_n| \to 0 (resp \to +\infty) \Leftrightarrow \rho_{p(\cdot)}(u_n) \to 0, (resp \to +\infty)$.
4. *the following statements are equivalent to one another:*
   - *(i)* $\lim_{n\to+\infty} |u_n - u|_{p(\cdot)} = 0$,
   - *(ii)* $\lim_{n\to+\infty} \rho_{p(\cdot)}(u_n - u) = 0$,
   - *(iii)* $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to+\infty} \rho_{p(\cdot)}(u_n) = \rho_{p(\cdot)}(u)$.

We recall also the following proposition, which will be needed later.

Proposition 2.2 ([4]). *Let p and q be measurable functions such that $p \in L^\infty(\Omega)$*

and $1 < p(x)q(x) < \infty$*, for a.e.* $x \in \Omega$*. Let* $u \in L^{q(\cdot)}(\Omega)$*,* $u \neq 0$*. Then*

$$|u|_{p(\cdot)q(\cdot)} \leq 1 \Rightarrow |u|_{p(\cdot)q(\cdot)}^{p^+} \leq \left| |u|^{p(\cdot)} \right|_{q(\cdot)} \leq |u|_{p(\cdot)q(\cdot)}^{p^-},$$

$$|u|_{p(\cdot)q(\cdot)} \geq 1 \Rightarrow |u|_{p(\cdot)q(\cdot)}^{p^-} \leq \left| |u|^{p(\cdot)} \right|_{q(\cdot)} \leq |u|_{p(\cdot)q(\cdot)}^{p^+}.$$

Note that the weak solutions of (P) are considered in the generalized Sobolev space $X := W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$.

Generally, we know that if $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are Banach spaces, we define the norm on the space $X := E \cap F$ as $\|u\|_X = \|u\|_E + \|u\|_F$. In our case, we have, for any $u \in X$, $\|u\|_X = \|u\|_{1,p(\cdot)} + \|u\|_{2,p(\cdot)}$, thus $\|u\|_X = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)} + \sum_{|\alpha|=2} |D^\alpha u|_{p(\cdot)}$.

In [18], the equivalence of the norms was proved, and it was even proved that the norm $|\Delta(\cdot)|_{p(\cdot)}$ is equivalent to the norm $\|\cdot\|_X$ (see [18, Theorem 4.4]).

To deal with the problem under consideration, we choose on X the norm defined by $\|u\| := |\Delta u|_{p(\cdot)}$. Note that, $(X, \|\cdot\|)$ is also a separable and reflexive Banach space.

Proposition 2.3. *For all* $u \in X$*, denote* $\Lambda_{p(\cdot)}(u) := \int_\Omega |\Delta u(x)|^{p(x)}\, dx$*. Then:*

1. For $u \in X$*, we have*

- *(i)* $\|u\| < 1$ *(*$= 1, > 1$*)* $\Leftrightarrow \Lambda_{p(\cdot)}(u) < 1$ *(*$= 1 > 1$*);*
- *(ii)* $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \Lambda_{p(\cdot)}(u) \leq \|u\|^{p^+}$*;*
- *(iii)* $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \Lambda_{p(\cdot)}(u) \leq \|u\|^{p^-}$*.*

2. If $u, u_n \in X, n = 1, 2, \ldots$*, then the following statements are equivalent:*

- *(i)* $\lim_{n\to\infty} \|u_n - u\| = 0$*;*
- *(ii)* $\lim_{n\to\infty} \Lambda_{p(\cdot)}(u_n - u) = 0$*;*
- *(iii)* $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \Lambda_{p(\cdot)}(u_n) = \Lambda_{p(\cdot)}(u)$*.*

Definition 2.4. For $p \in C_1^+(\overline{\Omega})$, let us define the so-called critical Sobolev exponent of p by

$$p_2^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)}, & \text{if } p(x) < \frac{N}{2}, \\ +\infty, & \text{if } p(x) \geq \frac{N}{2}, \end{cases}$$

for every $x \in \overline{\Omega}$.

Theorem 2.5 ([1, Theorem 3.2]). *Let* $p, q \in C_1^+(\overline{\Omega})$*. Assume that* $p(x) < \frac{N}{2}$ *and* $q(x) < p_2^*(x)$*. Then there is a continuous and compact embedding of* X *into* $L^{q(\cdot)}(\Omega)$*.*

The last mathematical tool needed in this paper is given in [6], in which the authors deal with the so defined property (S_+).

Lemma 2.6. *The following hold true:*

(i) $\Delta^2_{p(\cdot)} : X \to X^*$ *is a strictly monotone operator, that is,*

$$\left\langle \Delta^2_{p(\cdot)} u - \Delta^2_{p(\cdot)} v, u - v \right\rangle > 0, \qquad \textit{for all } u \neq v \in X.$$

(ii) $\Delta^2_{p(\cdot)} : X \to X^*$ *is a continuous, bounded homeomorphism.*

(iii) $\Delta^2_{p(\cdot)} : X \to X^*$ *is a mapping of type* (S_+), *that is, if* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty} \langle \Delta^2_{p(\cdot)} u_n, u_n - u \rangle \leq 0$, *then* $u_n \to u$ *in* X.

Notation 2.7. We introduce some notations that will clarify what follows:

- For simplicity we write $u_n \rightharpoonup u$ and $u_n \to u$ to denote the weak convergence and strong convergence of a sequence u_n in X, respectively.
- When we refer to a Banach space X, we denote by X^* its dual and by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X.
- By meas $(\cdot)$ we denote the Lebesgue measure of a set.

3. Hypotheses and basic lemmas

In this section, we investigate some basic lemmas. First, we will work under the following hypotheses on the problem (P),
(H1) $1 < \alpha^- \leq \alpha^+ < \beta^- \leq \beta^+ < p^- \leq p^+ < \frac{N}{2}$.

(H2) ζ and ξ are two weight functions satisfying

$$\zeta \in C(\overline{\Omega}) \ \text{ with } \ \zeta^- := \min_{x \in \overline{\Omega}} \zeta(x) > 0,$$

and

$$\xi \in L^{r(\cdot)}(\Omega), \quad r(x) > \frac{N}{2} \ \text{ and } \ \xi(x) > 0 \ \text{ a.e. in } \Omega. \tag{4}$$

Remark 3.1. We will denote by r' the conjugate exponent of the function r, and put

$$s(x) = \frac{r(x)\beta(x)}{r(x) - \beta(x)}.$$

Thus, by hypotheses (H1) and (H2)–(4) on the functions p, α, β and r, a straightforward computation gives $\alpha(x) < p_2^*(x)$, $s(x) < p_2^*(x)$, and $p^- < p_2^*(x)$, for $x \in \overline{\Omega}$. Then, from Theorem 2.5, the embeddings

$$X \hookrightarrow L^{\alpha(\cdot)}(\Omega), \quad X \hookrightarrow L^{s(\cdot)}(\Omega), \quad \text{and} \quad X \hookrightarrow L^{p^-}(\Omega), \tag{5}$$

are compact and continuous. Therefore, there exists a positive constant C such that $|u|_{\alpha(\cdot)} \leq C\|u\|$, $|u|_{s(\cdot)} \leq C\|u\|$, $|u|_{p^-} \leq C\|u\|$, for $u \in X$. Without any loss of generality, we can suppose that $C > 1$.

Definition 3.2. We recall that, for a fixed real λ, $\mu = \mu(\lambda)$ is an eigenvalue of (P)

if and only if there exists $u \in X \setminus \{0\}$, such that

$$\int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x)dx$$
$$= \int_\Omega \big(\lambda\zeta(x)|u(x)|^{\alpha(x)-2}u(x) + \mu\xi(x)|u(x)|^{\beta(x)-2}u(x)\big)v(x)\,dx \quad .$$

for all $v \in X$. u is then called an eigenfunction associated to μ.

In the rest of this section, we will prove some lemmas which allow us to prove our main results. To this end, consider the energy functional corresponding to problem (P) defined on X as $H(u) := \Phi(u) - \lambda\Psi(u) - \mu J(u)$, where

$$\Phi(u) = \int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}dx, \quad \Psi(u) = \int_\Omega \frac{\zeta(x)}{\alpha(x)}|u(x)|^{\alpha(x)}dx, \quad J(u) = \int_\Omega \frac{\xi(x)}{\beta(x)}|u(x)|^{\beta(x)}dx,$$

and set $\mathcal{V} := \big\{u \in X \;\big|\; J(u) = 1\big\}$.

LEMMA 3.3. *The following hold true:*
(a) Φ, Ψ *and* J *are even, and of class* C^1 *on* X.

(b) $\mathcal{V}$ *is a closed* C^1*-manifold.*

Proof. (a) It is clear that Φ, Ψ and J are even. Standard arguments imply that $\Phi, \Psi, J \in C^1(X, \mathbb{R})$ and their derivative functions are given by

$$\big\langle d\Phi(u), v\big\rangle = \int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x)\,dx,$$
$$\big\langle d\Psi(u), v\big\rangle = \int_\Omega \zeta(x)|u(x)|^{\alpha(x)-2}u(x)v(x)\,dx,$$
$$\big\langle dJ(u), v\big\rangle = \int_\Omega \xi(x)|u(x)|^{\beta(x)-2}u(x)v(x)\,dx,$$

for all $u, v \in X$.

(b) $\mathcal{V} = J^{-1}\{1\}$. Thus $\mathcal{V}$ is closed. For all $x \in \overline{\Omega}$, we have $\beta^- \le \beta(x) \le \beta^+$, and then for all $u \in \mathcal{V}$

$$\big\langle dJ(u), u\big\rangle = \int_\Omega \xi(x)|u(x)|^{\beta(x)}\,dx \ge \beta^- > 0.$$

The derivative operator dJ satisfies $dJ(u) \neq 0\;\forall u \in \mathcal{V}$, i.e., $dJ(u)$ is onto for all $u \in \mathcal{V}$. Hence J is a submersion, which proves that $J^{-1}\{1\}$ is a C^1-manifold of X. □

LEMMA 3.4. $d\Psi$ *is completely continuous, namely,* $u_n \rightharpoonup u$ *in* X *implies* $d\Psi(u_n) \to d\Psi(u)$ *in* X^*.

Proof. Let $u_n \rightharpoonup u$ in X. For any $v \in X$, by Hölder-type inequality (2) and continuous embedding of X into $L^{\alpha(\cdot)}(\Omega)$, it follows that

$$\big|\big\langle d\Psi(u_n) - d\Psi(u), v\big\rangle\big| = \left|\int_\Omega \zeta(x)\big(|u_n|^{\alpha(x)-2}u_n - |u|^{\alpha(x)-2}u\big)v\,dx\right|$$
$$\le 2\zeta^+\big||u_n|^{\alpha(x)-2}u_n - |u|^{\alpha(x)-2}u\big|_{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}|v|_{\alpha(\cdot)}$$

$$\leq 2C\zeta^+ \big||u_n|^{\alpha(x)-2}u_n - |u|^{\alpha(x)-2}u\big|_{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}} \|v\|.$$

On the other hand, using the compact embedding of X into $L^{\alpha(\cdot)}(\Omega)$, we have $u_n \to u$ in $L^{\alpha(\cdot)}(\Omega)$. Due to the fact that the map $L^{\alpha(\cdot)}(\Omega) \ni u \mapsto |u|^{\alpha(x)-2}u \in L^{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}(\Omega)$, is continuous, we get $|u_n|^{\alpha(x)-2}u_n \to |u|^{\alpha(x)-2}u$ in $L^{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}(\Omega)$. That is, $d\Psi(u_n) \to d\Psi(u)$ in $L^{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}(\Omega)$. Recall that the embedding $L^{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}(\Omega) \hookrightarrow X^*$, is compact. Thus $d\Psi(u_n) \to d\Psi(u)$ in X^*. □

Lemma 3.5. *dJ is completely continuous.*

Proof. Let $u_n \rightharpoonup u$ in X. For any $v \in x$, by Hölder-type inequality (3) and continuous embedding of X into $L^{s(\cdot)}(\Omega)$, it follows that

$$\begin{aligned}
\big|\langle dJ(u_n) - dJ(u), v\rangle\big| &= \left|\int_\Omega \xi(x)\big(|u_n|^{\beta(x)-2}u_n - |u|^{\beta(x)-2}u\big)v\,dx\right| \\
&\leq 3\big|\xi(x)\big|_{r(\cdot)}\big||u_n|^{\beta(x)-2}u_n - |u|^{\beta(x)-2}u\big|_{\frac{\beta(\cdot)}{\beta(\cdot)-1}}|v|_{s(\cdot)} \\
&\leq 3C\big|\xi(x)\big|_{r(\cdot)}\big||u_n|^{\beta(x)-2}u_n - |u|^{\beta(x)-2}u\big|_{\frac{\beta(\cdot)}{\beta(\cdot)-1}}\|v\|.
\end{aligned}$$

On the other hand, using the compact embedding of X into $L^{\beta(\cdot)}(\Omega)$, we have $u_n \to u$ in $L^{\beta(\cdot)}(\Omega)$. Due to the fact that the map $L^{\beta(\cdot)}(\Omega) \ni u \mapsto |u|^{\beta(x)-2}u \in L^{\frac{\beta(\cdot)}{\beta(\cdot)-1}}(\Omega)$, is continuous, we get $|u_n|^{\beta(x)-2}u_n \to |u|^{\beta(x)-2}u$ in $L^{\frac{\beta(\cdot)}{\beta(\cdot)-1}}(\Omega)$. Therefore, the above inequality ends the proof. □

Lemma 3.6. *There exists a constant $\lambda_1 > 0$ such that*

$$\lambda_1 = \inf_{u\in X, \|u\|>1} \frac{\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}\,dx}{\int_\Omega |u|^{p^-}\,dx}. \tag{6}$$

Proof. By (5), there exists a positive constant C such that $|u|_{p^-} \leq C\|u\|$ for all $u \in X$. On the other hand, we have $\int_\Omega |\Delta u|^{p(x)}\,dx \geq \|u\|^{p^-}$ for all $u \in X$ with $\|u\| > 1$.

From the above two inequalities, we obtain that

$$\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}\,dx \geq \frac{1}{C^{p^-}p^+}\int_\Omega |u|^{p^-}\,dx.$$

Thus, there exists $\lambda_1 > 0$ satisfying (6). □

Next, we write $\Phi_\lambda(\cdot) = \Phi(\cdot) - \lambda\Psi(\cdot)$.

Lemma 3.7. *For any $\lambda \in \mathbb{R}^+$, Φ_λ is bounded from below.*

Proof. Since $\alpha^+ < p^-$ we have $\lim_{|u|\to+\infty} \frac{\frac{\zeta^+}{\alpha^-}|u|^{\alpha(x)}}{|u|^{p^-}} = 0$ uniformly on $\overline{\Omega}$. Then, for any $\lambda > 0$, there exists a positive constant K such that

$$\frac{\lambda\zeta^+}{\alpha^-}|u|^{\alpha(x)} \leq \frac{\lambda_1}{2}|u|^{p^-} + K, \quad \forall x \in \overline{\Omega}. \tag{7}$$

where λ_1 is defined by (6).
For any $u \in X$ with $\|u\| > 1$, from (6), and (7), we have

$$\Phi_\lambda(u) \geq \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)} - \frac{\lambda\zeta^+}{\alpha^+}\int_\Omega |u|^{\alpha(x)}dx$$
$$\geq \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)} - \frac{\lambda_1}{2}\int_\Omega |u|^{p^-}dx - K\text{meas}(\Omega).$$

This implies that
$$\Phi_\lambda(u) \geq \frac{1}{2}\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)} - K\text{meas}(\Omega). \quad (8)$$

Thus
$$\Phi_\lambda(u) \geq \frac{1}{2p^+}\|u\|^{p^-} - K\text{meas}(\Omega).$$

As $p^- > 1$, Φ_λ is bounded from below and coercive since $\Phi_\lambda(u) \to \infty$ as $\|u\| \to \infty$. □

4. Existence of infinitely many eigengraph sequences

In this section, we show that the problem (P) has infinitely many eigengraph sequences, by using the results of Ljusternik-Schnirelmann [17]. Let

$$\Gamma_j = \left\{H \subset \mathcal{V} \;\middle|\; H \text{ is compact, } H = -H \text{ and } \gamma(H) \geq j\right\},$$

where $\gamma(H) = j$ is the Krasnoselskii genus of the set H, i.e.,

$$\gamma(H) = \inf\left\{j \;\middle|\; \text{ there exists an odd continuous map from } H \text{ to } \mathbb{R}^j \setminus \{0\}\right\}.$$

Let us now state the main result in this section.

Theorem 4.1. *For any integer $j \in \mathbb{N}^*$ and for any $\lambda \in \mathbb{R}^+$, $\mu_j(\lambda) := \inf_{H\in\Gamma_j}\max_{u\in H}\Phi_\lambda(u)$ is a critical value of Φ_λ restricted on $\mathcal{V}$. More precisely, there exists $u_j \in H$ such that $\mu_j(\lambda) = \Phi_\lambda(u_j) = \sup_{u\in H}\Phi_\lambda(u)$, and u_j is an eigenfunctin associated to the positive eigenvalue $(\lambda, \mu_j(\lambda))$. Moreover, $\mu_j(\lambda) \to \infty$, as $j \to \infty$.*

To obtain the proof of Theorem 4.1, we must show the functional Φ_λ satisfies the Palais-Smale condition [in short the (PS) condition] on $\mathcal{V}$, in the first place.

Proposition 4.2. *The functional Φ_λ satisfies the (PS) condition on $\mathcal{V}$ for every $\lambda > 0$. Namely, we will prove that if a sequence $\{u_n\}_{n\geq 1} \subset \mathcal{V}$ satisfies*

$$|\Phi_\lambda(u_n)| \leq d \quad \textit{for some } d > 0 \textit{ and all } n \geq 1, \quad (9)$$
$$d\Phi_\lambda(u_n) \to 0 \quad \textit{in } X^*, \textit{ as } n \to \infty. \quad (10)$$

then $\{u_n\}_{n\geq 1}$ has a convergent subsequence in X.

Proof. Let $\{u_n\}_{n\geq 1}$ be a sequence of Palais-Smale of Φ_λ in X. Since

$$\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx \geq \frac{1}{p^+}\int_\Omega |\Delta u_n|^{p(x)}\,dx = \frac{1}{p^+}\Lambda_{p(\cdot)}(u_n),$$

this fact, combined with (8) and (9), implies that

$$d_2 \leq \Lambda_{p(\cdot)}(u_n) \leq 2p^+\big(d + K\text{meas}(\Omega)\big) \leq d_1, \quad d, d_1, d_2 > 0.$$

That is $\Lambda_{p(\cdot)}(u_n)$ is bounded in $\mathbb{R}$.

Thus, without loss of generality, we can assume that u_n converges weakly in X to some function $u \in X$ and $\Lambda_{p(\cdot)}(u_n) \to \ell$.

If $\ell = 0$, then u_n converges strongly to 0 in X. Otherwise, then we argue as follows.

From (10), $d\Phi_\lambda(u_n) \to 0$. i.e.,

$$\alpha_n = d\Phi_\lambda(u_n) - \beta_n dJ(u_n) \to 0 \quad \text{as} \quad n \to +\infty, \tag{11}$$

where
$$\beta_n = \frac{\langle d\Phi_\lambda(u_n), u_n\rangle}{\langle dJ(u_n), u_n\rangle}.$$

The idea is to prove that $\limsup_{n\to\infty}\langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle \le 0$. Indeed, notice that

$$\langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle = \Lambda_{p(\cdot)}(u_n) - \langle \Delta^2_{p(\cdot)} u_n, u\rangle.$$

Applying α_n of (11) to u, we deduce that

$$\lim_{n\to\infty} \theta_n = \langle \Delta^2_{p(\cdot)} u_n, u\rangle - \lambda\langle d\Psi(u_n), u\rangle - \beta_n\langle dJ(u_n), u\rangle = 0.$$

Therefore,
$$\begin{aligned}\langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle =& \Lambda_{p(\cdot)}(u_n) - \lambda\langle d\Psi(u_n), u\rangle \\ &- \theta_n - \frac{\langle d\Phi_\lambda(u_n), u_n\rangle}{\langle dJ(u_n), u_n\rangle}\langle dJ(u_n), u\rangle.\end{aligned}$$

That is,
$$\begin{aligned}\langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle =& \frac{\Lambda_{p(\cdot)}(u_n)}{\langle dJ(u_n), u_n\rangle}\Big(\langle dJ(u_n), u_n\rangle - \langle dJ(u_n), u\rangle\Big) \\ &- \theta_n - \lambda\langle d\Psi(u_n), u\rangle + \lambda\frac{\langle d\Psi(u_n), u_n\rangle}{\langle dJ(u_n), u_n\rangle}\langle dJ(u_n), u\rangle.\end{aligned}$$

On the other hand, from Lemma 3.4, $d\Psi$ is completely continuous. Thus $d\Psi(u_n) \to d\Psi(u)$, $\langle d\Psi(u_n), u_n\rangle \to \langle d\Psi(u), u\rangle$ and $\langle d\Psi(u_n), u\rangle \to \langle d\Psi(u), u\rangle$. From Lemma 3.5, dJ is also completely continuous. So $dJ(u_n) \to dJ(u)$, $\langle dJ(u_n), u_n\rangle \to \langle dJ(u), u\rangle$ and $\langle dJ(u_n), u\rangle \to \langle dJ(u), u\rangle$.

Then
$$\begin{aligned}\big|\langle dJ(u_n), u_n\rangle - \langle dJ(u_n), u\rangle\big| \le& \big|\langle dJ(u_n), u_n\rangle - \langle dJ(u), u\rangle\big| \\ &+ \big|\langle dJ(u_n), u\rangle - \langle dJ(u), u\rangle\big|,\end{aligned}$$

and
$$\begin{aligned}\big|\langle dJ(u_n), u_n\rangle - \langle dJ(u_n), u\rangle\big| \le& \big|\langle dJ(u_n), u_n\rangle - \langle dJ(u), u\rangle\big| \\ &+ \big\|dJ(u_n) - dJ(u)\big\|_* \big\|u\big\|,\end{aligned}$$

where $\|\cdot\|_*$ is the dual norm associated to the norm $\|\cdot\|$. This implies that $\langle dJ(u_n), u_n\rangle - \langle dJ(u_n), u\rangle \to 0$ as $n \to \infty$. Combining with the above equalities, we obtain

$$\limsup_{n\to+\infty} \langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle \le \frac{\ell}{\langle dJ(u), u\rangle} \limsup_{n\to\infty} \big[\langle dJ(u_n), u_n\rangle - \langle dJ(u_n), u\rangle\big].$$

We deduce $\limsup_{n\to\infty} \langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle \le 0$. In view of Lemma 2.6, $u_n \to u$ strongly in X. $\square$

We are now in a position to prove Theorem 4.1.

Proof (of Theorem 4.1). We prove the theorem in two steps.
Step 1. We will show for any $j \in \mathbb{N}^*$, $\Gamma_j \neq \emptyset$.

Indeed, let $j \in \mathbb{N}^*$ be given and let $x_1 \in \Omega$ and $r_1 > 0$ be small enough such that $\overline{B(x_1, r_1)} \subset \Omega$ and $\text{meas}(\overline{B(x_1, r_1)}) < \frac{\text{meas}(\Omega)}{2}$. First, we take $\theta_1 \in C_0^\infty(\Omega)$ with $\text{supp}(\theta_1) = \overline{B(x_1, r_1)}$. Put $B_1 := \Omega \backslash \overline{B(x_1, r_1)}$, then $\text{meas}(B_1) > \frac{\text{meas}(\Omega)}{2}$. Let $x_2 \in B_1$ and $r_2 > 0$ such that $\overline{B(x_2, r_2)} \subset B_1$ and $\text{meas}(\overline{B(x_2, r_2)}) < \frac{\text{meas}(B_1)}{2}$. Next, we take $\theta_2 \in C_0^\infty(\Omega)$ with $\text{supp}(\theta_2) = \overline{B(x_2, r_2)}$. Continuing the process described above we can construct by recurrence a sequence of functions $\theta_1, \theta_2, \ldots, \theta_j \in C_0^\infty(\Omega)$ such that

$$\begin{cases} \text{supp}(\theta_i) \cap \text{supp}(\theta_j) = \emptyset & \text{if} \quad i \neq j, \\ \text{meas}(\text{supp}(\theta_i)) > 0 & \text{for} \quad i \in \{1, 2, \ldots, j\}. \end{cases}$$

Let $X_j = \text{span}\{\theta_1, \theta_2, \cdots, \theta_j\}$ be the vector subspace of X generated by j vectors $\{\theta_1, \theta_2, \ldots, \theta_j\}$. Then, it is clear that $\dim \ X_j = j$ and $\int_\Omega \frac{\xi(x)}{\beta(x)} |u(x)|^{\beta(x)} dx > 0$ for all $u \in X_j \setminus \{0\}$.

Note that $X_j \subset L^{\beta(\cdot)}(\Omega)$ because $X_j \subset X \subset L^{\beta(\cdot)}(\Omega)$. Thus the norms $\| \cdot \|$ and $|.|_{\beta(\cdot)}$ are equivalent on X_j because X_j is a finite dimensional space. Consequently the map $u \mapsto |u|_{\beta(\cdot)} := \inf \left\{ \alpha > 0 \mid \int_\Omega \left| \frac{u(x)}{\alpha} \right|^{\beta(x)} dx \leq 1 \right\}$, defines a norm on X_j. Putting $S_1 := \left\{ u \in X_j \middle| |u|_{\beta(\cdot)} = 1 \right\}$ the unit sphere of X_j.

Let us introduce the functional $g : \mathbb{R}^+ \times X_j \longrightarrow \mathbb{R}$, $(s, u) \mapsto J(su)$. On one hand, it is clear that $g(0, u) = 0$ and $g(s, u)$ is non-decreasing with respect to s. Moreover, for $s > 1$ we have $g(s, u) \geq s^{\beta^-} J(u)$, so that $\lim_{s \to +\infty} g(s, u) = +\infty$. Therefore, for every $u \in S_1$ fixed, there is a unique value $s = s(u) > 0$ such that $g(s(u), u) = 1$.

On the other hand, since

$$\frac{\partial g}{\partial s}(s(u), u) = \int_\Omega (s(u))^{\beta(x)-1} \xi(x) |u|^{\beta(x)} \, dx \geq \frac{\beta^-}{s(u)} g(s(u), u) = \frac{\beta^-}{s(u)} > 0,$$

the implicit function theorem implies that the map $u \mapsto s(u)$ is continuous and even by uniqueness. Now, take the compact $H_j := \mathcal{V} \cap X_j$. Since the map $h : S_1 \longrightarrow H_j$ defined by $h(u) = s(u) \cdot u$ is continuous and odd, it follows by the property of genus that $\gamma(H_j) = j$. Therefore $H_j \in \Gamma_j$.

Step 2. We claim that $\mu_j(\lambda) \to \infty$ as $j \to \infty$.

Let $(e_n, e_k^*)_{n,k}$ be a bi-orthogonal system such that $e_n \in X$ and $e_k^* \in X^*$, the $(e_n)_n$ are linearly dense in X and the $(e_k^*)_k$ are total for the dual X^*. For $j \in \mathbb{N}^*$, set $X_j = \text{span}\{e_1, \ldots, e_j\}$ and $X_j^\perp = \text{span}\{e_{j+1}, e_{j+2}, \ldots\}$. By a property of genus, we have for any $H \in \Gamma_j$, it is $H \cap X_{j-1}^\perp \neq \emptyset$.

We claim that $t_j = \inf_{H \in \Gamma_j} \sup_{u \in H \cap X_{j-1}^\perp} \Phi_\lambda(u) \to \infty$ as $j \to \infty$.

Indeed, if not, for large j there exists $u_j \in X_{j-1}^\perp$ with $\int_\Omega \frac{\xi(x)}{\beta(x)} |u_j(x)|^{\beta(x)} dx = 1$ such that $t_j \leq \Phi_\lambda(u_j) \leq M$, for some $M > 0$ independent of j. Thus in view of (8), we get $\|u_j\| \leq \left(2p^+(M + K \ \text{meas}(\Omega))\right)^{\frac{1}{p^-}}$. This implies that $(u_j)_j$ is bounded in X.

For a subsequence of $\{u_j\}$ if necessary, we can assume that $\{u_j\}_{j\geq 1}$ converges weakly in X and strongly in $L^{p(\cdot)}(\Omega)$.

By our choice of $X_{j-1}^{\perp}$, we have $u_j \rightharpoonup 0$ in X because $\langle e_k^*, e_n\rangle = 0$, for any $n > k$. This contradicts the fact that $\int_\Omega \frac{\xi(x)}{\beta(x)}|u_j(x)|^{\beta(x)}dx = 1$ for all j.

Indeed, from Lemma 3.5, dJ is completely continuous, so $\langle dJ(u_j), u_j\rangle \to 0$. On the other hand, since $\int_\Omega \frac{\xi(x)}{\beta(x)}|u_j(x)|^{\beta(x)}dx = 1$, and

$$\langle dJ(u_j), u_j\rangle = \int_\Omega \xi(x)|u_j(x)|^{\beta(x)}dx \geq \beta^- \int_\Omega \frac{\xi(x)}{\beta(x)}|u_j(x)|^{\beta(x)}dx \geq \beta^- \geq 1,$$

and since $\langle dJ(u_j), u_j\rangle \geq 1$, for all j, $\langle dJ(u_j), u_j\rangle \to l \geq 1$. Therefore, $l \neq 0$. Since $\mu_j(\lambda) \geq t_j$, we get $\mu_j(\lambda) \to \infty$ as $j \to \infty$, the claim is proved. □

5. Existence of the principal eigengraph

Our purpose in this section is to derive an existence result concerning principal eigengraph $\mu_1(\lambda)$, and we give a variational formulation of $\mu_1(\lambda)$ involving a mini-max argument over sets of genus greater than k.

Definition 5.1. We denote by $\mu_1(\lambda)$ the first principal eigenvalue of (P) and letting the parameter λ to vary, one gets the graph of the function $\lambda \to \mu_1(\lambda)$ from $\mathbb{R}^+$ into $\mathbb{R}$ which is called, in the literature, the principal eigengraph of (P) and sets as

$$\mu_1(\lambda) := \inf\left\{\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}\,dx - \lambda\int_\Omega \frac{\zeta(x)}{\alpha(x)}\big|u\big|^{\alpha(x)}\,dx \Big| \int_\Omega \frac{\xi(x)}{\beta(x)}|u|^{\beta(x)}\,dx = 1\right\}. \quad (12)$$

Remark 5.2. Clearly, $\mu_1(\lambda)$ defined by (12) can be equivalently written as

$$\mu_1(\lambda) := \inf_{H\in\Gamma_j}\max_{u\in H} \frac{\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}\,dx - \lambda\int_\Omega \frac{\zeta(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx}{\int_\Omega \frac{\xi(x)}{\beta(x)}|u|^{\beta(x)}\,dx},$$

or equivalently to,

$$\frac{1}{\mu_1(\lambda)} := \sup_{H\in\Gamma_j}\min_{u\in H} \frac{\int_\Omega \frac{\xi(x)}{\beta(x)}|u|^{\beta(x)}\,dx}{\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}\,dx - \lambda\int_\Omega \frac{\zeta(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx}.$$

In the following corollary we give some properties of the principal eigengraph μ_1.

Corollary 5.3. *The following properties hold true:*
a) $\mu_1(\lambda) = \inf\left\{\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx - \lambda\int_\Omega \frac{\zeta(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx \big| \int_\Omega \frac{\xi(x)}{\beta(x)}|u|^{\beta(x)}\,dx = 1\right\}$.
b) $\mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots \leq \mu_n(\lambda) \to +\infty$.

Proof. a) For $u \in \mathcal{V}$, set $H_1 = \{u, -u\}$. It is clear that $\gamma(H_1) = 1$, Φ_λ is even and

$$\Phi_\lambda(u) = \max_{H_1}\Phi_\lambda \geq \inf_{H\in\Gamma_1}\max_{u\in H}\Phi_\lambda(u).$$

Thus

$$\inf_{u\in\mathcal{V}}\Phi_\lambda(u) \geq \inf_{H\in\Gamma_1}\max_{u\in H}\Phi_\lambda(u) = \mu_1(\lambda).$$

On the other hand, for all $H \in \Gamma_1$ and $u \in H$, we have $\max_{u\in H} \Phi_\lambda \geq \Phi_\lambda(u) \geq \inf_{u\in\mathcal{V}} \Phi_\lambda(u)$. It follows that $\inf_{H\in\Gamma_1} \max_H \Phi_\lambda = \mu_1(\lambda) \geq \inf_{u\in\mathcal{V}} \Phi_\lambda(u)$. Then

$$\mu_1(\lambda) = \inf\left\{ \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx - \lambda\int_\Omega \frac{\zeta(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx \Big| \int_\Omega \frac{\xi(x)}{\beta(x)}|u|^{\beta(x)}\,dx = 1\right\}.$$

b) For all $i \geq j$, we have $\Gamma_i \subset \Gamma_j$ and in view of the definition of $\mu_i(\lambda), i \in \mathbb{N}^*$, we get $\mu_i(\lambda) \geq \mu_j(\lambda)$. As regards $\mu_n(\lambda) \to \infty$, it has been proved in Theorem 4.1. □

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