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Corrected proof
Available online 28.03.2021

research paper
оригинални научни рад

CONVERGENCE AND STABILITY OF PICARD S-ITERATION PROCEDURE FOR CONTRACTIVE-LIKE OPERATORS

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Abstract. Let $(X, \|.\|)$ be a normed linear space. Let K be a nonempty closed convex subset of X. Let $T : K \to K$ be a contractive-like operator with a nonempty fixed point set $F(T)$. We prove the strong convergence and T-stability of Picard S-iteration procedure with respect to the contractive-like operator T which are independent for any arbitrary choices of the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0, 1]$.

1. Introduction

Let K be a nonempty closed convex subset of a normed linear space $(X, \|.\|)$ and $T : K \to K$ be a selfmap of K. A point $x \in K$ is called a fixed point of T if $Tx = x$ and we denote the set of all fixed points of T by $F(T)$.

Harder and Hicks [3] initiated the concept of T-stability of a general fixed point iteration procedure. In the following, we state the definition of T-stability of Harder and Hicks as in Berinde [1].

Definition 1.1 ([1]). Let (X, d) be a metric space, $T : X \to X$ a mapping, $x_0 \in X$ and let us assume that the iteration procedure $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \dots \tag{1}$$

converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set $\epsilon_n = d(y_{n+1}, f(T, y_n))$ for $n = 0, 1, 2, \dots$. We say that the fixed point iteration procedure (1) is T-stable or stable with respect to T if $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} y_n = p$.

Let $(X, \|.\|)$ be a normed linear space, K a nonempty subset of X. A map $T : K \to K$ is called a contractive-like operator [4] if there exist $\delta \in [0, 1)$, a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$, such that for each $x, y \in K$,

$$\|Tx - Ty\| \le \delta\|x - y\| + \varphi(\|x - Tx\|), \tag{2}$$

2020 Mathematics Subject Classification: 47H10, 54H25.
Keywords and phrases: Fixed point; contractive-like operator; Picard S-iteration procedure; T-stability.

where $\mathbb{R}^+$ denote $[0, \infty)$.

In order to prove some stability results, the contractive inequality condition (2) was proposed and employed by Imoru and Olatinwo [4]. Let K be a nonempty convex subset of a normed linear space X and $T : K \to K$ be a map. In 1953, Mann [7] introduced an iteration procedure as follows: For $x_0 \in K$, the Mann iteration procedure $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \ , \ n = 0, 1, 2, \ldots$$

where $\{\alpha_n\} \subset [0, 1]$.

In 1974, Ishikawa [5] developed an iteration procedure in the following way: For $x_0 \in K$, the Ishikawa iteration procedure $\{x_n\}$ in K is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n \end{cases} \tag{3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$.

Based on these iteration procedures, several iteration procedures were developed. In 2014, Gürsoy and Karakaya [2] introduced Picard S-iteration procedure as follows:

$$\begin{cases} u_0 \in K \\ w_n = (1 - \beta_n)u_n + \beta_n T u_n \\ v_n = (1 - \alpha_n)T u_n + \alpha_n T w_n \\ u_{n+1} = T v_n \end{cases} \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$.

Zeana and Ahmed [15] proved that the sequence generated by Picard S-iteration procedure (4) is convergent for a contractive-like operator T having a fixed point under certain conditions on α_n and β_n. In fact, the following theorem was proved.

Theorem 1.2 ([15, Theorem 2.1]). *Let K be a nonempty closed convex subset of a Banach space X and $T : K \to K$ be a contractive-like operator with a fixed point p. Then for all $x_0 \in K$, the Picard S-iteration procedure* (4) *converges to the unique fixed point of T if $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$.*

We use the following lemma in our further discussion.

Lemma 1.3 ([6]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers. Assume that there exists a constant $0 \le h < 1$ such that $a_{n+1} \le h a_n + b_n$ for all n, and $\lim_{n\to\infty} b_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.*

Remark 1.4. Lemma 1.3 is also contained in Berinde [1] and the articles of the authors [4, 9, 11].

For more literature on the convergence and T-stability of a general fixed point iteration procedure, we refer to [1, 8–14] and related references therein.

In this paper, we prove the strong convergence of Picard S-iteration procedure of a contractive-like operator with a fixed point defined on a nonempty closed convex subset of a Banach space X. Also, we show that conditions $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ of Theorem 1.2 are redundant. Further, we prove the Picard S-iteration procedure (4) is T-stable for any arbitrary choices of the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$.

2. Convergence and T-stability of Picard S-iteration procedure

In the following we prove that the convergence of Picard S-iteration procedure (4) for contractive-like operators is independent of α_n and β_n.

THEOREM 2.1. *Let K be a nonempty closed convex subset of an arbitrary Banach space X and $T : K \to K$ be a contractive-like operator. Suppose that $F(T) \neq \emptyset$. Let $\{u_n\}_{n=0}^{\infty}$ be the sequence generated by Picard S-iteration procedure with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$. Then $\{u_n\}_{n=0}^{\infty}$ converges to a unique fixed point of T.*

Proof. Since $F(T) \neq \emptyset$ and T is a contractive-like operator, we have $F(T) = \{p\}$.

First, we consider

$$\begin{aligned}\|w_n - p\| &= \|(1-\beta_n)u_n + \beta_n T u_n - p\| \leq (1-\beta_n)\|u_n - p\| + \beta_n\|Tu_n - Tp\| \\ &\leq (1-\beta_n)\|u_n - p\| + \beta_n[\delta\|u_n - p\| + \varphi(\|p - Tp\|)] \\ &\overset{\varphi(0)=0}{=} [1 - \beta_n(1-\delta)]\|u_n - p\| \leq \|u_n - p\|. \end{aligned} \tag{5}$$

Next,

$$\begin{aligned}\|v_n - p\| &= \|(1-\alpha_n)Tu_n + \alpha_n T w_n - p\| \leq (1-\alpha_n)\|Tu_n - Tp\| + \alpha_n\|Tw_n - Tp\| \\ &\leq (1-\alpha_n)[\delta\|u_n - p\| + \varphi(\|p - Tp\|)] + \alpha_n[\delta\|w_n - p\| + \varphi(\|p - Tp\|)] \\ &= \delta[(1-\alpha_n)\|u_n - p\| + \alpha_n\|w_n - p\|] \overset{(5)}{\leq} \delta\|u_n - p\|.\end{aligned}$$

Now, $\|u_{n+1} - p\| = \|Tv_n - Tp\| \leq \delta\|v_n - p\| + \varphi(\|p - Tp\|) \leq \delta^2\|u_n - p\|$. Therefore, by induction on n, it is easy to see that $\|u_{n+1} - p\| \leq \delta^{2(n+1)}\|u_0 - p\|$ for $n = 0, 1, 2\dots$ and by letting $n \to \infty$, we have $\lim_{n\to\infty} u_n = p$. □

REMARK 2.2. From the previous proof, it follows that $\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n = p$.

In the following example, we consider all the possible choices of the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ and show that the sequence generated by Picard S-iteration procedure (4) is convergent for contractive-like operators.

EXAMPLE 2.3. Let $K = [1,3]$ be a closed convex subset of the normed linear space $X = \mathbb{R}$ equipped with the usual norm.

Define $T : [1,3] \to [1,3]$ by $Tx = 2 + \frac{1}{x}$. Then T is a contractive-like operator with $\delta = \frac{2}{3}$ and $\varphi(x) = \frac{3x^2}{4}$, $x \geq 0$. Observe that $1 + \sqrt{2}$ is the unique fixed point of T.

Case (i): Let $\alpha_n = \frac{n+1}{n+2}$, $\beta_n = \frac{1}{n^2+1}$ so that $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $\sum\limits_{n=0}^{\infty} \beta_n < \infty$ and $\sum\limits_{n=0}^{\infty} \alpha_n\beta_n < \infty$. For any $u_0 \in [1,3]$,

$$w_n = (1-\beta_n)u_n+\beta_n Tu_n = \frac{n^2u_n^2+2u_n+1}{(n^2+1)u_n},$$

$$\begin{aligned}
v_n &= (1-\alpha_n)Tu_n+\alpha_n Tw_n = \frac{2u_n+1}{(n+2)u_n}+\frac{n+1}{n+2}\Big(2+\frac{(n^2+1)u_n}{n^2u_n^2+2u_n+1}\Big)\\
&= \frac{2u_n+1}{(n+2)u_n}+\frac{n+1}{n+2}\Big(\frac{2n^2u_n^2+4u_n+2+(n^2+1)u_n}{n^2u_n^2+2u_n+1}\Big),\\
&= \frac{(2u_n^3+u_n^2)n^3+(4u_n^3+2u_n^2)n^2+(5u_n^2+2u_n)n+9u_n^2+6u_n+1}{n^3u_n^3+2n^2u_n^3+(2u_n^2+u_n)n+2u_n},
\end{aligned}$$

$$\begin{aligned}
u_{n+1} &= Tv_n = 2+\frac{n^3u_n^3+2n^2u_n^3+(2u_n^2+u_n)n+2u_n}{(2u_n^3+u_n^2)n^3+(4u_n^3+2u_n^2)n^2+(5u_n^2+2u_n)n+9u_n^2+6u_n+1}\\
&= \frac{(5u_n^3+2u_n^2)n^3+(10u_n^3+4u_n^2)n^2+(12u_n^2+5u_n)n+(22u_n^2+14u_n+2)}{(2u_n^3+u_n^2)n^3+(4u_n^3+2u_n^2)n^2+(5u_n^2+2u_n)n+9u_n^2+6u_n+1} = \frac{5u_n+2}{2u_n+1}+A_n,
\end{aligned}$$

where $A_n = \frac{(-u_n^3+2u_n^2+u_n)(n+1)}{(2u_n+1)[(2u_n^3+u_n^2)n^3+(4u_n^3+2u_n^2)n^2+(5u_n^2+2u_n)n+9u_n^2+6u_n+1]}$. Therefore, $|A_n| \le \frac{24(n+1)}{9n^3+18n^2+21n+48} = \frac{8(n+1)}{3n^3+6n^2+7n+16}$, for $n = 0,1,2\ldots$, and hence, $\lim\limits_{n\to\infty} A_n = 0$.

Now, $$\begin{aligned}
u_{n+1}-(1+\sqrt{2}) &= \frac{5u_n+2}{2u_n+1}-(1+\sqrt{2})+A_n = \frac{(3-2\sqrt{2})u_n+1-\sqrt{2}}{2u_n+1}+A_n\\
&= \frac{(3-2\sqrt{2})[u_n-(1+\sqrt{2})]}{2u_n+1}+A_n.
\end{aligned}$$

Therefore, $|u_{n+1}-(1+\sqrt{2})| \le \frac{3-2\sqrt{2}}{3}|u_n-(1+\sqrt{2})|+|A_n|$, for $n = 0,1,2,\ldots$. By applying limit superior on both sides and using $\lim\limits_{n\to\infty} A_n = 0$, we have $\limsup|u_{n+1}-(1+\sqrt{2})| \le \frac{3-2\sqrt{2}}{3}\limsup|u_n-(1+\sqrt{2})|$, so that $\limsup|u_n-(1+\sqrt{2})| \le 0$. Hence, $\lim\limits_{n\to\infty} u_n = 1+\sqrt{2}$.

Case (ii): Let $\alpha_n = \frac{1}{2^n}$, $\beta_n = \frac{1}{3^n}$ so that $\sum\limits_{n=0}^{\infty} \alpha_n < \infty$, $\sum\limits_{n=0}^{\infty} \beta_n < \infty$ and $\sum\limits_{n=0}^{\infty} \alpha_n\beta_n < \infty$. For any $u_0 \in [1,3]$, we have:

$$w_n = (1-\beta_n)u_n+\beta_n Tu_n = \frac{(3^n-1)u_n^2+2u_n+1}{3^nu_n},$$

$$\begin{aligned}
v_n &= (1-\alpha_n)Tu_n+\alpha_n Tw_n = \Big(1-\frac{1}{2^n}\Big)\Big(2+\frac{1}{u_n}\Big)+\frac{1}{2^n}\Big(2+\frac{3^nu_n}{(3^n-1)u_n^2+2u_n+1}\Big)\\
&= \frac{6^n(2u_n^3+u_n^2)+2^n(-2u_n^3+3u_n^2+4u_n+1)+u_n^2-2u_n-1}{6^nu_n^3-2^nu_n^3+2.2^nu_n^2+2^nu_n},
\end{aligned}$$

$$u_{n+1} = Tv_n = 2+\frac{6^nu_n^3-2^nu_n^3+2.2^nu_n^2+2^nu_n}{6^n(2u_n^3+u_n^2)+2^n(-2u_n^3+3u_n^2+4u_n+1)+u_n^2-2u_n-1}$$

$$= \frac{6^n(5u_n^3+2u_n^2)+2^n(-5u_n^3+8u_n^2+9u_n+2)+2u_n^2-4u_n-2}{6^n(2u_n^3+u_n^2)+2^n(-2u_n^3+3u_n^2+4u_n+1)+u_n^2-2u_n-1} = \frac{5u_n+2}{2u_n+1} - B_n,$$

where $B_n = \frac{u_n(u_n^2-2u_n-1)}{(2u_n+1)[6^n(2u_n^3+u_n^2)+2^n(-2u_n^3+3u_n^2+4u_n+1)+u_n^2-2u_n-1]}$. It is easy to see that $\lim\limits_{n\to\infty} B_n = 0$. We consider

$$u_{n+1} - (1+\sqrt{2}) = \frac{(3-2\sqrt{2})u_n + (1-\sqrt{2})}{2u_n+1} + B_n = \frac{(3-2\sqrt{2})[u_n - (1+\sqrt{2})]}{2u_n+1} + B_n,$$

where $B_n = \frac{u_n(u_n^2-2u_n-1)}{(2u_n+1)[6^n(2u_n^3+u_n^2)+2^n(-2u_n^3+3u_n^2+4u_n+1)+u_n^2-2u_n+1]}$, so that $\lim\limits_{n\to\infty} B_n = 0$. Therefore, for $n = 0, 1, 2\ldots$,

$$|u_{n+1}-(1+\sqrt{2})| \leq \frac{(3-2\sqrt{2})}{2u_n+1}|u_n-(1+\sqrt{2})|+|B_n| \leq \frac{3-2\sqrt{2}}{3}|u_n-(1+\sqrt{2})|+|B_n|.$$

By applying limit superior on both sides, we have $\frac{2\sqrt{2}}{3}\limsup|u_n - (1+\sqrt{2})| \leq 0$, so that $\lim\limits_{n\to\infty} u_n = 1+\sqrt{2}$.

Case (iii): Let $\alpha_0 = \beta_0 = 0$, $\alpha_n = \beta_n = \frac{1}{\sqrt{n}}$ so that $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $\sum\limits_{n=0}^{\infty} \beta_n = \infty$ and $\sum\limits_{n=0}^{\infty} \alpha_n\beta_n = \infty$. For any $u_0 \in [1,3]$, we have:

$$w_n = (1-\beta_n)u_n + \beta_n Tu_n = \frac{(\sqrt{n}-1)u_n^2 + 2u_n + 1}{u_n\sqrt{n}}, \quad \text{for } n = 1,2,3\ldots,$$

$$\begin{aligned} v_n &= (1-\alpha_n)Tu_n + \alpha_n Tw_n = (1-\frac{1}{\sqrt{n}})(2+\frac{1}{u_n}) + \frac{1}{\sqrt{n}}(2+\frac{1}{w_n}) \\ &= \frac{\sqrt{n}-1}{\sqrt{n}}\frac{2u_n+1}{u_n} + \frac{1}{\sqrt{n}}(2 + \frac{u_n\sqrt{n}}{(\sqrt{n}-1)u_n^2+2u_n+1}) \\ &= \frac{n(2u_n^3+u_n^2) + \sqrt{n}[-2u_n^3+3u_n^2+4u_n+1] + u_n^2 - 2u_n - 1}{\sqrt{n}u_n(\sqrt{n}u_n^2 - u_n^2 + 2u_n + 1)}, \end{aligned}$$

$$\begin{aligned} u_{n+1} &= 2 + \frac{nu_n^3 + \sqrt{n}(-u_n^3+2u_n^2+u_n)}{n(2u_n^3+u_n^2) + \sqrt{n}[-2u_n^3+3u_n^2+4u_n+1] + u_n^2 - 2u_n - 1} \\ &= \frac{n(5u_n^3+2u_n^2) + \sqrt{n}[-5u_n^3+8u_n^2+9u_n+2] + 2u_n^2 - 4u_n - 2}{n(2u_n^3+u_n^2) + \sqrt{n}[-2u_n^3+3u_n^2+4u_n+1] + u_n^2 - 2u_n - 1} = \frac{5u_n+2}{2u_n+1} + C_n, \end{aligned}$$

where $C_n = \frac{-u_n^3+2u_n^2+u_n}{(2u_n+1)[n(2u_n^3+u_n^2)+\sqrt{n}(-2u_n^3+3u_n^2+4u_n+1)+u_n^2-2u_n-1]}$ for $n = 1,2,3\ldots$, so that $\lim\limits_{n\to\infty} C_n = 0$. We consider

$$u_{n+1} - (1+\sqrt{2}) = \frac{(3-2\sqrt{2})u_n + 1 - \sqrt{2}}{2u_n+1} + C_n = \frac{3-2\sqrt{2}}{2u_n+1}[u_n - (1+\sqrt{2})] + C_n,$$

so that $|u_{n+1} - (1+\sqrt{2})| \leq \frac{3-2\sqrt{2}}{3}|u_n - (1+\sqrt{2})| + |C_n|$, for $n = 1,2,3\ldots$. By applying limit superior on both sides, we have $\frac{2\sqrt{2}}{3}\limsup|u_n - (1+\sqrt{2})| \leq 0$, hence $\lim\limits_{n\to\infty} u_n = 1+\sqrt{2}$.

Case (iv): Let $\alpha_0 = 1$, $\beta_0 = 1$, $\alpha_n = \beta_n = \frac{1}{n}$, for $n = 1, 2, \ldots$, so that $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \beta_n = \infty$ and $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty$. Now, for any $u_0 \in [1, 3]$,

$$\begin{aligned} w_n &= \frac{n^2 u_n - n u_n + 2u_n + 1}{n u_n}, \\ v_n &= \frac{n^2(2u_n^3 + u_n^2) + n(-2u_n^3 + 3u_n^2 + 4u_n + 1) + (u_n^2 - 2u_n - 1)}{n^2 u_n^3 + n(-u_n^3 + 2u_n^2 + u_n)}, \\ u_{n+1} &= \frac{5u_n + 2}{2u_n + 1} + D_n, \end{aligned}$$

where $D_n = \frac{-u_n^3 + 2u_n^2 + u_n}{(2u_n+1)[n^2(2u_n^3+u_n^2)+n(-2u_n^3+3u_n^2+4u_n+1)+u_n^2-2u_n-1]}$, for $n = 1, 2, \ldots$. Clearly, $\lim_{n\to\infty} D_n = 0$ and $u_{n+1} - (1+\sqrt{2}) = \frac{3-2\sqrt{2}}{2u_n+1}[u_n - (1+\sqrt{2})] + D_n$, so that $\lim_{n\to\infty} u_n = 1 + \sqrt{2}$.

Case (v): Let $\alpha_0 = 1$, $\alpha_n = \frac{1}{n^2}$ for $n = 1, 2, \ldots$, and $\beta_n = \frac{1}{2}$ for all n. Therefore, $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\sum_{n=0}^{\infty} \beta_n = \infty$ and $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty$.
For any $u_0 \in [1, 3]$ and $n = 1, 2, \ldots$,

$$w_n = \frac{u_n^2+2u_n+1}{2u_n}, \quad v_n = \frac{n^2(2u_n+1)(u_n^2+2u_n+1)+u_n^2-2u_n}{n^2 u_n(u_n^2+2u_n+1)}, \quad u_{n+1} = \frac{5u_n + 2}{2u_n + 1} + E_n,$$

where $E_n = \frac{-u_n^3+2u_n^2}{(2u_n+1)[n^2(2u_n+1)(u_n^2+2u_n+1)+u_n^2-2u_n]}$. Clearly, $\lim_{n\to\infty} E_n = 0$. Therefore, $|u_{n+1} - (1+\sqrt{2})| \leq \frac{3-2\sqrt{2}}{2u_n+1}|u_n - (1+\sqrt{2})| + E_n$, for $n = 1, 2, 3, \ldots$. Now, by applying limit superior on both sides, we have $\limsup |u_{n+1} - (1+\sqrt{2})| \leq \frac{3-2\sqrt{2}}{3} \limsup |u_n - (1+\sqrt{2})|$, which implies that $\limsup |u_n - (1+\sqrt{2})| \leq 0$, so that $\lim_{n\to\infty} u_n = 1 + \sqrt{2}$.

By applying comparison test, it is easy to see that the following cases do not arise.

Case (vi): $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\sum_{n=0}^{\infty} \beta_n < \infty$ and $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

Case (vii): $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \beta_n < \infty$ and $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

Case (viii): $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\sum_{n=0}^{\infty} \beta_n = \infty$ and $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

Hence, by considering all the above cases, we conclude that the convergence of Picard S-iteration procedure is independent of the choices of $\{\alpha_n\}$ and $\{\beta_n\}$ for contractive-like operators.

REMARK 2.4. Theorem 2.1 and Example 2.3 suggest that the conditions $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ are redundant in Theorem 1.2.

Theorem 2.5. Let K be a nonempty closed convex subset of an arbitrary Banach space X and $T : K \to K$ be a contractive-like operator. Suppose that $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be arbitrary sequences in $[0, 1]$. Then the Picard S-iteration procedure (4) is T-stable.

Proof. Since T has a unique fixed point in K, we let it be p. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in K and $\epsilon_n = \|y_{n+1} - f(T, y_n)\|$ where $f(T, y_n) = T((1-\alpha_n)Ty_n + \alpha_n Tt_n)$, $t_n = (1 - \beta_n)y_n + \beta_n Ty_n$ for $n = 0, 1, 2, \ldots$.

First we prove that

$$\|f(T, y_n) - p\| \leq \delta^2 \|y_n - p\| \text{ for } n = 0, 1, 2 \ldots . \tag{6}$$

We consider

$$\begin{aligned}\|f(T, y_n) - p\| &= \|T((1 - \alpha_n)Ty_n + \alpha_n Tt_n) - Tp\| \\ &\overset{(2)}{\leq} \delta\|(1 - \alpha_n)Ty_n + \alpha_n Tt_n - Tp\| + \varphi(\|p - Tp\|) \\ &\leq \delta[(1 - \alpha_n)\|Ty_n - Tp\| + \alpha_n\|Tt_n - Tp\|] \\ &\leq \delta^2[(1 - \alpha_n)\|y_n - p\| + \alpha_n\|t_n - p\|] + \delta\varphi(\|p - Tp\|).\end{aligned}$$

Hence,

$$\|f(T, y_n) - p\| \leq \delta^2[(1 - \alpha_n)\|y_n - p\| + \alpha_n\|t_n - p\|]. \tag{7}$$

Now, since

$$\begin{aligned}\|t_n - p\| &= \|(1 - \beta_n)y_n + \beta_n Ty_n - p\| \leq (1 - \beta_n)\|y_n - p\| + \beta_n\|Ty_n - Tp\| \\ &\overset{(2)}{\leq} (1 - \beta_n)\|y_n - p\| + \beta_n\delta\|y_n - p\| + \beta_n\varphi(\|p - Tp\|) \\ &\leq (1 - \beta_n(1 - \delta))\|y_n - p\| \leq \|y_n - p\|,\end{aligned}$$

we have

$$\|t_n - p\| \leq \|y_n - p\|. \tag{8}$$

Therefore from (7) and (8), it follows that (6) holds.

We assume that $\lim_{n\to\infty} \epsilon_n = 0$ and consider

$$\|y_{n+1} - p\| \leq \|y_{n+1} - f(T, y_n)\| + \|f(T, y_n) - p\| \overset{(6)}{\leq} \epsilon_n + \delta^2\|y_n - p\|.$$

By applying Lemma 1.3, we have $\lim_{n\to\infty} y_n = p$.

Conversely, we assume that $\lim_{n\to\infty} y_n = p$ and consider

$$\epsilon_n = \|y_{n+1} - f(T, y_n)\| \leq \|y_{n+1} - p\| + \|f(T, y_n) - p\|.$$

It follows from (6) that $\epsilon_n \leq \|y_{n+1} - p\| + \delta^2\|y_n - p\|$, for $n = 0, 1, 2 \ldots$, so that $\lim_{n\to\infty} \epsilon_n = 0$.

Thus the Picard S-iteration procedure is T-stable. □

Acknowledgement. The authors are thankful to the referee for his/her careful reading and valuable suggestions which improved the presentation of the paper.

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(received 13.10.2019; in revised form 11.06.2020; available online 28.03.2021)

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