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Corrected proof
Available online 22.03.2021

research paper
оригинални научни рад

AN ABSTRACT APPROACH FOR THE STUDY OF THE DIRICHLET PROBLEM FOR AN ELLIPTIC SYSTEM ON A CONICAL DOMAIN

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Abstract. In this paper, we analyze the Dirichlet problem for an elliptic system on a conical domain. We essentially apply the results from the theory of sums of operators to achieve our main results. We work in the setting of little Hölder spaces.

1. Position of the problem

Let $\Omega_0 \subset \mathbb{R}^2$ be the truncated plane sector defined by

$$\Omega_0 := \left\{ re^{i\theta} : r \in \mathbb{R}^+, 0 < \theta < \omega \right\},$$

where $\omega \neq \pi$ and $\omega < 2\pi$. We assume that

$$\partial\Omega_0 = \Gamma'_0 \cup \Gamma'_1,$$

where $\quad \Gamma'_0 = \left\{ (r,0) : r \in \mathbb{R}^+ \right\} \quad \text{and} \quad \Gamma'_1 = \left\{ (r,\omega) : r \in \mathbb{R}^+ \right\}.$

Of concern is the solvability of the following elliptic system

$$\begin{cases} \Delta u_1 + y\dfrac{\partial u_1}{\partial x} - x\dfrac{\partial u_1}{\partial y} = f_1(x,y), & (x,y) \in \Omega_0, \\ \Delta u_2 - y\dfrac{\partial u_2}{\partial x} + x\dfrac{\partial u_2}{\partial y} = f_2(x,y), & (x,y) \in \Omega_0, \end{cases} \tag{1}$$

accompanied with the following boundary conditions

$$u_1 = u_2 = 0 \quad \text{on} \quad \Gamma'_0 \cup \Gamma'_1. \tag{2}$$

Furthermore, we assume that

$$(f_1, f_2) \in h^{2\alpha}(\Omega_0) \times h^{2\alpha}(\Omega_0),\ 0 < 2\alpha < 1,$$

with

$$f_1|_{\Gamma'_0 \cup \Gamma'_1} = f_2|_{\Gamma'_0 \cup \Gamma'_1} = 0. \tag{3}$$

2020 Mathematics Subject Classification: 47D06, 47D99
Keywords and phrases: Elliptic systems of second order; linear operators; sums of operators theory; little Hölder spaces.

Here, $h^{2\alpha}(\Omega_0)$ denotes the little Hölder space defined by

$$h^{2\alpha}(\Omega_0) := \left\{ \varphi \in C^{\alpha}(\Omega_0) \, ; \, \lim_{\delta \to 0} \sup_{0 < \|(x-x', y-y')\| \leq \delta} \frac{|\varphi(x,y) - \varphi(x',y')|}{\|(x-x', y-y')\|^{2\alpha}} = 0 \right\},$$

endowed with the norm

$$\|\varphi\| := \sup_{(x,y) \in \Omega_0} |\varphi(x,y)| + \sup_{0 < \|(r-r', \theta-\theta')\| \leq \delta} \frac{|\varphi(x,y) - \varphi(x',y')|}{\|(x-x', y-y')\|^{2\alpha}}, \quad \varphi \in h^{2\alpha}(\Omega_0).$$

Conducting our study in Hölder spaces is motivated by the fact that this framework allows us to use multipliers type theorems.

The theory of the Dirichlet problem for an elliptic system in domains with non smooth boundary is deeply developed; see e.g. the monographs [7, 8, 12] and the references therein. The solvability of such problems has been investigated with the help of several different techniques. For example, in the L^p setting for this kind of problems, the variational method and results from potential theory are undoubtedly the ones which are most commonly used.

At this level, we note that the study of system (1) can be viewed as a particular case of the following system:

$$\begin{cases} \mu \Delta u_1 + L_1 \left(x, y, \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} \right) u_1 = f_1 - \nabla p, & \text{in } \Omega_0, \\ \mu \Delta u_2 + L_2 \left(x, y, \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} \right) u_2 = f_2 - \nabla p, & \text{in } \Omega_0, \\ \dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial y} = 0, & \text{in } \Omega_0. \end{cases} \tag{4}$$

Such a system appears in the theory of incompressible Newtonian fluids where L_1 and L_2 are two linear differential operators of the first order with smooth coefficients on Ω_0, $u = (u_1, u_2)$ is the velocity of the fluid, ∇p is the gradient of the pressure, and $f = (f_1, f_2)$ is an applied body force. The solvability of (4) will be the subject of a forthcoming work.

As already mentioned before, we consider the solvability of system (1)-(2) from an abstract point of view. We are going to follow and adapt to the present problem the ideas for the functional setting considered in [3–5]. The techniques used here are essentially based on the theory of the sums of linear operators in Banach spaces developed in [6]. The choice of such techniques is justified by the fact that they ensure the existence and uniqueness of a solution for our problem and allow us to obtain the maximal regularity properties of the resulting solution. Our main result is formulated as follows.

THEOREM 1.1. *Let* $(f_1, f_2) \in h^{2\alpha}(\Omega_0) \times h^{2\alpha}(\Omega_0)$, $0 < 2\alpha < 1$, *and let the condition* (3) *be satisfied. Then, under conditions* (2), *Problem* (1) *has a unique solution* $(u_1, u_2) \in C^2(\Omega_0)$, *such that*

$$\left(\frac{\partial^2 u_1}{\partial x^2}, \frac{\partial^2 u_2}{\partial x^2} \right) \in h^{2\alpha}(\Omega_0) \times h^{2\alpha}(\Omega_0), \quad \textit{and} \quad \left(\frac{\partial^2 u_1}{\partial y^2}, \frac{\partial^2 u_2}{\partial y^2} \right) \in h^{2\alpha}(\Omega_0) \times h^{2\alpha}(\Omega_0).$$

2. Change of variables

First, using the classical polar coordinates

$$\begin{aligned} \Pi : \Omega_0 &\to \Omega, \\ (x,y) &\mapsto (r\cos\theta, r\sin\theta), \end{aligned}$$

the system (1) set on Ω is given by

$$\begin{cases} \dfrac{\partial^2 u_1}{\partial r^2}(r,\theta) - \dfrac{1}{r}\dfrac{\partial u_1}{\partial r}(r,\theta) + \dfrac{1}{r^2}\dfrac{\partial^2 u_1}{\partial \theta^2}(r,\theta) - \dfrac{\partial u_1}{\partial \theta}(r,\theta) = f_1(r,\theta), \\ \\ \dfrac{\partial^2 u_2}{\partial r^2}(r,\theta) - \dfrac{1}{r}\dfrac{\partial u_2}{\partial r}(r,\theta) + \dfrac{1}{r^2}\dfrac{\partial^2 u_2}{\partial \theta^2}(r,\theta) + \dfrac{\partial u_2}{\partial \theta}(r,\theta) = f_2(r,\theta). \end{cases} \tag{5}$$

We will first focus ourselves on the study of (5). Multiplying (5) by r^2, we obtain

$$\begin{cases} \left[r\dfrac{\partial u_1}{\partial r}\right]^2(r,\theta) - r\dfrac{\partial u_1}{\partial r}(r,\theta) + \dfrac{\partial^2 u_1}{\partial \theta^2}(r,\theta) - r^2\dfrac{\partial u_1}{\partial \theta} = r^2 f_1(r,\theta), \\ \left[r\dfrac{\partial u_2}{\partial r}\right]^2(r,\theta) - r\dfrac{\partial u_2}{\partial r}(r,\theta) + \dfrac{\partial^2 u_2}{\partial \theta^2}(r,\theta) + r^2\dfrac{\partial u_2}{\partial \theta}(r,\theta) = r^2 f_1(r,\theta). \end{cases} \tag{6}$$

Using the natural change of variable $r = e^t$ and $r\partial_r = \partial_t$, the sector Ω_0 is transformed into the infinite strip $\Omega = \{(t,\theta) : t \in \mathbb{R},\, 0 < \theta < \omega\}$, while Γ_0' and Γ_1' are transformed into $\Gamma_0 = \{(t,0) : r \in \mathbb{R}\}$ and $\Gamma_1 = \{(t,\omega) : r \in \mathbb{R}\}$.

Let us introduce the following change of functions

$$\begin{cases} v_i(t,\theta) = e^{-(2+\alpha)t} u_i(e^t,\theta), & i = 1,2, \\ g_i(t,\theta) = e^{-\alpha t} f_i(e^t,\theta), & i = 1,2, \end{cases} \tag{7}$$

and set $m = 2+\alpha$. Then, (6) is written in Ω as follows

$$\begin{cases} \dfrac{\partial^2 v_1}{\partial \theta^2}(t,\theta) - \dfrac{\partial v_1}{\partial \theta}(t,\theta) + \dfrac{\partial^2 v_1}{\partial t^2}(t,\theta) + 2m\dfrac{\partial v_1}{\partial t}(t,\theta) + m^2 v_1(t,\theta) = g_1(t,\theta), \\ \dfrac{\partial^2 v_2}{\partial \theta^2}(t,\theta) + \dfrac{\partial v_1}{\partial \theta}(t,\theta) + \dfrac{\partial^2 v_2}{\partial t^2}(t,\theta) + 2m\dfrac{\partial v_2}{\partial t}(t,\theta) + m^2 v_2(t,\theta) = g_2(t,\theta). \end{cases} \tag{8}$$

Observe here that another compact form of (8) is given by

$$\begin{cases} \dfrac{\partial^2 v_1}{\partial \theta^2}(t,\theta) - \dfrac{\partial v_1}{\partial \theta}(t,\theta) + \left(\dfrac{\partial}{\partial t} + m\right)^2 v_1(t,\theta) = g_1(t,\theta), \\ \dfrac{\partial^2 v_2}{\partial \theta^2}(t,\theta) + \dfrac{\partial v_1}{\partial \theta}(t,\theta) + \left(\dfrac{\partial}{\partial t} + m\right)^2 v_1(t,\theta) = g_2(t,\theta), \end{cases} \tag{9}$$

subject to new boundary conditions given by

$$\begin{cases} v_i(\cdot,0) = 0, & i = 1,2, \\ v_i(\cdot,\omega) = 0. & i = 1,2, \\ v_i(0,\cdot) = 0. & i = 1,2, \end{cases} \tag{10}$$

and conditions (3) become

$$\begin{cases} g_i(\cdot,0)=0, & i=1,2,\\ g_i(\cdot,\omega)=0, & i=1,2,\\ g_i(0,\cdot)=0, & i=1,2. \end{cases} \tag{11}$$

As a consequence of the change of variables (7), one has the following lemma, the proof of which can be found in [5, Proposition 4].

LEMMA 2.1. *Let* $\alpha \in (0,1/2)$. *Then,* $(f_1,f_2) \in h^{2\alpha}(\Omega_0) \times h^{2\alpha}(\Omega_0) \Leftrightarrow (g_1,g_2) \in h^{2\alpha}(\Omega) \times h^{2\alpha}(\Omega)$.

REMARK 2.2. Taking into account the anisotropic properties of the little Hölder space $h^{2\alpha}(\Omega), \alpha \in (0,1/2)$, as in [5] and [11], our transformed problem (9)-(10) will be studied in two situations:

(a) case when $(g_1,g_2) \in X_1 \times X_1$ with $X_1 = C([0,\omega]\,;h_0^{2\sigma}(\mathbb{R}))$;

(b) case when $(g_1,g_2) \in X_2 \times X_2$ with $X_2 = h^{2\sigma}([0,\omega]\,;C_0(\mathbb{R}))$.

Here,

$$C_0(\mathbb{R}) := \left\{\varphi \in C(\mathbb{R}) : \lim_{t\to\pm\infty} \varphi(t) = 0\right\} \text{ and } h_0^{2\sigma}(\mathbb{R}) := \left\{\varphi \in h^{2\sigma}(\mathbb{R}) : \lim_{t\to\pm\infty} \varphi(t) = 0\right\}.$$

3. On the sums of operators

In this section, we recall the essential ingredients of this technique. As in [9, Section 2], let E be a complex Banach space and M, N two closed linear operators with domains $D(M)$, $D(N)$. Let S be the operator defined by

$$\begin{cases} Su = Mu + Nu,\\ u \in D(S) = D(N)\cap D(M), \end{cases} \tag{12}$$

where N and M satisfy the assumptions

$$(H.1)\ \begin{cases} i) & \rho(N) \supset \sum_N = \{z : |z| \geq r,\ |\mathrm{Arg}(z)| < \pi - \epsilon_N\},\\ & \forall z \in \sum_N \quad \left\|(N-zI)^{-1}\right\|_{L(E)} = O\left(\dfrac{1}{|z|}\right);\\ ii) & \rho(M) \supset \sum_M = \{z : |z| \geq r,\ |\mathrm{Arg}(z)| < \pi - \epsilon_M\},\\ & \forall z \in \sum_M \quad \left\|(M-zI)^{-1}\right\|_{L(E)} = O\left(\dfrac{1}{|z|}\right);\\ iii) & \epsilon_N + \epsilon_M < \pi;\\ iv) & \overline{D(N)+D(M)} = E; \end{cases}$$

and

$$(H.2)\begin{cases} & \forall z\in\rho(N),\forall z'\in\rho(M)\\ & (N-zI)^{-1}(M-z'I)^{-1}-(M-z'I)^{-1}(N-zI)^{-1}\\ = & \left[(N-zI)^{-1};\ (M-z'I)^{-1}\right]=0,\end{cases}$$

where $\rho(N)$ and $\rho(M)$ denote the resolvent sets of N and M.

Before stating the main result proved in [6], we recall the following definition of interpolation spaces:

Definition 3.1. For any $\varrho\in(0,1)$, we introduce two families of real Banach interpolation spaces between $D(N)$ and E:

$$D_N(\varrho,+\infty):=\left\{\zeta\in E:\sup_{r>0}\left\|r^{\varrho}N(N-rI)^{-1}\xi\right\|_E<\infty\right\},$$

and its subspace

$$D_N(\rho):=\left\{\xi\in X:\lim_{r\to0^+}\left\|r^{\rho}N(N-rI)^{-1}\xi\right\|_E=0\right\}.$$

For more details, see [10] and [13]. Now, we give the main result furnished by the sums of operators technique as it is formulated in [9].

Theorem 3.2. *Let $\varrho\in(0,1)$. Assume $(H.1)$, $(H.2)$ and $f\in D_N(\varrho)$. Then, the problem $Nu+Mu=f$, has a unique solution $u\in D(N)\cap D(M)$, given by*

$$u=-\frac{1}{2i\pi}\int_\gamma(M+z)^{-1}(N-z)^{-1}f\,dz,$$

where γ is a sectorial curve lying in $(\sum_N)\cap(\sum_{-M})$ oriented from $\infty e^{+i\theta_0}$ to $\infty e^{-i\theta_0}$ with $\epsilon_M<\theta_0<\pi-\epsilon_N$. Moreover, one has $Nu,\ Mu\in D_N(\varrho)$.

4. Optimal results for the first case

In order to state the abstract version of (9)-(10), we choose first θ as a principal variable. Let $E=h_0^{2\sigma}(\mathbb{R})$, $\alpha\in(0,1/2)$. First, we must state the abstract version of our concrete Problem (9)-(10). We need to introduce the following vectorial functions:

$$v_i:[0,\omega]\to E;\theta\longrightarrow v_i(\theta);\qquad v_i(\theta)(t)=v_i(\theta,t), i=1,2,$$

$$g_i:[0,\omega]\to E;\theta\longrightarrow g_i(\theta);\qquad g_i(\theta)(t)=g_i(\theta,t), i=1,2,$$

and consider the following linear operators A_1,A_2 and B defined by

$$\begin{cases}D(A_1):=\{\psi\in W^{2,p}((0,\omega);E)\cap C^2([0,\omega];E):u(0)=u(\omega)=0\},\\(A_1\psi)(\theta):=\psi''(\theta)+\psi'(\theta),\quad\theta\in(0,\omega),\end{cases}\tag{13}$$

and

$$\begin{cases}D(A_2):=\{\psi\in W^{2,p}((0,\omega);E)\cap C^2([0,\omega];E):u(0)=u(\omega)=0\},\\(A_2\psi)(\theta):=\psi''(\theta)-\psi'(\theta),\quad\theta\in(0,\omega),\end{cases}\tag{14}$$

$$\begin{cases} D(B) := \{\psi \in h^{2\alpha}\left([0,\omega]\,;h^{2\alpha}(\mathbb{R})\right) : \psi(\theta) \in D(C), \quad \theta \in [0,\omega]\}, \\ (B\psi)\,(\theta) := C(\psi(\theta)), \quad \theta \in (0,\omega)\,, \end{cases} \tag{15}$$

where
$$\begin{cases} D(C) := \{\phi \in W^{2,p}\,(\mathbb{R}) \cap C^2(\mathbb{R}) : \phi(0) = 0\}, \\ (C\phi)\,(t) := \varphi''\,(t) + 2m\varphi'\,(t) + m^2\varphi\,(t)\,, \end{cases} \tag{16}$$

Then, we conclude that a new version of (9) is given by

$$\mathcal{A}V\,(\theta) + \mathcal{B}V\,(\theta) = \mathcal{G}\,(\theta)\,, \tag{17}$$

where
$$\begin{cases} \mathcal{A} := \begin{bmatrix} A_1 & 0 \\ & \\ 0 & A_2 \end{bmatrix}, \\ \\ D\,(\mathcal{A}) := D\,(A_1) \times D\,(A_2)\,, \end{cases} \tag{18}$$

and
$$\begin{cases} \mathcal{B} := \begin{bmatrix} B & 0 \\ & \\ 0 & B \end{bmatrix} \\ \\ D\,(\mathcal{B}) := D\,(B) \times D\,(B)\,, \end{cases} \tag{19}$$

while
$$V\,(\theta) := \begin{bmatrix} v_1\,(\theta) \\ \\ v_2\,(\theta) \end{bmatrix}, \qquad \mathcal{G}\,(\theta) := \begin{bmatrix} g_1\,(\theta) \\ \\ g_2\,(\theta) \end{bmatrix}$$

are two vector-valued functions defined on $E \times E$ endowed with its usual norm. Now, let us state some useful results concerning the operator A_1 defined by (13).

Lemma 4.1. *There exists $\epsilon_{A_1} > 0$ such that*

$$\rho(A_1) \supset \sum_{A_1} \{\lambda : |\lambda| \geq r,\ |\mathrm{Arg}(\lambda)| < \pi - \epsilon_{A_1}\}\,,$$

and
$$\forall \lambda \in \sum_{A_1} : \quad \left\| (A_1 - \lambda I)^{-1} \right\|_{L(E)} = O\left(\frac{1}{|\lambda|}\right). \tag{20}$$

Proof. Recall that

$$\begin{cases} D(A_1) = \{\psi \in W^{2,p}((0,\omega)\,;E) \cap C^2([0,\omega];E) : u(0) = u(\omega) = 0\}, \\ (A_1\psi)(\theta) = \psi''(\theta) + \psi'(\theta), \quad \theta \in (0,\omega)\,. \end{cases}$$

First of all, we will solve explicitly the equation

$$\begin{cases} v''\,(\theta) + v'(\theta) - \lambda v\,(\theta) = \phi\,(\theta)\,, \\ v\,(0) = v\,(\omega) = 0. \end{cases} \tag{21}$$

For $\lambda > 0$ the unique solution v is given by

$$v\,(\theta) = (A - \lambda)^{-1}\,\phi\,(\theta) = \int_0^1 K_{\sqrt{\triangle(\lambda)}}(\theta, s)\varphi(s)\,ds,$$

where

$$K_{\sqrt{\triangle(\lambda)}}(\theta,s)=\begin{cases}\frac{\sinh\sqrt{\triangle(\lambda)}(\omega-\theta)\ \sinh\sqrt{\triangle(\lambda)}s}{\sqrt{\triangle(\lambda)}\sinh\omega\sqrt{\triangle(\lambda)}}, & \text{if } 0\le s\le\theta,\\ \frac{\sinh\sqrt{\triangle(\lambda)}(\omega-s)\ \sinh\sqrt{\triangle(\lambda)}\theta}{\sqrt{\triangle(\lambda)}\sinh\omega\sqrt{\triangle(\lambda)}}, & \text{if } \theta\le s\le\omega,\end{cases}$$

with $\triangle(\lambda)=1+4\lambda$. Concerning the estimate (20), the proof is purely technical and we will only outline the main details. Observe that for $0<\theta'<\theta<\omega$, one has

$$(A-\lambda)^{-1}\phi(\theta)-(A-\lambda)^{-1}\phi(\theta')=I_1+I_2+I_3,$$

where

$$I_1=\int_0^{\theta'}\frac{\sinh\sqrt{\triangle(\lambda)}(\omega-\theta)\ \sinh\sqrt{\triangle(\lambda)}s}{\sqrt{\triangle(\lambda)}\sinh\omega\sqrt{\triangle(\lambda)}}\varphi(s)\,ds-\int_0^{\theta'}\frac{\sinh\sqrt{\triangle(\lambda)}\left(\omega-\theta'\right)\sinh\sqrt{\triangle(\lambda)}s}{\sqrt{\triangle(\lambda)}\sinh\omega\sqrt{\triangle(\lambda)}}\varphi(s)\,ds,$$

$$I_2=\int_{\theta'}^{\theta}\frac{\sinh\sqrt{\triangle(\lambda)}(\omega-\theta)\ \sinh\sqrt{\triangle(\lambda)}s}{\sqrt{\triangle(\lambda)}\sinh\omega\sqrt{\triangle(\lambda)}}\varphi(s)\,ds-\int_{\theta'}^{\theta}\frac{\sinh\sqrt{\triangle(\lambda)}(\omega-s)\ \sinh\sqrt{\triangle(\lambda)}\theta}{\sqrt{\triangle(\lambda)}\sinh\omega\sqrt{\triangle(\lambda)}}\varphi(s)\,ds,$$

$$I_3=\int_{\theta}^{1}\frac{\sinh\sqrt{\triangle(\lambda)}(\omega-s)\sinh\sqrt{\triangle(\lambda)}\theta}{\sqrt{\triangle(\lambda)}\sinh\omega\sqrt{\triangle(\lambda)}}\varphi(s)\,ds-\int_{\theta}^{1}\frac{\sinh\sqrt{\triangle(\lambda)}(\omega-s)\ \sinh\sqrt{\triangle(\lambda)}\theta'}{\sqrt{\triangle(\lambda)}\sinh\omega\sqrt{\triangle(\lambda)}}\varphi(s)\,ds.$$

Using the methods established in [1] and [2], we obtain $\|I_j\|_{L(E)}=O\left(\frac{1}{|\lambda|}\right)$, $j=1,2,3$, which implies that $\left\|(A_1-\lambda I)^{-1}\right\|_{L(E)}=O\left(\frac{1}{|\lambda|}\right)$.

Now, using a classical argument of analytic continuation of the resolvent, we deduce that there exists an $\epsilon_{A_1}\in(0,\pi)$ such that the previous estimate holds true in the sector $\sum_A=\{\lambda:|\lambda|\ge r,|\operatorname{Arg}(\lambda)|<\pi-\epsilon_A\}$. □

By the same argument, we obtain a similar result for the operator A_2 defined by (14).

LEMMA 4.2. *There exists $\epsilon_{A_2}>0$ such that*

$$\rho(A_2)\supset\sum_{A_2}=\{\lambda:|\lambda|\ge r,\ |\operatorname{Arg}(\lambda)|<\pi-\epsilon_{A_2}\},\tag{22}$$

and

$$\forall\lambda\in\sum_{A_2}:\quad\left\|(A_2-\lambda I)^{-1}\right\|_{L(E)}=O\left(\frac{1}{|\lambda|}\right).\tag{23}$$

Concerning the operator B, we have the following lemma.

LEMMA 4.3. *There exists $\epsilon_B>0$ such that*

$$\rho(B)\supset\sum_{B}=\{\lambda:|\lambda|\ge r,\ |\operatorname{Arg}(\lambda)|<\pi-\epsilon_B\},$$

and

$$\forall\lambda\in\sum_{B}:\quad\left\|(B-\lambda I)^{-1}\right\|_{L(E)}=O\left(\frac{1}{|\lambda|}\right).\tag{24}$$

Proof. It is necessary to note that B has the same properties as its realization C. We have $C=P(K)$, where P is the polynomial $P(z)=z^2+2mz+m^2=(z+m)^2$ and

$$\begin{cases}D(K):=\{\psi\in W^{1,p}(\mathbb{R})\cap C^1(\mathbb{R}):u(0)=0\},\\(K\psi)(t):=\psi'(t),\quad t\in\mathbb{R}.\end{cases}\tag{25}$$

On the other hand, observe that the equation $P(z) = \lambda$, has two complex roots $z_\pm(\lambda) = -m \pm \sqrt{\lambda}$, which implies that $(C - \lambda I)^{-1} = (K - z_+(\lambda))^{-1}(K - z_-(\lambda))^{-1}$. However, we know that $\sigma(K) = i\mathbb{R}$ and, for every $\varphi \in E$,

$$\left[(K + zI)^{-1}\varphi\right](t,\sigma) = \begin{cases} -\int_t^{+\infty} e^{z(s-t)}\varphi(s,\sigma)\, ds \text{ if } \Re z < 0, \\ \int_{-\infty}^t e^{z(t-s)}\varphi(s,\sigma)\, ds \text{ if } \Re z > 0, \end{cases} \tag{26}$$

from which we easily obtain the estimate $\forall z \notin i\mathbb{R} : \left\|(K + zI)^{-1}\right\|_{L(E)} = O\left(\frac{1}{|\Re z|}\right)$. □

LEMMA 4.4. *For all $z \in \rho(A), \forall z' \in \rho(B)$, one has*

$$(A - zI)^{-1}(B - z'I)^{-1} - (B - z'I)^{-1}(A - zI)^{-1} = 0.$$

Proof. We just check the commutativity between the operator A_1 and the operator K defined by (25). Let $z \in \rho(A)$ and $z' \in \rho(B)$. Then we have

$$\left[(K - zI)^{-1}(A_1 - z'I)^{-1}\varphi\right](\theta) = \left\{-\int_t^{+\infty} e^{z(s-t)}\left((A_1 - z'I)^{-1}\varphi\right)(s)\, ds\right\}$$

$$= \left\{-\int_t^{+\infty} e^{z(s-t)}\int_0^1 K_{\sqrt{\triangle(z')}}(\theta,\tau)\,[\varphi(s)](\tau)\, d\tau\, ds\right\}$$

$$= \int_0^1 K_{\sqrt{\triangle(z')}}(\theta,\tau)\left(\left\{(K - zI)^{-1}\varphi\right\}(\theta)\right)(\tau)\, d\tau$$

$$= \left\{(A_1 - z'I)^{-1}\left((K - zI)^{-1}\varphi\right)(t)\right\}(\theta).$$

Then, we conclude that $(A - zI)^{-1}(B - z'I)^{-1} = (B - z'I)^{-1}(A - zI)^{-1}$. □

REMARK 4.5. Keeping in mind the conditions (3) and [10, Theorem 3.1.12], we can show that these spaces are well defined and have the following characterization: $D_{A_1}(\alpha) = D_{A_2}(\alpha) = h^{2\alpha}([0,\omega]; E)$, $\alpha \in (0, 1/2)$, and $D_B(\alpha) = h_0^{2\alpha}(\mathbb{R})$, $\alpha \in (0, 1/2)$.

By collecting all the previous results, we are able to state some useful spectral properties of the matrix differential operators $\mathcal{A}$ and $\mathcal{B}$.

LEMMA 4.6. *The closed linear operators $\mathcal{A}$ and $\mathcal{B}$ defined respectively by* (18) *and* (19) *satisfy the following properties:*

1) $\exists \epsilon_{\mathcal{A}} > 0 : \rho(\mathcal{A}) \supset \sum_{\mathcal{A}} = \{\lambda : |\lambda| \geq r,\ |\text{Arg}(\lambda)| < \pi - \epsilon_{\mathcal{A}}\}$ *and* $\forall \lambda \in \sum_{\mathcal{A}}$, *one has*

$$\left\|(\mathcal{A} - \lambda I)^{-1}\right\|_{L(E \times E)} = O\left(\frac{1}{|\lambda|}\right).$$

2) $\exists \epsilon_{\mathcal{B}} > 0 : \rho(\mathcal{A}) \supset \sum_{\mathcal{A}} = \{\lambda : |\lambda| \geq r,\ |\text{Arg}(\lambda)| < \pi - \epsilon_{\mathcal{B}}\}$ *and* $\forall \lambda \in \sum_{B}$, *one has*

$$\left\|(\mathcal{B} - \lambda I)^{-1}\right\|_{L(E \times E)} = O\left(\frac{1}{|\lambda|}\right).$$

From Theorem 3.2, we have also the following result.

Theorem 4.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be the two operators defined respectively by* (18) *and* (19)*. Then, the problem* (17) *has a unique solution $V \in D(\mathcal{A}) \cap D(\mathcal{B})$, given by*

$$V = -\frac{1}{2i\pi} \int_{\gamma} (\mathcal{A} + \mu)^{-1} (\mathcal{B} - \mu)^{-1} \mathcal{G} \, dz,$$

where γ is a sectorial curve lying in $\left(\sum_{\mathcal{A}}\right) \cap \left(\sum_{-\mathcal{B}}\right)$ oriented from $\infty e^{+i\theta_0}$ to $\infty e^{-i\theta_0}$ with $\epsilon_{\mathcal{A}} < \theta_0 < \pi - \epsilon_{\mathcal{B}}$.

Moreover, one has $\mathcal{A}V, \ \mathcal{B}V \in D_{\mathcal{A}}(\alpha) := D_{\mathcal{A}}(\alpha) \times D_{\mathcal{A}}(\alpha)$.

Applying this result to our concrete problem (9)-(10), we obtain the following

Theorem 4.8. *Let $(g_1, g_2) \in X_1 \times X_1$. Then, under conditions* (11)*, Problem* (9)-(10) *has a unique strict solution $(v_1, v_2) \in C^2\left([0,\omega]\,; h_0^{2\alpha}(\mathbb{R})\right) \times C^2\left([0,\omega]\,; h_0^{2\alpha}(\mathbb{R})\right)$, such that $\left(\frac{\partial^2 v_1}{\partial \theta^2}, \frac{\partial^2 v_2}{\partial \theta^2}\right) \in X_1 \times X_1$ and $\left(\frac{\partial^2 v_1}{\partial t^2}, \frac{\partial^2 v_2}{\partial t^2}\right) \in X_1 \times X_1$.*

5. Optimal results for the second case

In the second case, the abstract formulation of problem (9)-(10) is handled by choosing $E = C_0(\mathbb{R})$. Arguing in the same way as in the previous section, the application of the sum of operators theory allows us to formulate the following result:

Theorem 5.1. *Let $(g_1, g_2) \in X_2 \times X_2$. Then, under conditions* (11)*, Problem* (9)-(10) *has a unique strict solution $(v_1, v_2) \in C^2\left([0,\omega]\,; C_0(\mathbb{R})\right) \times C^2\left([0,\omega]\,; C_0(\mathbb{R})\right)$, such that $\left(\frac{\partial^2 v_1}{\partial \theta^2}, \frac{\partial^2 v_2}{\partial \theta^2}\right) \in X_2 \times X_2$ and $\left(\frac{\partial^2 v_1}{\partial t^2}, \frac{\partial^2 v_2}{\partial t^2}\right) \in X_2 \times X_2$.*

6. Global optimal results for Problem (9)-(10)

Now, we are in a position to summarize all the results concerning our Problem (9)-(10) with $(g_1, g_2) \in h^{2\alpha}(\Omega) \times h^{2\alpha}(\Omega)$. Using the results obtained in the former sections, one obtains the following theorem justifying our main result:

Theorem 6.1. *Let $(g_1, g_2) \in h^{2\alpha}(\Omega) \times h^{2\alpha}(\Omega)$, $0 < 2\alpha < 1$ satisfying* (11)*. Then, under conditions* (2)*, Problem* (1) *has a unique strict solution $(u_1, u_2) \in C^2(\Omega)$, such that $\left(\frac{\partial^2 u_1}{\partial \theta^2}, \frac{\partial^2 u_2}{\partial \theta^2}\right) \in h^{2\alpha}(\Omega) \times h^{2\alpha}(\Omega)$ and $\left(\frac{\partial^2 u_1}{\partial r^2}, \frac{\partial^2 u_2}{\partial r^2}\right) \in h^{2\alpha}(\Omega) \times h^{2\alpha}(\Omega)$.*

Acknowledgement. The authors would like to express their sincere gratitude to the reviewers for their valuable suggestions.

The second author is partially supported by grant 451-03-68/2020/14/200156 of Ministry of Science and Technological Development, Republic of Serbia.

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(received 18.12.2019; in revised form 03.05.2020; available online 22.03.2021)

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