

## NEW CONGRUENCES MODULO SMALL POWERS OF 2 FOR OVERPARTITIONS INTO ODD PARTS

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**Abstract.** In this article, we establish several infinite families of Ramanujan-type congruences modulo 16, 32 and 64 for  $\bar{p}_o(n)$ , the number of overpartitions of  $n$  in which only odd parts are used.

### 1. Introduction

An overpartition of the nonnegative integer  $n$  is a partition of  $n$  where the first occurrence of parts of each size may be overlined. For example, there are 8 overpartitions of the integer 3:

$$3, \bar{3}, 2 + 1, \bar{2} + 1, 2 + \bar{1}, \bar{2} + \bar{1}, 1 + 1 + 1, \bar{1} + 1 + 1.$$

We denote the number of overpartitions of  $n$  by  $\bar{p}(n)$ . The generating function of  $\bar{p}(n)$  is given by

$$\sum_{n=0}^{\infty} \bar{p}(n)q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}},$$

where

$$(a; q)_n = \begin{cases} 1, & \text{for } n = 0; \\ \prod_{k=1}^n (1 - aq^{k-1}), & \text{for } n > 0 \end{cases}$$

is  $q$ -shifted factorial,  $(a; q)_{\infty} = \lim_{n \rightarrow \infty} (a; q)_n$ ,  $|q| < 1$  and let  $f_k := (q^k; q^k)_{\infty}$ . Many mathematicians have extensively studied overpartitions to obtain properties analogous to ordinary partitions, see, for example [4–6, 9].

In this context, we consider the number of overpartitions into odd parts. Let  $\bar{p}_o(n)$  denote the number of such partitions. It is evident that

$$\sum_{n=0}^{\infty} \bar{p}_o(n)q^n = \frac{(-q; q^2)_{\infty}}{(q; q^2)_{\infty}} = \frac{f_2^3}{f_1^2 f_4}. \quad (1)$$

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The generating function of  $\bar{p}_o(n)$  appears in the following series-product identity of Lebesgue [8],

$$\sum_{j=0}^{\infty} \frac{(-1; q)_j q^{j(j+1)/2}}{(q; q)_j} = \frac{(-q; q^2)_{\infty}}{(q; q^2)_{\infty}}.$$

We assert that the sequence  $\{\bar{p}_o(n)\}_{n \geq 0}$  is known and can be seen in [11, A080054]. Hirschhorn and Sellers [7] considered  $\bar{p}_o(n)$  in arithmetic point of view and obtained many congruences modulo 8 and 16 for  $\bar{p}_o(n)$ , for example they proved that  $\bar{p}_o(2^\alpha(8n+5)) \equiv 0 \pmod{8}$  and

$$\bar{p}_o(8n+7) \equiv 0 \pmod{16}, \quad (2)$$

for all nonnegative integers  $\alpha$  and  $n$ , while Chen [3] showed that

$$\sum_{n=0}^{\infty} \bar{p}_o(16n+14)q^n = 112 \frac{f_2^{27}}{f_1^{25} f_4^2} + 256q \frac{f_2^3 f_4^{14}}{f_1^{17}} \equiv 48 \frac{f_2^{11}}{f_1} \pmod{64}. \quad (3)$$

which implies that  $\bar{p}_o(16n+14) \equiv 0 \pmod{16}$ . Using elementary theory of modular forms, Chen [3] extended these congruences to modulo 32 and 64. In particular, Chen showed the following theorem.

**THEOREM 1.1.** *Let  $t$  be an integer,  $p \equiv 1 \pmod{8}$  be a prime. Then for all integers  $n$  with  $n \not\equiv -\frac{7}{8} \pmod{p}$ ,*

$$\begin{aligned} \bar{p}_o(p^{2t+1}(16n+14)) &\equiv 0 \pmod{32}, \\ \bar{p}_o(p^{4t+3}(16n+14)) &\equiv 0 \pmod{64}. \end{aligned}$$

*Suppose that  $p_1, p_2 \equiv 1 \pmod{8}$  are two distinct primes. Then for all nonnegative integers  $n$  satisfying  $n \not\equiv -\frac{7}{8} \pmod{p_1}$  and  $n \not\equiv -\frac{7}{8} \pmod{p_2}$ ,  $\bar{p}_o(p_1 p_2 (16n+14)) \equiv 0 \pmod{64}$ .*

Recently, C. Ray and R. Barman [10] obtained identities for  $\bar{p}_o(n)$  and as a consequence derived many congruences modulo 8 and 16 for  $\bar{p}_o(n)$ .

With this motivation, we prove several infinite families of congruences modulo 16, 32 and 64 for the partition function  $\bar{p}_o(n)$ .

## 2. Congruences modulo small powers of 2 for $\bar{p}_o(n)$

Jacobi's triple product identity can be stated in terms of the Ramanujan's theta function [1, p. 34] as follows:

$$(-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty} = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \quad (4)$$

The following lemma plays a vital role in proving our main results.

**LEMMA 2.1.** *The following 2-dissections hold:*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8},$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \quad (5)$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q \frac{f_2^2 f_8^4}{f_4^2} \quad (6)$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}. \quad (7)$$

*Proof.* Lemma 2.1 is an immediate consequence of dissection formulas of Ramanujan, collected in Berndt's book [1, Entry 25, p. 40].  $\square$

The main results are the following.

**THEOREM 2.2.** For all integers  $n \geq 0$  and  $\alpha \in \{2, 3, 4, 5\}$ ,

$$\bar{p}_o(2^{\alpha+1}n) \equiv \bar{p}_o(2^\alpha n) \pmod{2^{\alpha+2}}. \quad (8)$$

**THEOREM 2.3.** If  $n$  cannot be represented as a sum of a triangular number and twice a triangular number, then for any nonnegative integer  $\alpha$ ,

$$\bar{p}_o(2^\alpha(8n+3)) \equiv 0 \pmod{32}. \quad (9)$$

**COROLLARY 2.4.** For any positive integer  $k$ , let  $p_j \geq 3$ ,  $1 \leq j \leq k$  be primes. If  $(-2/p_j) = -1$  for every  $j$ , then for all nonnegative integers  $\alpha$  and  $n$  with  $p_k \nmid n$ ,

$$\bar{p}_o(2^\alpha(8p_1^2 p_2^2 \cdots p_{k-1}^2 p_k n + 3p_1^2 p_2^2 \cdots p_{k-1}^2 p_k^2)) \equiv 0 \pmod{32}. \quad (10)$$

**THEOREM 2.5.** For all integers  $\alpha \geq 0$  and  $n \geq 0$ ,

$$\bar{p}_o(2^\alpha(8n+7)) \equiv 0 \pmod{16}. \quad (11)$$

**THEOREM 2.6.** Let  $t \geq 0$  be an integer and  $p \equiv 1 \pmod{8}$  be a prime. Then for all nonnegative integers  $\alpha$  and  $n$  with  $8n \not\equiv -7 \pmod{p}$ ,

$$\bar{p}_o(2^\alpha p^{2t+1}(8n+7)) \equiv 0 \pmod{32},$$

$$\bar{p}_o(2^\alpha p^{4t+3}(8n+7)) \equiv 0 \pmod{64}$$

and

$$\bar{p}_o(2^\alpha p_1 p_2 (8m+7)) \equiv 0 \pmod{64}$$

where  $p_1, p_2 \equiv 1 \pmod{8}$  are two distinct primes and  $m$  is any integer satisfying  $8m \not\equiv -7 \pmod{p_1}$  and  $8m \not\equiv -7 \pmod{p_2}$ .

Note that Theorem 1.1 is the special case of Theorem 2.6 for  $\alpha = 2$ .

**THEOREM 2.7.** If  $n$  cannot be represented as a sum of a triangular number and four times a triangular number, then for any nonnegative integer  $\alpha$ ,

$$\bar{p}_o(2^\alpha(8n+5)) \equiv 0 \pmod{32}. \quad (12)$$

**COROLLARY 2.8.** For any positive integer  $k$ , let  $p_j \geq 3$ ,  $1 \leq j \leq k$  be primes. If  $(-4/p_j) = -1$  for every  $j$ , then for all nonnegative integers  $\alpha$  and  $n$  with  $p_k \nmid n$ ,

$$\bar{p}_o(2^\alpha(8p_1^2 p_2^2 \cdots p_{k-1}^2 p_k n + 5p_1^2 p_2^2 \cdots p_{k-1}^2 p_k^2)) \equiv 0 \pmod{32}.$$

By the binomial theorem, it is easy to see that for all positive integers  $k$  and  $m$ ,

$$f_m^{2^k} \equiv f_{2^m}^{2^{k-1}} \pmod{2^k}. \quad (13)$$

*Proof* (Proof of Theorem 2.2). Consider the generating function (1),

$$\sum_{n=0}^{\infty} \bar{p}_o(n)q^n = \frac{f_2^3}{f_1^2 f_4}. \quad (14)$$

Substituting (5) in (14) and extracting the terms involving even and odd powers of  $q$ , we find that

$$\sum_{n=0}^{\infty} \bar{p}_o(2n)q^n = \frac{f_4^5}{f_1^2 f_2 f_8^2} \quad (15)$$

and

$$\sum_{n=0}^{\infty} \bar{p}_o(2n+1)q^n = 2 \frac{f_2 f_8^2}{f_1^2 f_4}. \quad (16)$$

Again, employing (5) in (15),

$$\sum_{n=0}^{\infty} \bar{p}_o(2n)q^n = \frac{f_4^5 f_8^3}{f_6^2 f_{16}^2} + 2q \frac{f_4^7 f_{16}^2}{f_2^6 f_8^3},$$

which yields

$$\sum_{n=0}^{\infty} \bar{p}_o(4n)q^n = \frac{f_2^5 f_4^3}{f_1^6 f_8^2} \quad (17)$$

and

$$\sum_{n=0}^{\infty} \bar{p}_o(4n+2)q^n = 2 \frac{f_2^7 f_8^2}{f_1^6 f_4^3}. \quad (18)$$

Employing (5) in (17) and extracting the terms involving even powers of  $q$ ,

$$\sum_{n=0}^{\infty} \bar{p}_o(8n)q^n = \frac{f_2^3 f_4^{13}}{f_1^{10} f_8^6} + 12q \frac{f_2^7 f_4 f_8^2}{f_1^{10}}. \quad (19)$$

By (6),

$$\frac{f_2^5 f_4^3}{f_1^6 f_8^2} = \frac{f_2^3 f_4^{13}}{f_1^{10} f_8^6} - 4q \frac{f_2^7 f_4 f_8^2}{f_1^{10}}. \quad (20)$$

In view of (19) and (20),

$$\sum_{n=0}^{\infty} \bar{p}_o(8n)q^n = \frac{f_2^5 f_4^3}{f_1^6 f_8^2} + 16q \frac{f_2^7 f_4 f_8^2}{f_1^{10}}. \quad (21)$$

From (17) and (21),

$$\sum_{n=0}^{\infty} \bar{p}_o(8n)q^n = \sum_{n=0}^{\infty} \bar{p}_o(4n)q^n + 16q \frac{f_2^7 f_4 f_8^2}{f_1^{10}}.$$

Let

$$\sum_{n=0}^{\infty} a(n)q^n = q \frac{f_2^7 f_4 f_8^2}{f_1^{10}} \equiv q \frac{f_2^3 f_4 f_8^2}{f_1^2} \pmod{8}.$$

Substituting (5) in the previous line and extracting the terms involving even powers of  $q$ ,

$$\sum_{n=0}^{\infty} a(2n)q^n \equiv 2q \frac{f_2^3 f_4 f_8^2}{f_1^2} \pmod{8}.$$

Last three identities yield,

$$\sum_{n=0}^{\infty} \bar{p}_o(2^{\alpha+1}n)q^n \equiv \sum_{n=0}^{\infty} \bar{p}_o(2^\alpha n)q^n + 2^{\alpha+2} q \frac{f_2^3 f_4 f_8^2}{f_1^2} \pmod{128} \quad (22)$$

for all integers  $\alpha \geq 2$ . Congruence (8) follows from (22).  $\square$

*Proof* (Proof of Theorem 2.3). Using (5), we can rewrite (16) as

$$\sum_{n=0}^{\infty} \bar{p}_o(2n+1)q^n = 2 \frac{f_8^7}{f_2^4 f_4 f_{16}^2} + 4q \frac{f_4 f_8 f_{16}^2}{f_2^4} \quad (23)$$

which yields 
$$\sum_{n=0}^{\infty} \bar{p}_o(4n+1)q^n = 2 \frac{f_4^7}{f_1^4 f_2 f_8^2} \quad (24)$$

and 
$$\sum_{n=0}^{\infty} \bar{p}_o(4n+3)q^n = 4 \frac{f_2 f_4 f_8^2}{f_1^4}. \quad (25)$$

Invoking (7) in (25) and extracting the terms involving odd and even powers of  $q$ ,

$$\sum_{n=0}^{\infty} \bar{p}_o(8n+7)q^n = 16 \frac{f_2^3 f_4^6}{f_1^9} \equiv 16 \frac{f_2^{11}}{f_1} \pmod{64}. \quad (26)$$

and 
$$\sum_{n=0}^{\infty} \bar{p}_o(8n+3)q^n = 4 \frac{f_2^{15}}{f_1^{13} f_4^2} = 4 \frac{f_1^3 f_4^2}{f_2} \frac{f_2^{16}}{f_1^{16} f_4^4} \equiv 4 \frac{f_1^3 f_4^2}{f_2} \pmod{32}. \quad (27)$$

From [2, Theorem 1.3.9, p.14], (4) and (27),

$$\sum_{n=0}^{\infty} \bar{p}_o(8n+3)q^n = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^k (2k+1) q^{\frac{k(k+1)}{2} + m(m+1)} \pmod{32}. \quad (28)$$

Substituting (5) and (7) in (18) and extracting the terms involving even and odd powers of  $q$ ,

$$\sum_{n=0}^{\infty} \bar{p}_o(8n+2)q^n = 2 \frac{f_2^{11} f_4^3}{f_1^{12} f_8^2} + 16q \frac{f_2 f_4^5 f_8^2}{f_1^8} \quad (29)$$

and 
$$\sum_{n=0}^{\infty} \bar{p}_o(8n+6)q^n = 4 \frac{f_2^{13} f_8^2}{f_1^{12} f_4^3} + 8 \frac{f_4^{11}}{f_8^8 f_2^2} \equiv 12 \frac{f_2 f_4 f_8^2}{f_1^4} \pmod{32}. \quad (30)$$

In view of (25) and (30), we see that

$$\bar{p}_o(8n+6) \equiv 3 \bar{p}_o(4n+3) \pmod{32}. \quad (31)$$

Substituting (5) and (7) in (17) and extracting the terms involving odd powers of  $q$ ,

$$\sum_{n=0}^{\infty} \bar{p}_o(8n+4)q^n = 2 \frac{f_2^{19} f_8^2}{f_1^{14} f_4^7} + 4 \frac{f_2^5 f_4^7}{f_1^{10} f_8^2}. \quad (32)$$

Employing (5) and (7) in (32) and extracting terms involving even and odd powers of  $q$ ,

$$\sum_{n=0}^{\infty} \bar{p}_o(16n+4)q^n = 6 \frac{f_2^{35}}{f_1^{28} f_4^5 f_8^2} + 160q \frac{f_2^{11} f_4^{11}}{f_1^{20} f_8^2} + 112q \frac{f_2^{25} f_8^2}{f_1^{24} f_4^3} + 256q^2 \frac{f_2 f_4^{13} f_8^2}{f_1^{16}} \quad (33)$$

and 
$$\begin{aligned} \sum_{n=0}^{\infty} \bar{p}_o(16n+12)q^n &= 12 \frac{f_2^{37} f_8^2}{f_1^{28} f_4^{11}} + 320q \frac{f_2^{13} f_4^5 f_8^2}{f_1^{20}} + 56 \frac{f_2^{23} f_4^3}{f_1^{24} f_8^2} + 128q \frac{f_4^{19}}{f_1^{16} f_8^2 f_2} \\ &\equiv 12 \frac{f_2^{21} f_8^2 f_4^4}{f_1^{11}} + 56 \frac{f_2^3 f_8^2}{f_4} \pmod{64} \end{aligned} \quad (34)$$

$$\equiv 4 \frac{f_2 f_4 f_8^2}{f_1^4} \pmod{32}. \quad (35)$$

From (25) and (35),

$$\bar{p}_o(16n+12) \equiv \bar{p}_o(4n+3) \pmod{32}. \quad (36)$$

Employing (5) in (22) and extracting odd powers of  $q$ ,

$$\sum_{n=0}^{\infty} \bar{p}_o(2^{\alpha+1}(2n+1))q^n \equiv \sum_{n=0}^{\infty} \bar{p}_o(2^\alpha(2n+1))q^n + 2^{\alpha+2} \frac{f_2 f_4^7}{f_1^2 f_8^2} \pmod{128}. \quad (37)$$

Again, extracting the terms involving even and odd powers of  $q$  from the last expression,

$$\sum_{n=0}^{\infty} \bar{p}_o(2^{\alpha+1}(4n+1))q^n \equiv \sum_{n=0}^{\infty} \bar{p}_o(2^\alpha(4n+1))q^n + 2^{\alpha+2} \frac{f_2^7 f_4^3}{f_1^4 f_8^2} \pmod{128} \quad (38)$$

$$\text{and } \sum_{n=0}^{\infty} \bar{p}_o(2^{\alpha+1}(4n+3))q^n \equiv \sum_{n=0}^{\infty} \bar{p}_o(2^\alpha(4n+3))q^n + 2^{\alpha+3} \frac{f_2^9 f_8^2}{f_1^4 f_4^3} \pmod{128} \quad (39)$$

for all  $\alpha \geq 2$ . In view of (31), (36) and (39), we see that

$$\bar{p}_o(2^\alpha(4n+3)) \equiv 3^\beta \bar{p}_o(4n+3) \pmod{32} \quad (40)$$

for all  $\alpha > 0, n \geq 0$  and  $\beta = \begin{cases} 1, & \text{if } \alpha = 1; \\ 0, & \text{otherwise.} \end{cases}$

Congruence (9) follows from (28) and (40).  $\square$

*Proof* (Proof of Corollary 2.4). By (28)

$$\sum_{n=0}^{\infty} \bar{p}_o(8n+3)q^{8n+3} \equiv 4 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^k (2k+1)q^{(2k+1)^2 + 2(2m+1)^2} \pmod{32},$$

which implies that if  $8n+3$  is not of the form  $(2k+1)^2 + 2(2m+1)^2$ , then  $\bar{p}_o(8n+3) \equiv 0 \pmod{32}$ . Let  $k \geq 1$  be an integer and let  $p_i \geq 3, 1 \leq i \leq k$  be primes with  $\left(\frac{-2}{p_i}\right) = -1$ .

If  $N$  is of the form  $x^2 + 2y^2$ , then  $v_{p_i}(N)$  is even since  $\left(\frac{-2}{p_i}\right) = -1$ . Let

$$\begin{aligned} N &= 8 \left( p_1^2 p_2^2 \cdots p_{k-1}^2 p_k n + 3 \frac{p_1^2 p_2^2 \cdots p_{k-1}^2 p_k^2 - 1}{8} \right) + 3 \\ &= 8 p_1^2 p_2^2 \cdots p_{k-1}^2 p_k n + 3 p_1^2 p_2^2 \cdots p_{k-1}^2 p_k^2. \end{aligned}$$

If  $p_k \nmid n$ , then  $v_{p_k}(N)$  is an odd number and hence  $N$  is not of the form  $x^2 + 2y^2$ .

$$\text{Thus, } \bar{p}_o(8 p_1^2 p_2^2 \cdots p_{k-1}^2 p_k n + 3 p_1^2 p_2^2 \cdots p_{k-1}^2 p_k^2) \equiv 0 \pmod{32}. \quad (41)$$

Congruence (10) follows from (41) and (40).  $\square$

*Proof* (Proof of Theorems 2.5 and 2.6). Applying (6) in (34) and extracting the terms involving odd powers of  $q$ ,

$$\sum_{n=0}^{\infty} \bar{p}_o(32n+28)q^n \equiv -48 \frac{f_1^{23} f_4^6}{f_2^{13}} \equiv 16 \frac{f_2^{11}}{f_1} \pmod{64}. \quad (42)$$

From (39),

$$\bar{p}_o(2^{\alpha+1}(8n+7)) \equiv \bar{p}_o(2^\alpha(8n+7)) \pmod{128} \quad (43)$$

for all  $\alpha \geq 2$  and  $n \geq 0$ . In view of (26), (3), (42) and (43),

$$\bar{p}_o(2^\alpha(8n+7)) \equiv 3^\beta \bar{p}_o(8n+7) \pmod{64} \quad (44)$$

for all  $\alpha > 0, n \geq 0$  and  $\beta = \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$

Congruence (11) follows from (2) and (44). Theorem 2.6 follows from Theorem 1.1 and (44).  $\square$

*Proof* (Proof of Theorem 2.7). By employing (7) in (23) and extracting the terms involving odd powers of  $q$ ,

$$\sum_{n=0}^{\infty} \bar{p}_o(8n+5)q^n = 8 \frac{f_2^9 f_4^2}{f_1^{11}} \equiv 8 \frac{f_1 f_8^2}{f_2} \pmod{32}. \quad (45)$$

Using (13) in (29),

$$\sum_{n=0}^{\infty} \bar{p}_o(8n+2)q^n \equiv 2 \frac{f_1^4 f_2^3 f_4^3}{f_8^2} + 16q \frac{f_2 f_{16}^2}{f_4} \pmod{32}. \quad (46)$$

Using (6) in the last expression and extracting the terms involving odd powers of  $q$ ,

$$\sum_{n=0}^{\infty} \bar{p}_o(16n+10)q^n \equiv -8f_1^5 f_2 f_4^2 + 16 \frac{f_1 f_8^2}{f_2} \equiv 8 \frac{f_1 f_8^2}{f_2} \pmod{32}. \quad (47)$$

Using (13) in (33),  $\sum_{n=0}^{\infty} \bar{p}_o(16n+4)q^n \equiv 6 \frac{f_1^4 f_2^3 f_4^3}{f_8^2} + 48q \frac{f_2 f_{16}^2}{f_4} \pmod{32}$ . Extracting coefficients of odd powers of  $q$  from (38),

$$\bar{p}_o(2^{\alpha+1}(8n+5)) \equiv \bar{p}_o(2^\alpha(8n+5)) \pmod{64} \quad (48)$$

for all  $\alpha \geq 2$  and  $n \geq 0$ . From (45)- (48) it is evident that

$$\bar{p}_o(2^\alpha(8n+5)) \equiv 3^\beta \bar{p}_o(8n+5) \pmod{32} \quad (49)$$

for all  $\alpha > 0, n \geq 0$  and  $\beta = \begin{cases} 0 & \text{if } \alpha = 1, \\ 1 & \text{otherwise.} \end{cases}$

Furthermore,

$$\begin{aligned} \frac{f_1 f_8^2}{f_2} &= (q; q^2)_\infty (q^8; q^8)_\infty (q^4; q^8)_\infty (-q^4; q^8)_\infty (q^{16}; q^{16})_\infty \\ &= (q; q^2)_\infty (q^4; q^4)_\infty (-q^4; q^8)_\infty (q^{16}; q^{16})_\infty \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{\lceil k/2 \rceil} q^{\frac{k(k+1)}{2} + 2m(m+1)}. \end{aligned}$$

Last equality follows from (4). Thus, congruence (12) follows from (45), (49) and the last equality.  $\square$

Proof of Corollary 2.8 is similar to Corollary 2.4, so we skip the details.

#### REFERENCES

- [1] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, 1991.

- [2] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Student Mathematical Library, AMS, 2006.
- [3] S. C. Chen, *On the number of overpartitions into odd parts*, *Discrete Math.*, **325** (2014), 32–37.
- [4] S. Corteel, J. Lovejoy, *Overpartitions*, *Trans. Amer. Math. Soc.*, **356** (2004), 1623–1635.
- [5] M. D. Hirschhorn, J. A. Sellers, *An infinite family of overpartition congruences modulo 12*, *Integers*, **5** (2005), #A20.
- [6] M. D. Hirschhorn, J. A. Sellers, *Arithmetic relations for overpartitions*, *J. Combin. Math. Combin. Comput.*, **53** (2005), 65–73.
- [7] M. D. Hirschhorn, J. A. Sellers, *Arithmetic properties of overpartitions into odd parts*, *Ann. Comb.*, **10** (2006), 353–367.
- [8] V. A. Lebesgue, *Sommation de quelques séries*, *J. Math. Pure. Appl.*, **5** (1840), 42–71.
- [9] K. Mahlburg, *The overpartition function modulo small powers of 2*, *Discrete Math.*, **286** (2014), 263–267.
- [10] C. Ray, R. Barman, *New congruences for overpartitions into odd parts*, *Integers*, **18** (2018), #A50.
- [11] M. Somos, *The on-line encyclopedia of integer sequences*, published electronically at <http://oeis.org>.

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