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GRÖBNER BASES FOR IDEALS IN UNIVARIATE POLYNOMIAL RINGS OVER VALUATION RINGS

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Abstract. Let V be a valuation ring such that $\dim(V) = 0$ and the annihilator of each element in V is finitely generated. In this paper it is proved that if I is a finitely generated ideal in the polynomial ring $V[X]$, then there is a Gröbner basis for I. Also, an example of a zero-dimensional non-Noetherian valuation ring R_M is presented, together with an example of finding a Gröbner basis for a certain ideal in a polynomial ring $R_M[X]$.

1. Introduction

The Gröbner basis theory appeared in relation to algebraic geometry, which, among other things, deals with solutions of systems of polynomial equations. It was presented by Bruno Buchberger in 1965., and since then, it develops and broadens its field of applications. The first notion that appears when introducing the idea of a Gröbner basis is the ideal membership problem, that is, how to test whether a given polynomial belongs to a certain ideal. Along with elementary applications, the Gröbner basis theory includes numerous advanced applications in commutative algebra and algebraic geometry. Also, the computational approach is what is highly appreciated in this theory.

This theory was primarily developed for ideals in rings of polynomials over a field (see, e.g., [1]), but it is extended to the cases of polynomial rings over Noetherian commutative rings, mostly principal ideal domains and Dedekind domains (e.g., [1], [2]). Outside the class of Noetherian rings, valuation domains were considered; for example, in [4], it is proved that a finitely generated ideal in a polynomial ring over a valuation domain with one indeterminate has a Gröbner basis iff $\dim(V) \leq 1$. According to [5], the same result holds for multivariate polynomial rings over valuation domains, with lexicographic monomial order.

Here, in Theorem 3.8, it is proved that a finitely generated ideal in $V[X]$, where V is a a valuation ring, has a minimal strong Gröbner basis, provided that $\dim(V) = 0$

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and that the annihilator of each element in V is finitely generated. An example of a zero-dimensional, non-Noetherian valuation ring is also given, in which we also demonstrated the method for finding a Gröbner basis for an ideal in a polynomial ring with one indeterminate over the presented ring.

In [3], the authors proved that each finitely generated ideal in a univariate polynomial ring over a valuation zero-dimensional ring has a Gröbner basis, a result which came to our knowledge during the final preparation of this paper. However, the proofs of the main results in that paper are quite different from what is presented here. In [3], the authors start with Lemma 2.3, which is stated very similarly to Lemma 3.5 of this paper, and then use it repeatedly in the following steps to obtain the main result. While here, before the proof of Lemma 3.5, we prove several introductory results, which actually do not assume the ring to be zero-dimensional and which are independent of the mentioned lemma. Then we use all these results together to prove the main result. Besides, all the proofs in this paper contain the methods which allow us to find a Gröbner basis. In addition, the example mentioned in the previous paragraph illustrates and further clarifies all the steps in the proofs.

2. Preliminaries

We suppose that all the rings in this paper are commutative and with identity. The Jacobson radical and the nilradical of a ring R will be denoted by $\mathrm{J}(R)$ and $\mathrm{Nil}(R)$, respectively. The annihilator of a module M is denoted by $\mathrm{Ann}(M)$, and the set of invertible elements in a ring R by $\mathrm{U}(R)$. Let us recall the definition of a valuation ring.

Definition 2.1. A ring V is called a valuation ring if at least one of the following relations is true: $a \mid b$ or $b \mid a$, for all $a, b \in V \setminus \{0\}$.

So, if we have a finite set of elements $\{a_1, \dots, a_s\}$ of a valuation ring V, there is $i \in \{1, \dots, s\}$ such that a_i divides all the elements of the given set. This fact will be used throughout the text.

The following proposition is well known and we state it without proof.

Lemma 2.2. *Any valuation ring is local.*

Here, the maximal ideal in a valuation ring V will be denoted by M.

We now present an example of a valuation ring of dimension zero which is not Noetherian.

Example 2.3. Let R be the following quotient:

$$R = \mathbb{Q}[X_1, X_2, X_3, \dots]/\langle X_1^2, X_2^2 - X_1, X_3^2 - X_2, \dots\rangle.$$

Observe the localization of R with respect to the maximal ideal $M = \langle x_1, x_2, \dots\rangle$, where x_i stands for the class of the element X_i in R. We will prove that R_M is a dimension zero valuation ring that is not Noetherian.

Each element in R is of the form $f(x_m)$ for some polynomial f with rational coefficients and for some $m \in \mathbb{N}$. Namely, for $r \in R$, we have that $r = g(x_1, \ldots, x_m)$. Since the relations $x_1 = x_2^2 = \cdots = x_m^{2^{m-1}}$ hold in R, it follows that $r = g(x_m^{2^{m-1}}, \ldots, x_m) = f(x_m)$. Also, each $r \in R_M$ can be seen as $f(x_m) = x_m^s u$, where s is the highest possible degree of x_m which appears in each term and u is invertible, since it does not belong to M.

Let $f_1(x_m)$ and $f_2(x_m)$ be some elements of R_M. Because of the given relations, we can pick the same m. These elements can be seen as $f_1(x_m) = x_m^s u$ and $f_2(x_m) = x_m^t v$, where u and v are invertible. So, the comparison of elements f_1 and f_2 comes down to comparison of the degrees s and t. Therefore, R_M is a valuation ring.

We have that $0 = x_1^2 = x_2^4 = x_3^8 = \cdots$ which implies that each element in R_M is nilpotent or invertible. If we denote by M^e the maximal ideal in R_M, then it follows that

$$M^e = \mathrm{J}(R_M) = \mathrm{Nil}(R_M) = \bigcap_{P \text{ prime}} P.$$

Namely, the first equality holds because R_M is local, and the second because of the fact that each non-invertible is nilpotent. So, we have that $M^e \subseteq P$, for each prime ideal P in R_M, which implies that R_M is a zero-dimensional ring.

For the proof that R_M is not Noetherian, let us suppose that the chain of ideals $\langle x_1 \rangle \subseteq \langle x_2 \rangle \subseteq \ldots$ is stationary. Then we have $\langle x_{n-1} \rangle = \langle x_n \rangle$, for some n. So, $x_n \in \langle x_n^2 \rangle$ and $x_n = p x_n^2$. It follows that $x_n(1 - p x_n) = 0$. The latter factor is invertible, since $p x_n \in M^e$. Consequently, $x_n = 0$, which is a contradiction.

3. Gröbner bases

Let us introduce all the notions that will be needed in this paper. If $f = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$, where $a_n \neq 0$, then the leading term of f is $\mathrm{LT}(f) = a_n X^n$. Also, for a non-zero ideal $I \lhd R[X]$, $\mathcal{LT}(I) = \{\mathrm{LT}(f) \mid f \in I \setminus \{0\}\}$ and $\mathrm{LT}(I) = \langle \mathcal{LT}(I) \rangle$.

Let us recall the definitions of (strong) Gröbner bases for an ideal in $R[X]$ (see, e.g., [1]).

Definition 3.1. Let I be a non-zero ideal in $R[X]$. A subset $G = \{g_1, \ldots, g_r\}$ of I is a Gröbner basis for I if $\mathrm{LT}(I) = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_r) \rangle$.

Definition 3.2. Let I be a non-zero ideal in $R[X]$. A subset $G = \{g_1, \ldots, g_r\}$ of I is a strong Gröbner basis for I if for any $f \in I \setminus \{0\}$, there exists $g_i \in G$ such that $\mathrm{LT}(g_i) \mid \mathrm{LT}(f)$. This basis is minimal if for all $i \neq j$: $\mathrm{LT}(g_i) \nmid \mathrm{LT}(g_j)$.

Let us denote by I_k the submodule $I_k = I \cap R[X]_k$ of $R[X]$, where R is a ring and $R[X]_k$ a submodule of $R[X]$ generated by $1, X, X^2, \ldots, X^k$. In the following two lemmas we consider an ideal which contains a monic polynomial of the degree n and observe the submodules $I_{n-1}, I_{n-2}, \ldots, I_0$.

Lemma 3.3. *Let I be an ideal in the ring $R[X]$, which is generated by the polynomials $f, f_1, \dots, f_s$, where f is a monic polynomial of degree n. Then I_{n-1} is a finitely generated R-module.*

Proof. Let us prove that the set of the following elements generates I_{n-1}:

$$\rho(X^t f_i, f), \qquad 0 \le t \le n-1, \qquad 1 \le i \le s,$$

where $\rho(g, h)$ stands for the remainder in the division of g by h.

Let $g \in I_{n-1}$. Since $g \in I$, we have $g = hf + h_1 f_1 + \cdots + h_s f_s$, $h, h_1, \dots, h_s \in R[X]$ and we can suppose that $\deg(h_1), \dots, \deg(h_s) < n$. If not, we could write: $h_i = q_i f + t_i$, $\deg(t_i) < n$, $1 \le i \le s$. Consequently, we would have $g = (h + q_1 f_1 + \cdots + q_s f_s) f + t_1 f_1 + \cdots + t_s f_s$. So, let

$$h_i = \sum_{t=0}^{k_i} \alpha_t^{(i)} X^t, \qquad k_i < n, \qquad 1 \le i \le s,$$

$$X^t f_i = f r_t^{(i)} + \rho(X^t f_i, f), \qquad 0 \le t \le k_i, \qquad 1 \le i \le s.$$

It follows that
$$g = f\left(h + \sum_{i=1}^{s} \sum_{t=0}^{k_i} \alpha_t^{(i)} r_t^{(i)}\right) + \sum_{i=1}^{s} \sum_{t=0}^{k_i} \alpha_t^{(i)} \rho(X^t f_i, f).$$

So, the polynomial g is of the form $fp + q$, $p \in R[X]$ and q being the R-linear combination of the polynomials $\rho(X^t f_i, f)$, $0 \le t \le k_i$, $1 \le i \le s$. Since $g \in I_{n-1}$, then $\deg(g) < n$. From the fact that f is monic, we can conclude that $p = 0$. Also from $\rho(X^t f_i, f) = X^t f_i - f r_t^{(i)}$ it follows that all the remainders $\rho(X^t f_i, f)$ belong to I, and consequently to I_{n-1}. Therefore, these polynomials form the generating set for the R-module I_{n-1}. □

For the next lemma, we depart from the discussion of an arbitrary ring and consider the valuation ring in which the annihilator of every element is finitely generated.

Lemma 3.4. *Let V be a valuation ring in which the annihilator of every element is finitely generated and I be an ideal in $V[X]$ generated by polynomials $f, f_1, \dots, f_s$, where f is a monic polynomial of the degree n. Then $I_{n-2}, I_{n-3}, \dots, I_0$ are finitely generated V-modules.*

Proof. According to Lemma 3.3, the V-module I_{n-1} is finitely generated. Let $\pi_k : V[X]_k \to V$ be the homomorphisms such that $\pi_k(p)$ is the coefficient of the monomial X^k in the polynomial p. It follows that the image $\pi_{n-1}(I_{n-1})$ is also a finitely generated submodule of V, that is, a finitely generated ideal. Since V is a valuation ring, this image must be a principal ideal $\pi_{n-1}(I_{n-1}) = \langle c_{n-1} \rangle$. Without loss of generality, we can suppose that at least one of $f_1, \dots, f_s$ is not a multiple of f. Then $c_{n-1} \ne 0$. Let $h_{n-1} \in I_{n-1}$ be the polynomial such that $\mathrm{LT}(h_{n-1}) = c_{n-1} X^{n-1}$. (This polynomial is actually one of the remainders $\rho(X^t f_i, f)$, as in the proof of the previous lemma.)

Let us prove that I_{n-2} is a finitely generated V-module by explicitly finding its generating set. Since $g \in I_{n-2} \subseteq I_{n-1}$, then obviously $g = \sum r_{ij} \rho(X^i f_j, f)$, $r_{ij} \in V$,

$1 \leq j \leq s$, $0 \leq i \leq n-1$. Since all the remainders $\rho(X^i f_j, f)$ are in I_{n-1}, we can reduce them by h_{n-1}. So, $\rho(X^i f_j, f) = a_{ij}h_{n-1} + f_{ij}$. Note that it may happen that $a_{ij} = 0$. We now have that $g = \sum r_{ij}(a_{ij}h_{n-1} + f_{ij}) = (\sum r_{ij}a_{ij})\, h_{n-1} + \sum r_{ij}f_{ij}$. Since $g \in I_{n-2}$, then $\sum r_{ij}a_{ij}$ belongs to $\mathrm{Ann}(c_{n-1})$. This annihilator is finitely generated, that is, principal, so let $\mathrm{Ann}(c_{n-1}) = \langle d_{n-1}\rangle$. It follows that each element in I_{n-2} is a V-linear combination of $d_{n-1}h_{n-1}$ and $f_{ij} = \rho(\rho(X^i f_j, f), h_{n-1})$, where $1 \leq j \leq s$, $0 \leq i \leq n-1$.

We can repeat this procedure. If $I_{n-2}{=}\{0\}$, then $I_0{=}\cdots{=}I_{n-3}{=}\{0\}$. If $I_{n-2}{\neq}\{0\}$, then $\pi_{n-2}(I_{n-2})$ is a finitely generated submodule of V, there is $c_{n-2} \in V$ such that $\pi_{n-2}(I_{n-2}){=}\langle c_{n-2}\rangle$. Let $h_{n-2}{\in}I_{n-2}$ be the polynomial such that $\mathrm{LT}(h_{n-2}){=}c_{n-2}X^{n-2}$. The generating set for I_{n-3} consists of the remainders in the division of the generators for I_{n-2} by h_{n-2} together with $d_{n-2}h_{n-2}$, where $\mathrm{Ann}(\mathrm{LC}(h_{n-2})){=}\langle d_{n-2}\rangle$. We can continue in the same manner to prove that all these modules are finitely generated. □

Obviously, we consider as important the existence of a monic polynomial in the ideal. If $I = \langle f_1, \ldots, f_s\rangle \triangleleft V[X]$, then there is a coefficient $\alpha \in V$ of some of the generators which divides all the other coefficients of all the polynomials which generate I. Let $f_i = \alpha g_i$, for $i \in \{1, \ldots, s\}$. Then $I = \alpha\langle g_1, \ldots, g_s\rangle$ and at least one coefficient among all the coefficients of $g_1, \ldots, g_s$ is invertible. Under the assumption that the dimension of the valuation ring is zero, in the following lemma we prove that $\langle g_1, \ldots, g_s\rangle$ contains a monic polynomial.

LEMMA 3.5. *Let $I = \langle f_1, \ldots, f_s\rangle$ be an ideal in $V[X]$, where V is a valuation ring of dimension zero. If at least one coefficient among all the coefficients of the generators of I is invertible, then there is a monic polynomial f that belongs to I.*

Proof. Without loss of generality, we can suppose that f_1 has at least one invertible coefficient, and also, that this invertible coefficient is 1. If $\mathrm{LC}(f_1) = 1$, then we are done. If not, f_1 is a sum of a monic polynomial and a polynomial with all the coefficients in M (non-invertible elements). The latter can be seen as a product of an element in M and a polynomial with one coefficient equal to 1. So $f_1(X) = bp(X) + q(X)$, $b \in M$, q monic in $V[X]$. Since V is a local ring of dimension zero, we have that $\mathrm{Nil}(V) = \mathrm{J}(V) = M$. It follows that each element in V is nilpotent or invertible. So, there is $m \in \mathbb{N}$ such that $b^m = 0$. According to the formula $(y+q)^m - y^m = (y+q)q_1 + (-1)^{m+1}q^m$, for polynomials y, q, q_1 and $m \geq 1$, we have that

$$\begin{aligned} I \ni f_1(X)^m &= (bp(X) + q(X))^m = (bp(X) + q(X))^m - (bp(X))^m \\ &= (bp(X) + q(X))q_1(X) + (-1)^{m+1}q(X)^m \\ &= f_1(X)q_1(X) + (-1)^{m+1}q(X)^m. \end{aligned}$$

So, the monic polynomial q^m belongs to I. □

REMARK 3.6. Actually, there is a monic polynomial of the degree $\deg(q) = k$ in I. Let us divide the polynomial $X^s f_1$ by q^m, where $s = (m-1)k-1$. Now, the coefficient with $X^s X^k = X^{mk-1}$ in $X^s f_1$ is equal to 1 and we are dividing by the monic polynomial q^m, which is of the degree mk. Since $f_1(X) = bp(X) + q(X)$, $b \in M$, then all the coefficients in $X^s f_1$ higher than $mk-1$ are also in M. Therefore, the quotient in this division is also a multiple of some element in M. The leading coefficient of the remainder is then of the form $1-\mu$, with $\mu \in M$, and consequently, invertible. So, we have proved that I contains a monic polynomial r_1 of the degree $mk-1$. We can repeat this procedure:

$$\begin{aligned} X^s f_1 &= q^m q_1 + r_1 \\ X^{s-1} f_1 &= r_1 q_2 + r_2 \\ &\cdots \\ X^2 f_1 &= r_{s-2} q_{s-1} + r_{s-1} \\ X f_1 &= r_{s-1} q_s + r_s \\ f_1 &= r_s q_{s+1} + r_{s+1}. \end{aligned}$$

Here we have that $r_i \in I$, $\deg(r_i) = mk - i$ and, although these are not monic, their leading coefficients are invertible. So, r_{s+1} is a polynomial in I of the degree k whose leading coefficient is invertible.

Now, we can return to the question of existence of a Gröbner basis for an ideal I in $V[X]$. First, it is clear than any strong Gröbner basis is also a Gröbner basis. Let us prove that, in the case of valuation rings, these two notions actually coincide.

LEMMA 3.7. *Let $G = \{g_1, \ldots, g_r\}$ be a Gröbner basis for an ideal I in $V[X]$, where V is a valuation ring. Then G is also a strong Gröbner basis for I.*

Proof. Let $f \in I$. Since G is a Gröbner basis there exists polynomials $p_1(X), \ldots, p_r(X)$ such that $\mathrm{LT}(f) = p_1(X)\mathrm{LT}(g_1) + \cdots + p_r(X)\mathrm{LT}(g_r)$. Let $p_i(X) = b_0^{(i)} + b_1^{(i)} X + \cdots + b_{s_i}^{(i)} X^{s_i}$, $\mathrm{LT}(g_i) = a_i X^{n_i}$ and $\mathrm{LT}(f) = aX^n$. So

$$aX^n = \sum_{i=1}^{r} \sum_{j=0}^{s_i} a_i X^{n_i} b_j^{(i)} X^j.$$

We can conclude that for some subset $K \subseteq \{1, \ldots, r\}$, we have

$$aX^n = \sum_{k \in K} a_k X^{n_k} b_{n-n_k}^{(k)} X^{n-n_k} = \left(\sum_{k \in K} a_k b_{n-n_k}^{(k)} \right) X^n.$$

It follows that $a = \sum_{k \in K} a_k b_{n-n_k}^{(k)}$. Let $k_0 \in K$ be such that $a_{k_0} \mid a_k$ for all $k \in K$. Then we have that $a_{k_0} X^{n_{k_0}} \mid aX^n$ and we are done. □

Now we prove the main theorem.

THEOREM 3.8. *Let V be a dimension zero valuation ring in which the annihilator of every element is finitely generated. If I is a finitely generated ideal in $V[X]$, then there is a minimal strong Gröbner basis for I.*

Proof. Let $I = \langle f_1, \ldots, f_s \rangle$ and $\alpha \in V$ a coefficient of some of the generators which divides all the other coefficients of all the polynomials which generate I. Let $f_i = \alpha g_i$, for $i \in \{1, \ldots, s\}$. Then $I = \alpha \langle g_1, \ldots, g_s \rangle$. So it suffices to prove that $J = \langle g_1, \ldots, g_s \rangle$ has a Gröbner basis.

We can suppose that at least one coefficient among all the coefficients of $g_1, \dots, g_s$ is equal to 1, and according to Lemma 3.5, there is a monic polynomial in J. Let f be a monic polynomial of the lowest degree that belongs to J and suppose $\deg(f) = n$. Then by Lemma 3.4 $J_{n-1}, \dots, J_0$ are finitely generated V-modules. As in the proof of Lemma 3.4, let $\pi_k : V[X]_k \to V$ be the homomorphisms such that $\pi_k(p)$ is the coefficient of the monomial X^k in the polynomial p. It follows that the images $\pi_k(J_k)$ are principal ideals in V. Let $\pi_k(J_k) = \langle c_k \rangle$ and let $h_k \in J_k$ be such that $\mathrm{LT}(h_k) = c_k X^k$. If we denote by $\mathcal{S}$ the set of indices $i \in \{1, \dots, n-1\}$ such that $c_i \neq 0$ and $c_j \nmid c_i$, for all $j < i$, then a strong Gröbner basis for J is given by $G = \{f, h_i \,|\, i \in \mathcal{S}\}$, that is $\mathrm{LT}(J) = \langle X^n, c_i X^i \,|\, i \in \mathcal{S} \rangle$. Namely, suppose that $p \in J \setminus \{0\}$. If $\deg(p) \geq n$, then $X^n \mid \mathrm{LT}(p)$. If $\deg(p) = k < n$, then $p \in J_k$; so, $c_k X^k \mid \mathrm{LT}(p)$. Let $c_i X^i$ for $i \in \mathcal{S}$ be such that $c_i | c_k$ and $i \leq k$. Then $c_i X^i \mid \mathrm{LT}(p)$. So, G is a strong Gröbner basis for J. It is also minimal: $\mathrm{LT}(h_i) \nmid \mathrm{LT}(h_j)$ for all $i \neq j$, $i, j \in \mathcal{S}$ and none of the h_i, $i \in \mathcal{S}$ is monic since f is chosen to be a monic polynomial of the lowest degree that belongs to J. If we set $G' = \{\alpha g \,|\, g \in G\}$, we obtain a minimal strong Gröbner basis G' for the starting ideal I. ☐

Example 3.9. Let us use Example 2.3 to illustrate the previous results. Let I be an ideal in the ring $R_M[X]$ generated by

$$f_1(X) = x_1 X^4 + x_1 x_2 x_3 X^3 + X^2 + x_2 X - x_2 x_3$$
$$f_2(X) = x_2 X^2 + (x_1 + x_3) X + x_1$$
$$f_3(X) = x_2 X^2 + x_1 X.$$

If we use the given relations, we can see these polynomials as

$$f_1(X) = x_3^4 X^4 + x_3^7 X^3 + X^2 + x_3^2 X - x_3^3$$
$$f_2(X) = x_3^2 X^2 + (x_3^4 + x_3) X + x_3^4$$
$$f_3(X) = x_3^2 X^2 + x_3^4 X.$$

For the sake of simplicity, let us denote x_3 with α. Since $f_1(X) = \alpha^4(X^4 + \alpha^3 X^3) + X^2 + \alpha^2 X - \alpha^3$ and the fact that $(\alpha^4)^2 = 0$, according to Lemma 3.5, the polynomial $q(X)^2 = (X^2 + \alpha^2 X - \alpha^3)^2$ belongs to I. According to the remark, a monic polynomial of degree $k = 2$ belongs to I also. We proceed to find this polynomial. Since $s = k(m-1) - 1 = 1$, we divide Xf_1 by q^2. After this division, we get that the monic polynomial $r_1(X) = X^3 + \alpha^2 X^2 - \alpha^3 X$ is in I. The following division of f_1 by r_1 allows us to conclude that the polynomial $r_2(X) = X^2 + \alpha^2 X - \alpha^3$ is a monic polynomial in I.

Let $h = r_2$ and since $\rho(f_1, h) = 0$, $\rho(f_2, h) = \alpha X + \alpha^4(1 + \alpha)$, $\rho(f_3, h) = \alpha^5$, we can subtract the third remainder from the second to get that $I = \langle h, h_1, h_2 \rangle$, where $h_1 = \alpha X + \alpha^4$ and $h_2 = \alpha^5$.

Following the algorithm, we have that

$$\rho(Xh_1, h) = \alpha^3(\alpha - 1)X + \alpha^4, \qquad \rho(Xh_2, h) = \alpha^5 X,$$
$$\rho(h_1, h) = h_1 = \alpha X + \alpha^4, \qquad \rho(h_2, h) = h_2 = \alpha^5,$$

which means that $\pi_1(I_1)$ is generated by α. Also, h_1 is the polynomial such that its leading coefficient generates $\pi_1(I_1)$. Towards the final step of the calculation, we determine

$$\rho(\alpha^3(\alpha-1)X+\alpha^4,h_1)=\alpha^4(1-\alpha^3+\alpha^2),$$
$$\rho(\alpha^5 X,h_1)=0,\qquad \rho(h_2,h_1)=h_2=\alpha^5.$$

Also, let $p(x_m)\in \mathrm{Ann}(\alpha)$. Then $0=\alpha\cdot p(x_m)=x_3\cdot x_m^k u$, $u\in \mathrm{U}(R_M)$, $k\geq 1$. It follows that $x_m^{2^{m-3}}x_m^k=x_m^{2^{m-3}+k}=0=x_m^{2^m}$, and so $k\geq 2^m-2^{m-3}=7\cdot 2^{m-3}$. Therefore, $\mathrm{Ann}(\alpha)=\langle\alpha^7\rangle$. Since $\alpha^7 h_1=0$, it follows that $\pi_0(I_0)=I_0$ is generated by α^4.

Finally, we get that $\mathrm{LT}(I)=\langle X^2,x_3X,x_3^4\rangle=\langle X^2,x_3X,x_1\rangle$, and a minimal strong Gröbner basis for I is given by the polynomials $g_1=X^2+x_2X-x_2x_3$, $g_2=x_3X+x_1$ and $g_3=x_1$.

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