MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
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FIBONACCI NUMBERS WHICH ARE CONCATENATIONS OF THREE REPDIGITS

Fatih Erduvan and Refik Keskin

Abstract. In this study, it is proved that the only Fibonacci numbers which are concatenations of three repdigits are 144, 233, 377, 610, 987, 17711.

1. Introduction

Let (F_n) be the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. Binet formula for the n^{th} Fibonacci number is

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$. It can be seen that $1 < \alpha < 2$, $-1 < \beta < 0$ and $\alpha\beta = -1$. A relation between n^{th} Fibonacci number F_n and α is given by

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{for } n \geq 1. \tag{1}$$

The inequality (1) can be proved by induction. Given $k \geq 1$, we say that N is a concatenations of k repdigits, if N can be written in the form

$$\overline{\underbrace{d_1 \dots d_1}_{m_1 \text{ times}} \underbrace{d_2 \dots d_2}_{m_2 \text{ times}} \dots \underbrace{d_k \dots d_k}_{m_k \text{ times}}}.$$

In [1], the authors solved the problem of finding the Fibonacci numbers which are concatenations of two repdigits. In [6, 7] Ddamulira determined all the tribonacci numbers that are concatenations of two repdigits, respectively. In [10], Trojovský considered Fibonacci numbers of the form

$$F_n = \overline{a\, \underbrace{b \dots b}_{m \text{ times}}\, \underbrace{c \dots c}_{k \text{ times}}},$$

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where $a \geq 1$, $0 \leq a, b, c \leq 9$ with $2 \leq k \leq m$. He showed that the largest Fibonacci numbers of the above form is 17711. Motivated by these studies, in this paper, we study the equation

$$F_n = \overline{\underbrace{d_1 \ldots d_1}_{m_1 \text{ times}} \underbrace{d_2 \ldots d_2}_{m_2 \text{ times}} \underbrace{d_3 \ldots d_3}_{m_3 \text{ times}}}, \tag{2}$$

where $d_1, m_1, m_2, m_3 \geq 1$ and $0 \leq d_1, d_2, d_3 \leq 9$. That is, we find all Fibonacci numbers which are concatenations of three repdigits. It is shown that the only Fibonacci numbers which are concatenations of three repdigits are 144, 233, 377, 610, 987, 17711. In Section 2, we introduce necessary lemmas. Then we prove our main theorem in Section 3.

2. Auxiliary results

Let η be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \ldots + a_d = a_0 \prod_{i=1}^{d} \left(x - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right) \tag{3}$$

is called the logarithmic height of η. In particularly, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b \geq 1$, then $h(\eta) = \log(\max\{|a|, b\})$.

For algebraic numbers η and γ, the function h has the following basic properties (see [4]):

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \quad h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \quad h(\eta^m) = |m| h(\eta).$$

Now, we give a theorem which is deduced from [9, Corollary 2.3] and provides a large upper bound for the subscript n in the equations (2) (also see [5, Theorem 9.4]).

Lemma 2.1. *Assume that $\gamma_1, \gamma_2, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, \ldots, b_t$ are rational integers, and $\Lambda := \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1$ is not zero. Then*

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 A_2 \ldots A_t\right),$$

where $B \geq \max\{|b_1|, \ldots, |b_t|\}$ and $A_i \geq \max\{D h(\gamma_i), |\log \gamma_i|, 0.16\}$ for all $i = 1, \ldots, t$.

The following lemma is given in [3]. This lemma is an immediate variation of the result due to Dujella and Pethő from [8], which is a version of a lemma of Baker and Davenport [2]. It will be used to reduce the upper bound for the subscript n in the equation (2). Let $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$ for any real number x.

Lemma 2.2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real* *numbers with $A > 0$ and $B > 1$. Let $\epsilon := \|\mu q\| - M\|\gamma q\|$. If $\epsilon > 0$, then there exists no solution to the inequality $0 < |u\gamma - v + \mu| < AB^{-w}$, in positive integers u, v, and w with $u \le M$ and $w \ge \frac{\log(Aq/\epsilon)}{\log B}$.*

The following lemma is given in [11].

LEMMA 2.3. *Let $a, x \in \mathbb{R}$. If $0 < a < 1$ and $|x| < a$; then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a} \cdot |x| \quad \textit{and} \quad |x| < \frac{a}{1-e^{-a}} \cdot |e^x - 1|.$$

3. Main theorem

THEOREM 3.1. *The only Fibonacci numbers which are concatenations of three repdigits are 144, 233, 377, 610, 987, 17711.*

Proof. Assume that the equation (2) holds. Since the number F_{149} has 31 digits, $1 \le m_1, m_2, m_3 \le 29$ for $n \le 149$. Then by using Mathematica program, we can find all the solutions to the equation (2) for $d_1 > 0$, $0 \le d_1, d_2, d_3 \le 9$, and $1 \le n \le 149$. In this case, we see that $F_n \in \{144, 233, 377, 610, 987, 17711\}$. From now on, we assume that $n \ge 150$ and we rule out $d_1 = d_2 \ne d_3$ and $d_1 \ne d_2 = d_3$ in equation (2) since these cases have been solved in [1]. Furthermore, the case $d_1 = d_2 = d_3$ in equation (2) is impossible since the largest repdigit in Fibonacci sequence is $F_{10} = 55$, which has two digits. As

$$F_n = \overline{\underbrace{d_1 \ldots d_1}_{m_1 \text{ times}}\underbrace{d_2 \ldots d_2}_{m_2 \text{ times}}\underbrace{d_3 \ldots d_3}_{m_3 \text{ times}}} = \underbrace{d_1 \ldots d_1}_{m_1 \text{ times}} \times 10^{m_2+m_3} + \underbrace{d_2 \ldots d_2}_{m_2 \text{ times}} \times 10^{m_3} + \underbrace{d_3 \ldots d_3}_{m_3 \text{ times}},$$

we get
$$F_n = \frac{d_1(10^{m_1}-1)}{9}10^{m_2+m_3} + \frac{d_2(10^{m_2}-1)}{9}10^{m_3} + \frac{d_3(10^{m_3}-1)}{9}, \tag{4}$$

i.e.
$$F_n = \frac{1}{9}\left(d_1 10^{m_1+m_2+m_3} - (d_1 - d_2)\,10^{m_2+m_3} - (d_2 - d_3)\,10^{m_3} - d_3\right). \tag{5}$$

Combining the right-hand side of inequality of (1) with (4), we obtain

$$10^{m_1+m_2+m_3-1} < F_n \le \alpha^{n-1} < 10^{n-1}.$$

From this, we get $m_1 + m_2 + m_3 < n$. Now, we can rewrite equation (5) as

$$\begin{aligned} &9\alpha^n - d_1\sqrt{5} \cdot 10^{m_1+m_2+m_3} \\ &= 9\beta^n - (d_1 - d_2)\sqrt{5} \cdot 10^{m_2+m_3} - (d_2 - d_3)\sqrt{5} \cdot 10^{m_3} - d_3\sqrt{5}. \end{aligned} \tag{6}$$

Taking absolute values of both sides of the equation (6), we get

$$\begin{aligned} &\left|9\alpha^n - d_1\sqrt{5} \cdot 10^{m_1+m_2+m_3}\right| \\ &\le 9\,|\beta|^n + |d_1 - d_2|\,\sqrt{5} \cdot 10^{m_2+m_3} + |d_2 - d_3|\,\sqrt{5} \cdot 10^{m_3} + d_3\sqrt{5} \\ &\le 9\alpha^{-n} + 9\sqrt{5} \cdot 10^{m_2+m_3} + 9\sqrt{5} \cdot 10^{m_3} + 9\sqrt{5} \\ &\le 9\alpha^{-n} + \sqrt{5}(9 \cdot 10^{m_2+m_3} + 9 \cdot 10^{m_3} + (0.9) \cdot 10^{m_3}) \end{aligned}$$

$$= 9\alpha^{-n} + \sqrt{5}\cdot 10^{m_3}(9\cdot 10^{m_2} + (9.9)) \le 9\alpha^{-n} + \sqrt{5}\cdot 10^{m_3}(9\cdot 10^{m_2} + (0.99)\cdot 10^{m_2})$$
$$< (0.09)\,\alpha^{-n}10^{m_2+m_3} + 22.34\cdot 10^{m_2+m_3} < 22.341\cdot 10^{m_2+m_3},$$

where we have used the fact that $n \ge 150$. Therefore, it is seen that

$$\left|9\alpha^n - d_1\sqrt{5}\cdot 10^{m_1+m_2+m_3}\right| < 22.341\cdot 10^{m_2+m_3}. \tag{7}$$

Dividing both sides of (7) by $d_1\sqrt{5}\cdot 10^{m_1+m_2+m_3}$, we obtain

$$\left|\frac{9}{d_1\sqrt{5}}\alpha^n 10^{-m_1-m_2-m_3} - 1\right| \le \frac{22.341\cdot 10^{m_2+m_3}}{d_1\sqrt{5}\cdot 10^{m_1+m_2+m_3}} < \frac{9.992}{10^{m_1}}. \tag{8}$$

Now, let us apply Lemma 2.1 with $\gamma_1 := \frac{9}{d_1\sqrt{5}}, \gamma_2 := \alpha, \gamma_3 := 10$ and $b_1 := 1, b_2 := n, b_3 := -m_1 - m_2 - m_3$. Note that the numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $K = \mathbb{Q}(\sqrt{5})$. The degree of the field K is 2. So $D = 2$. It can be seen that $\Lambda_1 := \frac{9}{d_1\sqrt{5}}\alpha^n 10^{-m_1-m_2-m_3} - 1$ is nonzero. Moreover, since

$$h(\gamma_1) = h\left(\frac{9}{d_1\sqrt{5}}\right) \le h\left(\frac{9}{d_1}\right) + h(\sqrt{5}) \le \log 9 + \frac{\log 5}{2} < 3.1,$$

and
$$h(\gamma_2) = h(\alpha) = \frac{\log\alpha}{2} < 0.25, \quad h(\gamma_3) = h(10) = \log 10 < 2.31$$

by (3), we can take $A_1 := 6.2, A_2 := 0.5$, and $A_3 := 4.62$. On the other hand, as $m_1 + m_2 + m_3 < n$ and $B \ge \max\{|1|, |n|, |-m_1 - m_2 - m_3|\}$, we can take $B := n$. Thus, taking into account the inequality (8) and using Lemma 2.1, we obtain

$$(9.992)\cdot 10^{-m_1} > |\Lambda_1| > \exp\left(C\cdot(1+\log n)\right),$$

where $C = -1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2)\cdot(6.2)\cdot(0.5)\cdot(4.62)$. By a simple computation, it follows that

$$m_1\log 10 < 1.39\cdot 10^{13}\cdot(1+\log n) + \log(9.992). \tag{9}$$

Rearranging the equation (5) as

$$\begin{aligned} &9\alpha^n - d_1\sqrt{5}\cdot 10^{m_1+m_2+m_3} \\ &= 9\beta^n - (d_1 - d_2)\sqrt{5}\cdot 10^{m_2+m_3} - (d_2 - d_3)\sqrt{5}10^{m_3} - d_3\sqrt{5} \end{aligned} \tag{10}$$

and taking absolute values of both sides of the equation (10), we get

$$\begin{aligned} &\left|9\alpha^n - (d_1 10^{m_1} - (d_1 - d_2))\sqrt{5}\cdot 10^{m_2+m_3}\right| \\ &\le 9|\beta|^n + |d_2 - d_3|\sqrt{5}\cdot 10^{m_3} + d_3\sqrt{5} \le 9\alpha^{-n} + 9\sqrt{5}\cdot 10^{m_3} + 9\sqrt{5} \\ &\le 9\alpha^{-n} + \sqrt{5}(9\cdot 10^{m_3} + 9) \le 9\alpha^{-n} + \sqrt{5}(9\cdot 10^{m_3} + (0.9)\cdot 10^{m_3}) < 22.15\cdot 10^{m_3}, \end{aligned}$$

i.e.,
$$\left|9\alpha^n - (d_1 10^{m_1} - (d_1 - d_2))\sqrt{5}\cdot 10^{m_2+m_3}\right| < 22.15\cdot 10^{m_3}. \tag{11}$$

Dividing both sides of (11) by $(d_1 10^{m_1} - (d_1 - d_2))\sqrt{5}\cdot 10^{m_2+m_3}$, we obtain

$$\left|1 - \left(\frac{9}{(d_1 10^{m_1} - (d_1 - d_2))\sqrt{5}}\right)\alpha^n 10^{-m_2-m_3}\right| < \frac{1.11}{10^{m_2}}. \tag{12}$$

Taking $\gamma_1 := \frac{9}{(d_1 10^{m_1} - (d_1 - d_2))\sqrt{5}}$, $\gamma_2 := \alpha$, $\gamma_3 := 10$ and $b_1 := 1$, $b_2 := n$, $b_3 :=$

$-m_2 - m_3$, we can apply Lemma 2.1. The numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $K = \mathbb{Q}(\sqrt{5})$ and so $D = 2$. One can verify that $\Lambda_2 := 1 - \left(\frac{9}{(d_1 10^{m_1} - (d_1 - d_2))\sqrt{5}}\right) \alpha^n 10^{-m_2 - m_3}$ is nonzero. By using (3) and the properties of the logarithmic height, we get

$$h(\gamma_1) = h\left(\frac{9}{(d_1 10^{m_1} - (d_1 - d_2))\sqrt{5}}\right)$$
$$\leq h(9) + h(\sqrt{5}) + h(d_1 10^{m_1}) + h(d_1 - d_2) + \log 2$$
$$\leq 3\log 9 + \frac{\log 5}{2} + m_1 \log 10 + \log 2 < 8.1 + m_1 \log 10,$$
$$h(\gamma_2) = h(\alpha) = \frac{\log \alpha}{2} < 0.25,$$

and
$$h(\gamma_3) = h(10) = \log 10 < 2.31.$$

So, we can take $A_1 := 16.2 + 2m_1 \log 10$, $A_2 := 0.5$, and $A_3 := 4.62$. As $m_2 + m_3 < n$ and $B \geq \max\{|1|, |-n|, |-m_2 - m_3|\}$, we can take $B := n$. Thus, taking into account the inequality (12) and using Lemma 2.1, we obtain

$$1.11 \cdot 10^{-m_2} > |\Lambda_2| > \exp\left(C \cdot (1 + \log n)(16.2 + 2m_1 \log 10)\right)$$

i.e.,
$$m_2 \log 10 < 2.25 \cdot 10^{12} \cdot (1 + \log n)(16.2 + 2m_1 \log 10) + \log(1.11), \tag{13}$$

where $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2) \cdot (0.5) \cdot (4.62)$. Rearranging the equation (5) as

$$9\alpha^n - \left(d_1 10^{m_1+m_2} - (d_1 - d_2)10^{m_2} - (d_2 - d_3)\right)\sqrt{5} \cdot 10^{m_3} = 9\beta^n - d_3 \tag{14}$$

and taking absolute values of both sides of the equation (14), we get

$$\left|9\alpha^n - \left(d_1 10^{m_1+m_2} - (d_1 - d_2)10^{m_2} - (d_2 - d_3)\right)\sqrt{5} \cdot 10^{m_3}\right| \leq 9|\beta|^n + d_3 = 9\alpha^{-n} + 9 < 9.1,$$

i,e.,
$$\left|9\alpha^n - \left(d_1 10^{m_1+m_2} - (d_1 - d_2)10^{m_2} - (d_2 - d_3)\right)\sqrt{5} \cdot 10^{m_3}\right| < 9.1. \tag{15}$$

If we divide both sides of (15) by $9\alpha^n$, we obtain

$$\left|1 - \left(\frac{(d_1 10^{m_1+m_2} - (d_1 - d_2)10^{m_2} - (d_2 - d_3))\sqrt{5}}{9}\right) \alpha^{-n} 10^{m_3}\right| \leq 1.02 \cdot \alpha^{-n}. \tag{16}$$

Taking $\gamma_1 := \left(\frac{(d_1 10^{m_1+m_2} - (d_1 - d_2)10^{m_2} - (d_2 - d_3))\sqrt{5}}{9}\right)$, $\gamma_2 := \alpha$, $\gamma_3 := 10$ and $b_1 := 1$, $b_2 := -n$, $b_3 := m_3$, we can apply Lemma 2.1. The numbers γ_1, γ_2 and γ_3 are positive real numbers and elements of the field $K = \mathbb{Q}(\sqrt{5})$ and so $D = 2$. One can verify that

$$\Lambda_3 := 1 - \left(\frac{(d_1 10^{m_1+m_2} - (d_1 - d_2)10^{m_2} - (d_2 - d_3))\sqrt{5}}{9}\right) \alpha^{-n} 10^{m_3}$$

is nonzero. By using (3) and the properties of the logarithmic height, we get

$$h(\gamma_1) = h\left(\frac{(d_1 10^{m_1+m_2} - (d_1 - d_2)10^{m_2} - (d_2 - d_3))\sqrt{5}}{9}\right)$$
$$\leq h(\sqrt{5}) + h(9) + h(d_1 10^{m_1+m_2}) + h((d_1 - d_2)10^{m_2}) + h(d_2 - d_3) + 2\log 2$$

$$\leq \frac{\log 5}{2} + 4\log 9 + (m_1+m_2)\log 10 + m_2 \log 10 + 2\log 2$$
$$< 10.98 + (m_1+m_2)\log 10 + m_2\log 10,$$
$$h(\gamma_2) = h(\alpha) = \frac{\log\alpha}{2} < 0.25, \quad \text{and} \quad h(\gamma_3) = h(10) = \log 10 < 2.31.$$

So, we can take $A_1 := 21.96 + 2m_1\log 10 + 4m_2\log 10$, $A_2 := 0.5$, and $A_3 := 4.62$. As $m_3 < n$ and $B \geq \max\{|1|, |-n|, |m_3|\}$, we can take $B := n$. Thus, taking into account the inequality (16) and using Lemma 2.1, we obtain

$$1.02 \cdot \alpha^{-n} > |\Lambda_3| > \exp\left(C \cdot (1+\log n)\left(21.96 + 2m_1\log 10 + 4m_2\log 10\right)\right)$$

i.e., $$n\log\alpha - \log(1.02) < 2.25\cdot 10^{12}(1+\log n)\left(21.96+2m_1\log 10+4m_2\log 10\right), \tag{17}$$

where $C = -1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2(1+\log 2)\cdot(0.5)\cdot(4.62)$. Using the inequalities (9), (13) and (17), a computer search with Mathematica gives us that $n < 1.36\cdot 10^{45}$.

Now, let us try to reduce the upper bound on n by applying Lemma 2.2. Let

$$z_1 := (m_1+m_2+m_3)\log 10 - n\log\alpha - \log\left(\frac{9}{\sqrt{5}d_1}\right).$$

From (8), it is seen that $|x| = |e^{-z_1} - 1| < \frac{9.992}{10^{m_1}} < 0.9995$ for $m_1 \geq 1$. Choosing $a := 0.9995$, by Lemma 2.3, we get the inequality

$$|z_1| = |\log(x+1)| < \frac{\log(2000)}{0.9995}\cdot\frac{9.992}{10^{m_1}} < \frac{75.99}{10^{m_1}}.$$

Thus, it follows that

$$0 < \left|(m_1+m_2+m_3)\log 10 - n\log\alpha - \log\left(\frac{9}{\sqrt{5}d_1}\right)\right| < \frac{75.99}{10^{m_1}}.$$

Dividing this inequality by $\log\alpha$, we get

$$0 < \left|(m_1+m_2+m_3)\frac{\log 10}{\log\alpha} - n - \left(\frac{\log(9/\sqrt{5}d_1)}{\log\alpha}\right)\right| < 157.92\cdot 10^{-m_1}. \tag{18}$$

Now, we can apply Lemma 2.2. Put $\gamma := \dfrac{\log 10}{\log\alpha} \notin \mathbb{Q}$, $\mu := -\frac{\log(9/\sqrt{5}d_1)}{\log\alpha}$, $A := 157.92$, $B := 10$ and $w := m_1$. Let $M := 1.36\cdot 10^{45}$. Then $M > m_1+m_2+m_3$ and denominator of the 97$^{\text{th}}$ convergent of γ exceeds $6M$. Furthermore, $\epsilon := \|\mu q_{97}\| - M\|\gamma q_{97}\| > 0.12$. Thus, the inequality (18) has no solutions for $\dfrac{\log(Aq_{97}/\epsilon)}{\log B} < 49.62 < m_1$. So $m_1 \leq 49$. Now replace (13) by (17). Substituting this upper bound for m_1 into (17), we obtain $n < 1.12\cdot 10^{32}$. Now, let

$$z_2 := (m_2+m_3)\log 10 - n\log\alpha - \log\left(\frac{9}{(d_1 10^{m_1} - (d_1-d_2))\sqrt{5}}\right).$$

From (12), it is seen that $|x| = |e^{-z_2} - 1| < 1.11\cdot 10^{-m_2} < 0.2$ for $m_2 \geq 1$. Choosing $a := 0.2$, we get the inequality

$$|z_2| = |\log(x+1)| < \frac{\log(10/8)}{0.2}\cdot\frac{1.11}{10^{m_2}} < \frac{1.24}{10^{m_2}}$$

by Lemma 2.3. Thus, it follows that

$$0 < \left|(m_2+m_3)\log 10 - n\log\alpha - \log\left(\frac{9}{(d_1 10^{m_1}-(d_1-d_2))\sqrt{5}}\right)\right| < 1.24\cdot 10^{-m_2}.$$

Dividing both sides of the above inequality by $\log\alpha$, shows that

$$0 < \left|(m_2+m_3)\frac{\log 10}{\log\alpha} - n + \frac{\log\left(9/(d_1 10^{m_1}-(d_1-d_2))\sqrt{5}\right)}{\log\alpha}\right| < 2.57\cdot 10^{-m_2}. \quad (19)$$

Putting $\gamma := \dfrac{\log 10}{\log\alpha}$ and taking $m_2+m_3 < M := 5.57\cdot 10^{31}$, we find that q_{73}, the denominator of the 73^{rd} convergent of γ exceeds $6M$. Taking $\mu := \frac{\log\left(9/(d_1 10^{m_1}-(d_1-d_2))\sqrt{5}\right)}{\log\alpha}$ and considering the fact that $m_1 \le 49$, $d_1 \neq d_2$, $1 \le d_1 \le 9$ and $0 \le d_2 \le 9$, a quick computation with Mathematica gives us the equality $\epsilon = \epsilon(\mu) := \|\mu q_{73}\| - M\|\gamma q_{73}\| > 0.00001$. Let $A := 2.57$, $B := 10$ and $w := m_2$ in Lemma 2.2. Then with the help of Mathematica, we can say that the inequality (19) has no solution for $\frac{\log(Aq_{73}/\epsilon)}{\log B} < 39.51 < m_2$. Therefore $m_2 \le 42$. Substituting this upper bound for m_1 and m_2 into (17), we obtain $n < 1.15\cdot 10^{17}$. Now, let

$$z_3 := m_3\log 10 - n\log\alpha + \log\left(\frac{(d_1 10^{m_1+m_2}-(d_1-d_2)\,10^{m_2}-(d_2-d_3))\sqrt{5}}{9}\right).$$

From (16), one can write $|x| = |e^{z_3}-1| < 1.02\cdot\alpha^{-n} < 0.01$ for $n \ge 150$. Choosing $a := 0.01$, we get the inequality

$$|z_3| = |\log(x+1)| < \frac{\log(100/99)}{0.01}\cdot\frac{1.02}{\alpha^n} < \frac{1.03}{\alpha^n}$$

by Lemma 2.3. Thus, it follows that

$$0 < \left|m_3\log 10 - n\log\alpha + \log\left(\frac{d_1 10^{m_1+m_2}-(d_1-d_2)\,10^{m_2}-(d_2-d_3)}{9}\right)\right| < \frac{1.03}{\alpha^n}.$$

Dividing both sides of the above inequality by $\log\alpha$, we get

$$0 < \left|\frac{m_3\log 10}{\log\alpha} - n + \frac{\log\left((d_1 10^{m_1+m_2}-(d_1-d_2)\,10^{m_2}-(d_2-d_3))\sqrt{5}/9\right)}{\log\alpha}\right| < \frac{2.15}{\alpha^n}. \quad (20)$$

Putting $\gamma := \dfrac{\log 10}{\log\alpha}$ and taking $M := 1.15\cdot 10^{17}$, it is seen that $M > m_3$ and q_{48}, the denominator of the 48^{th} convergent of γ exceeds $6M$.

Taking $\mu := \frac{\log\left((d_1 10^{m_1+m_2}-(d_1-d_2)10^{m_2}-(d_2-d_3))\sqrt{5}/9\right)}{\log\alpha}$ and considering the fact that $m_1 \le 49$, $m_2 \le 39$, $1 \le d_1 \le 9$ and $0 \le d_2, d_3 \le 9$, a quick computation with Mathematica gives us the equality $\epsilon = \epsilon(\mu) := \|\mu q_{48}\| - M\|\gamma q_{48}\| > 0$ except for the case $d_1 = d_2 \neq d_3$, $d_1 \neq d_2 = d_3$ and $d_1 = d_2 = d_3$. Let $A := 2.15$, $B := \alpha$ and $w := n$ in Lemma 2.2. Then with the help of Mathematica, we can say that the inequality (20) has no solution for $\frac{\log(Aq_{48}/\epsilon)}{\log B} < 147.17 < n$. Therefore $n \le 147$. This contradicts our assumption that $n \ge 150$. This completes the proof. $\square$

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Sakarya University, Mathematics Department, Sakarya, Turkey

E-mail: erduvanmat@hotmail.com

Sakarya University, Mathematics Department, Sakarya, Turkey

E-mail: rkeskin@sakarya.edu.tr