

TWO-WEIGHTED INEQUALITIES FOR RIESZ POTENTIAL AND ITS COMMUTATORS IN GENERALIZED WEIGHTED MORREY SPACES

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Abstract. In this paper we find the conditions for the boundedness of Riesz potential I^α and its commutators from the generalized weighted Morrey spaces $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ to the generalized weighted Morrey spaces $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)$, where $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$, φ_1, φ_2 are generalized functions and $b \in BMO(\mathbb{R}^n)$. Furthermore, we give some applications of our results.

1. Introduction

Let $1 \leq p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and ω be a non-negative measurable function on \mathbb{R}^n . We denote by $\mathcal{M}_{\omega}^{p,\varphi}$ the generalized weighted Morrey space, the space of all functions $f \in L_{p,\omega}^{loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{\mathcal{M}_{\omega}^{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi(x, r) \|\omega\|_{L_p(B(x, r))}} \|f\|_{L_{p,\omega}(B(x, r))},$$

where the supremum is taken over all balls $B(x, r)$ in \mathbb{R}^n and $L_{p,\omega}(B(x, r))$ denotes the weighted L_p -space of measurable functions f for which

$$\|f\|_{L_{p,\omega}(B(x, r))} \equiv \|f \chi_{B(x, r)}\|_{L_{p,\omega}(\mathbb{R}^n)} = \left(\int_{B(x, r)} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}}.$$

Moreover, by $W\mathcal{M}_{\omega}^{p,\varphi}$ we denote the weak generalized weighted Morrey space of all functions $f \in WL_{p,\omega}^{loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{W\mathcal{M}_{\omega}^{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi(x, r) \|\omega\|_{L_p(B(x, r))}} \|f\|_{WL_{p,\omega}(B(x, r))},$$

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where $WL_{p,\omega}(B(x,r))$ denotes the weak weighted L_p -space of measurable functions f for which

$$\|f\|_{WL_{p,\omega}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,\omega}(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{y \in B(x,r): |f(y)|>t} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}}.$$

Note that if $\omega(x) = \chi_{B(x,r)}$, then $\mathcal{M}_{\omega}^{p,\varphi}(\mathbb{R}^n) = \mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ is the generalized Morrey space and if $\varphi(x,r) = r^{\frac{n-\lambda}{p}}$, then $\mathcal{M}_{\omega}^{p,\varphi}(\mathbb{R}^n) = L_{p,\lambda}(\omega)$ is the weighted Morrey space.

Let f be a locally integrable function on \mathbb{R}^n . The so-called fractional maximal function is defined by the formula

$$M^{\alpha} f(x) = \sup_{r>0} |B(x,r)|^{-1+\alpha/n} \int_{B(x,r)} |f(y)| dy, \quad 0 \leq \alpha < n,$$

where $|B(x,r)| = \omega_n r^n$ is the Lebesgue measure of the ball $B(x,r)$. It coincides with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$. Maximal operators play an important role in the differentiability properties of functions, singular integrals and partial differential equations. They often provide a deeper and more simplified approach to understanding problems in these areas than other methods.

Fractional maximal operator is intimately related to the Riesz potential

$$I^{\alpha} f(x) = \int_{\mathbb{R}^n} \frac{f(y) dy}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n.$$

The aim of this paper is to find the conditions for the boundedness of Riesz potential I^{α} and its commutators $[b, I^{\alpha}]$ and $|b, I^{\alpha}|$ from the generalized weighted Morrey spaces $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ to the spaces $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)$, where $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$, ω_1, ω_2 are generalized functions and $b \in BMO(\mathbb{R}^n)$. Furthermore, we give some applications of our main results.

In 2012, Guliyev gave a concept of generalized weighted Morrey space, which could be viewed as extension of both generalized Morrey space and weighted Morrey space [9]. In [9] Guliyev also obtained the boundedness of sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in generalized weighted Morrey spaces (see also [12, 15]).

In this paper we aim to give a characterization of two-weighted inequalities for Riesz potential and its commutators in generalized weighted Morrey spaces. Two-weight norm inequalities for the operators of harmonic analysis on various function spaces were widely studied (see, for example [5, 8, 16, 17, 20]). The weighted norm inequalities with different types of weights on Morrey spaces were also studied (see, for example [13, 22, 24]). The two-weight norm inequality for the Hardy-Littlewood maximal function on Morrey spaces was obtained in [28]. Two-weight norm inequalities on weighted Morrey spaces for fractional maximal operators and fractional integral operators were obtained in [23]. Two-weight norm inequalities on generalized weighted Morrey spaces for maximal, Calderón-Zygmund operators and their commutators were obtained in [3].

In the sequel we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence. If $p \in [1, \infty]$, the conjugate

number p' is defined by $pp' = p + p'$. $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+; \uparrow)$ stand for the set of Lebesgue-measurable functions on \mathbb{R}_+ , and its subspaces of non-negative and non-negative non-decreasing functions, respectively.

2. Background material

Even though the A_p class is well-known, we offer the definition of A_p weight functions.

DEFINITION 2.1. The weight function ω belongs to the class $A_p(\mathbb{R}^n)$, for $1 \leq p < \infty$, if the following

$$\sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \left(\int_{B(x, r)} \omega^p(y) dy \right)^{\frac{1}{p}} \left(\int_{B(x, r)} \omega^{-p'}(y) dy \right)^{\frac{1}{p'}}$$

is finite and ω belongs to $A_1(\mathbb{R}^n)$, if there exists a positive constant C such that for any $x \in \mathbb{R}^n$ and $r > 0$

$$|B(x, r)|^{-1} \int_{B(x, r)} \omega(y) dy \leq C \operatorname{ess\,sup}_{y \in B(x, r)} \frac{1}{\omega(y)}.$$

DEFINITION 2.2. The weight functions (ω_1, ω_2) belong to the class $\tilde{A}_p(\mathbb{R}^n)$, for $1 \leq p < \infty$, if the following supremum is finite

$$\sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \left(\int_{B(x, r)} \omega_2^p(y) dy \right)^{\frac{1}{p}} \left(\int_{B(x, r)} \omega_1^{-p'}(y) dy \right)^{\frac{1}{p'}}.$$

DEFINITION 2.3. The weight functions (ω_1, ω_2) belong to the class $A_{p,q}(\mathbb{R}^n)$, for $1 \leq p, q < \infty$, if the following supremum is finite

$$\sup_{x \in \mathbb{R}^n, r > 0} |B(x, r)|^{\frac{1}{p} - \frac{1}{q} - 1} \left(\int_{B(x, r)} \omega_2^q(y) dy \right)^{\frac{1}{q}} \left(\int_{B(x, r)} \omega_1^{-p'}(y) dy \right)^{\frac{1}{p'}}.$$

The following theorem was proved in [21].

THEOREM 2.4. *Let $1 \leq p < \infty$, then*

- (i) $M : L_{p,\varphi}(\mathbb{R}^n) \rightarrow L_{p,\varphi}(\mathbb{R}^n)$ if and only if $\varphi \in A_p(\mathbb{R}^n)$,
- (ii) $M : L_{1,\varphi}(\mathbb{R}^n) \rightarrow WL_{1,\varphi}(\mathbb{R}^n)$ if and only if $\varphi \in A_1(\mathbb{R}^n)$.

Let M^\sharp be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r > 0} |B(x, r)|^{-1} \int_{B(x, r)} |f(y) - f_{B(x, r)}| dy,$$

where $f_{B(x, r)}(x) = |B(x, r)|^{-1} \int_{B(x, r)} f(y) dy$.

DEFINITION 2.5. We define the $BMO(\mathbb{R}^n)$ space as the set of all locally integrable functions f such that

$$\|f\|_{BMO} = \sup_{x \in \mathbb{R}^n, r > 0} |B(x, r)|^{-1} \int_{B(x, r)} |f(y) - f_{B(x, r)}| dy < \infty$$

or
$$\|f\|_{BMO} = \inf_C \sup_{x \in \mathbb{R}^n, r > 0} |B(x, r)|^{-1} \int_{B(x, r)} |f(y) - C| dy < \infty.$$

DEFINITION 2.6. We define the $BMO_{p, \omega}(\mathbb{R}^n)$ ($1 \leq p < \infty$) space as the set of all locally integrable functions f such that

$$\|f\|_{BMO_{p, \omega}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{\|(f(\cdot) - f_{B(x, r)})\chi_{B(x, r)}\|_{L_{p, \omega}(\mathbb{R}^n)}}{\|\omega\|_{L_p(B(x, r))}} < \infty$$

or
$$\|f\|_{BMO_{p, \omega}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \|(f(\cdot) - f_{B(x, r)})\chi_{B(x, r)}\|_{L_{p, \omega}(\mathbb{R}^n)} \|\omega^{-1}\|_{L_{p'}(B(x, r))} < \infty.$$

THEOREM 2.7 ([18]). *Let $1 \leq p < \infty$ and ω be a Lebesgue measurable function. If $\omega \in A_p(\mathbb{R}^n)$, then the norms $\|\cdot\|_{BMO_{p, \omega}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

Before proving the main theorems, we need the following lemma.

LEMMA 2.8 ([14]). *Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that $|b_{B(x, r)} - b_{B(x, t)}| \leq C\|b\|_{BMO} \ln \frac{t}{r}$ for $0 < 2r < t$, where C is independent of b, x, r and t .*

Let $L_{\infty, v}(\mathbb{R}_+)$ be the weighted L_{∞} -space with the norm $\|g\|_{L_{\infty, v}(\mathbb{R}_+)} = \operatorname{ess\,sup}_{t > 0} v(t)g(t)$.

We denote $\mathbb{A} = \{\varphi \in \mathfrak{M}^+(\mathbb{R}_+; \uparrow) : \lim_{t \rightarrow 0^+} \varphi(t) = 0\}$. Let u be a continuous and non-negative function on \mathbb{R}_+ . We define the supremal operator \bar{S}_u by $(\bar{S}_u g)(t) := \|u g\|_{L_{\infty}(0, t)}$, $t \in (0, \infty)$.

The following theorem was proved in [4].

THEOREM 2.9 ([4]). *Suppose that v_1 and v_2 are non-negative measurable functions such that $0 < \|v_1\|_{L_{\infty}(0, t)} < \infty$ for every $t > 0$. Let u be a continuous non-negative function on \mathbb{R} . Then the operator \bar{S}_u is bounded from $L_{v_1}^{\infty}(\mathbb{R}_+)$ to $L_{v_2}^{\infty}(\mathbb{R}_+)$ on the cone \mathbb{A} if and only if $\left\|v_2 \bar{S}_u \left(\|v_1\|_{L_{\infty}(0, \cdot)}^{-1}\right)\right\|_{L_{\infty}(\mathbb{R}_+)} < \infty$.*

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_0^t g(s)w(s)ds, \quad H_w^* g(t) := \int_t^{\infty} g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

THEOREM 2.10 ([10]). *Let v_1, v_2 and w be weights on $(0, \infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname{ess\,sup}_{t > 0} v_2(t) H_w^* g(t) \leq C \operatorname{ess\,sup}_{t > 0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

THEOREM 2.11 ([10,11]). *Let v_1, v_2 and w be weights on $(0, \infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname{ess\,sup}_{t>0} v_2(t) H_w g(t) \leq C \operatorname{ess\,sup}_{t>0} v_1(t) g(t) \quad (1)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \operatorname{ess\,sup}_{t>0} v_2(t) \int_0^t \frac{w(s)ds}{\operatorname{ess\,sup}_{0<\tau<s} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (1).

3. Two-weighted inequalities for Riesz potential and its commutators in generalized weighted Morrey spaces

In this section we prove two-weighted inequalities for Riesz potential and its commutators in generalized weighted Morrey spaces.

THEOREM 3.1 ([25]). *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and the $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. Then the operator I^α is bounded from $L_{p,\omega_1}(\mathbb{R}^n)$ to $L_{q,\omega_2}(\mathbb{R}^n)$.*

From the inequality $M^\alpha f(x) \leq \omega_n^{\frac{\alpha}{n}-1} (I^\alpha)|f|(x)$, we get the following corollary.

COROLLARY 3.2. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. Then the operator M^α is bounded from $L_{p,\omega_1}(\mathbb{R}^n)$ to $L_{q,\omega_2}(\mathbb{R}^n)$.*

THEOREM 3.3. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for an arbitrary $f \in L_{p,\omega_1}(B(x,t))$ the inequality*

$$\|I^\alpha f\|_{L_{q,\omega_2}(B(x,t))} \leq C \|\omega_2\|_{L_q(B(x,t))} \int_t^\infty s^{\alpha-n-1} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_q(B(x,s))}} ds \quad (2)$$

holds.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{B(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\mathbb{R}^n \setminus B(x,2t)}(y), \quad t > 0, \quad (3)$$

and have $I^\alpha f(x) = I^\alpha f_1(x) + I^\alpha f_2(x)$.

By Theorem 3.1 we obtain

$$\|I^\alpha f_1\|_{L_{q,\omega_2}(B(x,t))} \leq \|I^\alpha f_1\|_{L_{q,\omega_2}(\mathbb{R}^n)} \leq C \|f_1\|_{L_{p,\omega_1}(\mathbb{R}^n)} = C \|f\|_{L_{p,\omega_1}(B(x,2t))}.$$

Then

$$\|I^\alpha f_1\|_{L_{q,\omega_2}(B(x,t))} \leq C \|f\|_{L_{p,\omega_1}(B(x,2t))},$$

where the constant C is independent of f . Taking into account that, we get

$$\|I^\alpha f_1\|_{L_{q,\omega_2}(B(x,t))} \leq C \|\omega_2\|_{L_q(B(x,t))} \int_t^\infty s^{\alpha-n-1} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_q(B(x,s))}} ds. \quad (4)$$

When $|x-z| \leq t$, $|z-y| \geq 2t$, we have $\frac{1}{2}|z-y| \leq |x-y| \leq \frac{3}{2}|z-y|$, and therefore

$$|I^\alpha f_2(z)| \leq \int_{\mathbb{R}^n \setminus B(x,2t)} |z-y|^{\alpha-n} |f(y)| dy \leq C \int_{\mathbb{R}^n \setminus B(x,2t)} |x-y|^{\alpha-n} |f(y)| dy.$$

We choose $\beta > \frac{n}{q}$ and obtain

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B(x,2t)} |x-y|^{\alpha-n} |f(y)| dy &= \beta \int_{\mathbb{R}^n \setminus B(x,2t)} |f(y)| \left(\int_{|x-y|}^\infty s^{\alpha-n-1} ds \right) dy \\ &= \beta \int_{2t}^\infty s^{\alpha-n-1} \left(\int_{\{y \in \mathbb{R}^n : 2t \leq |x-y| \leq s\}} |f(y)| dy \right) ds \\ &\leq C \int_{2t}^\infty s^{\alpha-n-1} \|\chi_{B(x,s)} \omega_1^{-1}\|_{L_{p'}(\mathbb{R}^n)} \|f\|_{L_{p,\omega_1}(B(x,s))} ds \\ &\leq C \int_t^\infty s^{\alpha-n-1} \|f\|_{L_{p,\omega_1}(B(x,s))} \|\omega_1^{-1}\|_{L_{p'}(B(x,s))} ds. \end{aligned}$$

Hence

$$\|I^\alpha f_2\|_{L_{q,\omega_2}(B(x,t))} \leq C \int_t^\infty s^{\alpha-n-1} \|f\|_{L_{p,\omega_1}(B(x,s))} \|\omega_1^{-1}\|_{L_{p'}(B(x,s))} ds \|\chi_{B(x,t)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}.$$

Therefore we get

$$\|I^\alpha f_2\|_{L_{q,\omega_2}(B(x,t))} \leq C \|\omega_2\|_{L_q(B(x,t))} \int_t^\infty s^{\alpha-n-1} \|f\|_{L_{p,\omega_1}(B(x,s))} \|\omega_2\|_{L_q(B(x,s))}^{-1} ds \quad (5)$$

which together with (4) yields (2). \square

THEOREM 3.4. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. Let the functions $\varphi_1(x, r)$ and $\varphi_2(x, r)$ fulfill the condition*

$$\int_t^\infty s^{\alpha-n-1} \frac{\text{ess inf}_{s < r < \infty} \varphi_1(x, r) \|\omega_1\|_{L^p(B(x,r))}}{\|\omega_2\|_{L^q(B(x,s))}} ds \leq C \varphi_2(x, t). \quad (6)$$

Then the operator I^α is bounded from $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$. From the definition of the norm of generalized weighted Morrey spaces we write

$$\|I^\alpha f\|_{\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, t > 0} \frac{1}{\varphi_2(x, t) \|\omega_2\|_{L_q(B(x,t))}} \|I^\alpha f \chi_{B(x,t)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}. \quad (7)$$

We estimate $\|I^\alpha f \chi_{B(x,t)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}$ in (7) by means of Theorems 3.3, 2.11 and obtain

$$\|I^\alpha f\|_{\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)} \leq C \sup_{x \in \mathbb{R}^n, t > 0} \frac{\|\omega_2\|_{L_q(B(x,t))}}{\varphi_2(x, t) \|\omega_2\|_{L_q(B(x,t))}} \int_t^\infty s^{\alpha-n-1} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_q(B(x,s))}} ds$$

$$\leq C \sup_{x \in \mathbb{R}^n, t > 0} \frac{1}{\varphi_1(x, t) \|\omega_1\|_{L_p(B(x, t))}} \|f\|_{L_{p, \omega_1}(B(x, t))} = C \|f\|_{\mathcal{M}_{\omega_1}^{p, \varphi_1}(\mathbb{R}^n)}.$$

It remains to make use of condition (6). \square

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In this section we consider commutators of the Riesz potential defined by the following equality

$$[b, I^\alpha]f(x) = \int_{\mathbb{R}^n} (b(x) - b(y)) |x - y|^{\alpha-n} f(y) dy, \quad 0 < \alpha < n.$$

Given a measurable function b the operator $|b, I^\alpha|$ is defined by

$$|b, I^\alpha|f(x) = \int_{\mathbb{R}^n} |b(x) - b(y)| |x - y|^{\alpha-n} |f(y)| dy, \quad 0 < \alpha < n.$$

The maximal commutator is defined by

$$M_b(f)(x) := \sup_{r > 0} |B(x, r)|^{-1} \int_{B(x, r)} |b(x) - b(y)| |f(y)| dy$$

for all $x \in \mathbb{R}^n$.

LEMMA 3.5 ([6]). *Let $b \in BMO(\mathbb{R}^n)$, $1 < s < \infty$. Then*

$$M^\sharp(|b, I^\alpha|f(x)) \leq C \|b\|_{BMO} \left[(M|I^\alpha f(x)|^s)^{\frac{1}{s}} + (M^{s\alpha}|f(x)|^s)^{\frac{1}{s}} \right],$$

where $C > 0$ is independent of f and x .

LEMMA 3.6 ([27]). *Let $1 < p < \infty$ and $\omega \in A_p(\mathbb{R}^n)$. Then $\|f\|_{L_{p, \omega}} \leq C \|M^\sharp f\|_{L_{p, \omega}}$ with a constant $C > 0$ not depending on f .*

THEOREM 3.7 ([3]). *Let $1 < p < \infty$, $(\omega_1, \omega_2) \in \tilde{A}_p(\mathbb{R}^n)$ and the function $\varphi_1(x, r)$ and $\varphi_2(x, r)$ satisfy the condition*

$$\sup_{t > r} \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) \|\omega_1\|_{L_p(B(x, s))}}{\|\omega_2\|_{L_p(B(x, t))}} \leq C \varphi_2(x, r), \quad (8)$$

where C does not depend on x and t .

Then the operator M is bounded from the space $\mathcal{M}_{\omega_1}^{p, \varphi_1}(\mathbb{R}^n)$ to the space $\mathcal{M}_{\omega_2}^{p, \varphi_2}(\mathbb{R}^n)$.

THEOREM 3.8 ([3]). *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ and $(\omega_1, \omega_2) \in \tilde{A}_p(\mathbb{R}^n)$, $\omega_1 \in A_p(\mathbb{R}^n)$, then the operator M_b is bounded from $L_{p, \omega_1}(\mathbb{R}^n)$ to $L_{p, \omega_2}(\mathbb{R}^n)$.*

THEOREM 3.9 ([3]). *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ and $(\omega_1, \omega_2) \in \tilde{A}_p(\mathbb{R}^n)$, $\omega_1, \omega_2 \in A_p(\mathbb{R}^n)$. If the functions $\varphi_1(x, r)$ and $\varphi_2(x, r)$ satisfy the condition*

$$\sup_{t > r} \left(1 + \ln \frac{t}{r} \right) \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) \|\omega_1\|_{L_p(B(x, s))}}{\|\omega_2\|_{L_p(B(x, t))}} \leq C \varphi_2(x, r), \quad (9)$$

where C does not depend on x and t .

Then the operator M_b is bounded from the space $\mathcal{M}_{\omega_1}^{p, \varphi_1}(\mathbb{R}^n)$ to the space $\mathcal{M}_{\omega_2}^{p, \varphi_2}(\mathbb{R}^n)$.

THEOREM 3.10. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$ and $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$, $\omega_1 \in A_p(\mathbb{R}^n)$, $\omega_2 \in A_q(\mathbb{R}^n)$. The operator $|b, I^\alpha|$ is bounded from $L_{p,\omega_1}(\mathbb{R}^n)$ to $L_{q,\omega_2}(\mathbb{R}^n)$.*

Proof. Let $f \in L_{p,\omega_1}(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$. From Lemma 3.6 we have

$$\| |b, I^\alpha| f \|_{L_{q,\omega_2}(\mathbb{R}^n)} \leq C \| M^\sharp([b, I^\alpha]f) \|_{L_{q,\omega_2}(\mathbb{R}^n)}.$$

From Lemma 3.5, we have

$$\begin{aligned} \| M^\sharp(|b, I^\alpha|f) \|_{L_{q,\omega_2}(\mathbb{R}^n)} &\leq C \| b \|_{BMO} \left\| (M|I^\alpha f|^s)^{\frac{1}{s}} + (M^{\alpha s}|f|^s)^{\frac{1}{s}} \right\|_{L_{q,\omega_2}(\mathbb{R}^n)} \\ &\leq C \| b \|_{BMO} \left[\left\| (M|I^\alpha f|^s)^{\frac{1}{s}} \right\|_{L_{q,\omega_2}(\mathbb{R}^n)} + \left\| (M^{\alpha s}|f|^s)^{\frac{1}{s}} \right\|_{L_{q,\omega_2}(\mathbb{R}^n)} \right]. \end{aligned}$$

From Theorem 2.4 and Theorem 3.1, we get

$$\left\| (M|I^\alpha f|^s)^{\frac{1}{s}} \right\|_{L_{q,\omega_2}(\mathbb{R}^n)} \leq C \| I^\alpha f \|_{L_{q,\omega_2}(\mathbb{R}^n)} \leq C \| f \|_{L_{p,\omega_1}}.$$

From Corollary 3.2, we obtain

$$\left\| (M^{\alpha s}|f|^s)^{\frac{1}{s}} \right\|_{L_{q,\omega_2}(\mathbb{R}^n)} \leq C \| f \|_{L_{p,\omega_1}}.$$

Therefore we get

$$\| [b, I^\alpha]f \|_{L_{q,\omega_2}(\mathbb{R}^n)} \leq C \| b \|_{BMO} \| f \|_{L_{p,\omega_1}}.$$

Thus the theorem has been proved. \square

THEOREM 3.11. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$ and $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$, $\omega_1 \in A_p(\mathbb{R}^n)$, $\omega_2 \in A_q(\mathbb{R}^n)$. Then*

$$\begin{aligned} \| |b, I^\alpha| f \|_{L_{q,\omega_2}(B(x,t))} &\leq \\ C \| b \|_{BMO} \| \omega_2 \|_{L_q(B(x,t))} &\times \int_t^\infty \left(1 + \ln \frac{s}{t} \right) \| f \|_{L_{p,\omega_1}(B(x,s))} \| \omega_2 \|_{L_q(B(x,s))}^{-1} \frac{ds}{s}, \quad (10) \end{aligned}$$

where $t > 0$, C does not depend on f , x and t .

Proof. We represent f as (3) and have $|b, I^\alpha|f(x) \leq |b, I^\alpha|f_1(x) + |b, I^\alpha|f_2(x)$.

By Theorem 3.10 we obtain

$$\begin{aligned} \| |b, I^\alpha| f_1 \|_{L_{q,\omega_2}(B(x,t))} &\leq \| |b, I^\alpha| f_1 \|_{L_{q,\omega_2}(\mathbb{R}^n)} \\ &\leq C \| b \|_{BMO} \| f_1 \|_{L_{p,\omega_1}(\mathbb{R}^n)} = C \| b \|_{BMO} \| f \|_{L_{p,\omega_1}(B(x,2t))}. \end{aligned}$$

Then

$$\| |b, I^\alpha| f_1 \|_{L_{q,\omega_2}(B(x,t))} \leq C \| b \|_{BMO} \| f \|_{L_{p,\omega_1}(B(x,2t))},$$

where the constant C is independent of f . Taking into account that, we get

$$\begin{aligned} \| |b, I^\alpha| f_1 \|_{L_{q,\omega_2}(B(x,t))} &\leq \\ C \| b \|_{BMO} \| \omega_2 \|_{L_q(B(x,t))} &\times \int_t^\infty \left(1 + \ln \frac{s}{t} \right) \frac{\| f \|_{L_{p,\omega_1}(B(x,s))} ds}{\| \omega_2 \|_{L_q(B(x,s))} s}. \quad (11) \end{aligned}$$

When $|x - z| \leq t$, $|z - y| \geq 2t$, we have $\frac{1}{2}|z - y| \leq |x - y| \leq \frac{3}{2}|z - y|$. Therefore we get

$$\begin{aligned} |b, I^\alpha|f_2(z) &\leq \int_{\mathbb{R}^n \setminus B(x, 2t)} |b(y) - b(z)| |z - y|^{\alpha-n} |f(y)| dy \\ &\leq C \int_{\mathbb{R}^n \setminus B(x, 2t)} |b(y) - b(z)| |x - y|^{\alpha-n} |f(y)| dy. \end{aligned}$$

We obtain

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B(x, 2t)} |b(y) - b(z)| |x - y|^{\alpha-n} |f(y)| dy \\ &= \int_{\mathbb{R}^n \setminus B(x, 2t)} |b(y) - b(z)| |f(y)| \left(\int_{|x-y|}^{\infty} s^{\alpha-n-1} ds \right) dy \\ &\leq C \int_{2t}^{\infty} s^{\alpha-n-1} \left(\int_{\{y \in \mathbb{R}^n : 2t \leq |x-y| \leq s\}} |b(y) - b(z)| |f(y)| dy \right) ds \\ &\leq C \int_{2t}^{\infty} s^{\alpha-n-1} \left(\int_{\{y \in \mathbb{R}^n : 2t \leq |x-y| \leq s\}} |b(y) - b_{B(x,t)}| |f(y)| dy \right) ds \\ &+ C |b(z) - b_{B(x,t)}| \int_{2t}^{\infty} s^{\alpha-n-1} \left(\int_{\{y \in \mathbb{R}^n : 2t \leq |x-y| \leq s\}} |f(y)| dy \right) ds = J_1 + J_2. \end{aligned}$$

To estimate J_1 :

$$\begin{aligned} J_1 &= C \int_{2t}^{\infty} s^{\alpha-n-1} \left(\int_{\{y \in \mathbb{R}^n : 2t \leq |x-y| \leq s\}} |b(y) - b_{B(x,t)}| |f(y)| dy \right) ds \\ &\leq C \int_t^{\infty} s^{\alpha-n-1} \|b(\cdot) - b_{B(x,s)}\|_{L_{p', \omega^{-1}}(B(x,s))} \|f\|_{L_{p, \omega}(B(x,s))} ds \\ &+ C \int_t^{\infty} s^{\alpha-n-1} |b_{B(x,t)} - b_{B(x,s)}| \left(\int_{B(x,s)} |f(y)| dy \right) ds \\ &\leq C \|b\|_{BMO} \int_t^{\infty} s^{\alpha-n-1} \|\omega^{-1}\|_{L_{p'}(B(x,s))} \|f\|_{L_{p, \omega}(B(x,s))} ds \\ &+ C \|b\|_{BMO} \int_t^{\infty} s^{\alpha-n-1} \ln \frac{s}{t} \|\omega^{-1}\|_{L_{p'}(B(x,s))} \|f\|_{L_{p, \omega}(B(x,s))} ds \\ &\leq C \|b\|_{BMO} \int_t^{\infty} \left(1 + \ln \frac{s}{t}\right) \|\omega_2\|_{L_q(B(x,s))}^{-1} \|f\|_{L_{p, \omega}(B(x,s))} \frac{ds}{s}. \end{aligned} \quad (12)$$

To estimate J_2 :

$$\begin{aligned} J_2 &= C |b(z) - b_{B(x,t)}| \int_{2t}^{\infty} s^{\alpha-n-1} \left(\int_{\{y \in \mathbb{R}^n : 2t \leq |x-y| \leq s\}} |f(y)| dy \right) ds \\ &\leq C |B(x,t)|^{-1} \int_{B(x,t)} |b(z) - b(y)| dy \int_t^{\infty} s^{\alpha-n} \|\omega^{-1}\|_{L_{p'}(B(x,s))} \|f\|_{L_{p, \omega}(B(x,s))} \frac{ds}{s} \end{aligned}$$

$$\leq CM_b \chi_{B(x,t)}(z) \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|\omega_2\|_{L_q(B(x,s))}^{-1} \|f\|_{L_{p,\omega}(B(x,s))} \frac{ds}{s}, \quad (13)$$

where C does not depend on x, t . Then by Theorem 3.8 and (12), (13) we have

$$\begin{aligned} & \| |b, I^\alpha| f_2 \|_{L_{q,\omega_2}(B(x,t))} \leq \|J_1\|_{L_{q,\omega_2}(B(x,t))} + \|J_2\|_{L_{q,\omega_2}(B(x,t))} \\ & \leq C \|b\|_{BMO} \|\chi_{B(x,t)}\|_{L_{q,\omega_2}} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|\omega_2\|_{L_q(B(x,s))}^{-1} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s} \\ & \quad + C \|M_b \chi_{B(x,t)}\|_{L_{q,\omega_2}} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|\omega_2\|_{L_q(B(x,s))}^{-1} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s} \\ & \leq C \|b\|_{BMO} \|\omega_2\|_{L_q(B(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|\omega_2\|_{L_q(B(x,s))}^{-1} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s} \\ & \quad + C \|b\|_{BMO} \|\omega_2\|_{L_q(B(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|\omega_2\|_{L_q(B(x,s))}^{-1} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s} \\ & \leq C \|b\|_{BMO} \|\omega_2\|_{L_q(B(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|\omega_2\|_{L_q(B(x,s))}^{-1} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s}. \end{aligned}$$

Hence

$$\begin{aligned} & \| |b, I^\alpha| f_2 \|_{L_{q,\omega_2}(B(x,t))} \leq \\ & C \|b\|_{BMO} \|\omega_2\|_{L_q(B(x,t))} \times \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|\omega_2\|_{L_q(B(x,s))}^{-1} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s}, \end{aligned}$$

which together with (11) yields (10). \square

In the following two theorems we prove the boundedness of commutators of the Riesz potential operator $|b, I^\alpha|$ from the generalized weighted Morrey spaces $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ to the generalized weighted Morrey spaces $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)$. We find conditions on the functions $\varphi_1(x, r)$ and $\varphi_2(x, r)$ for the boundedness of $|b, I^\alpha|$ from $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)$.

THEOREM 3.12. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$, $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$, $\omega_1 \in A_p(\mathbb{R}^n)$, $\omega_2 \in A_q(\mathbb{R}^n)$ and the functions $\varphi_1(x, r)$ and $\varphi_2(x, r)$ fulfill the condition*

$$\int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{\operatorname{ess\,inf}_{s < t < 1} \varphi_1(x, r) \|\omega_1\|_{L_p(B(x,r))}}{\|\omega_2\|_{L_q(B(x,s))}} \frac{ds}{s} \leq C \varphi_2(x, t). \quad (14)$$

Then the operator $|b, I^\alpha|$ is bounded from $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$. From the definition of norm of generalized weighted Morrey spaces we write

$$\| |b, I^\alpha| f \|_{\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi_2(x, t) \|\omega_2\|_{L_q(B(x,t))}} \| |b, I^\alpha| f \chi_{B(x,r)} \|_{L_{q,\omega_2}(\mathbb{R}^n)}. \quad (15)$$

We estimate $\| |b, I^\alpha| f \chi_{B(x,r)} \|_{L_{q,\omega_2}(\mathbb{R}^n)}$ in (15) by means of Theorems 3.11, 2.11 and obtain

$$\| |b, I^\alpha| f \|_{\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)}$$

$$\begin{aligned}
&\leq C \|b\|_{BMO} \sup_{x \in \mathbb{R}^n, t > 0} \frac{\|\omega_2\|_{L_q(B(x,t))}}{\varphi_2(x,t) \|\omega_2\|_{L_q(B(x,t))}} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{\|f\|_{L_{p,\omega}(B(x,s))}}{\|\omega_2\|_{L_{p'}(B(x,s))}} \frac{ds}{s} \\
&\leq C \|b\|_{BMO} \sup_{x \in \mathbb{R}^n, t > 0} \frac{1}{\varphi_1(x,t) \|\omega_1\|_{L_p(B(x,t))}} \|f\|_{L_{p,\omega_1}(B(x,t))} = C \|b\|_{BMO} \|f\|_{\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)}.
\end{aligned}$$

It remains to make use of condition (14). \square

4. Some applications

In this section we give some applications of our main results. We apply the theorems of Section 3 to the operators which are estimated from above by Riesz potentials. Now we give some examples.

Let $0 < \alpha < n$. The fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}},$$

where L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x, y)$ satisfying the Gaussian upper bound, that is,

$$|p_t(x, y)| \leq \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \quad (16)$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, $c_1, c_2 > 0$ are independent of x, y and t .

If $L = -\Delta$ is the Laplacian on \mathbb{R}^n , then $L^{-\alpha/2}$ is the Riesz potential I^α (see [27]).

The following theorem states the boundedness of the operator $L^{-\alpha/2}$ from the spaces $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ to the spaces $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)$.

THEOREM 4.1. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. Let also (φ_1, φ_2) satisfy the condition (6). Then the operator $L^{-\alpha/2}$ is bounded from $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)$.*

Proof. From the condition (16), it follows that $|L^{-\alpha/2} f(x)| \leq CI^\alpha |f|(x)$ for all $x \in \mathbb{R}^n$ (see [7]).

Since the semigroup e^{-tL} has the kernel $p_t(x, y)$, from the above mentioned theorems we have $\|L^{-\alpha/2} f\|_{\mathcal{M}_{\omega_2}^{q,\varphi_2}} \leq C \|I^\alpha |f|\|_{\mathcal{M}_{\omega_2}^{q,\varphi_2}} \leq C \|f\|_{\mathcal{M}_{\omega_1}^{p,\varphi_1}}$, where the constant $C > 0$ is independent of f . \square

Various classes of differential operators also satisfy the inequality (16). Two of these operators are considered here:

(i) First we consider the magnetic potential \vec{a} , i.e., a real-valued vector potential $\vec{a} = (a_1, a_2, \dots, a_n)$, and an electric potential V . Let us assume that for any $k = 1, 2, \dots, n$, $a_k \in L_2^{loc}$ and $0 \leq V \in L_1^{loc}$. The operator L , which is given by $L = -(\nabla - i\vec{a})^2 + V(x)$ is called the magnetic Schrödinger operator.

From the well-known diamagnetic inequality (see [26]) we have the following pointwise estimate. For any $t > 0$ and $f \in L_2$, the following inequality $|e^{-tL} f| \leq e^{-t\Delta} |f|$

holds, which illustrates that the semigroup e^{-tL} has the kernel $p_t(x, y)$ that satisfies upper bound (16).

Furthermore, note that under the appropriate assumptions (see [2, 19, 27]) we can obtain similar results with Theorem 4.1 for a homogeneous elliptic operator L in L_2 of order $2m$ in the divergence form $Lf = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha (a_{\alpha\beta} D^\beta f)$. In this case

estimate (16) should be replaced by $|p_t(x, y)| \leq \frac{c_3}{t^{n/2m}} e^{-c_4 \left(\frac{|x-y|}{t^{1/(2m)}}\right)^{2m/(2m-1)}}$ for all $t > 0$ and all $x, y \in \mathbb{R}^n$.

(ii) Now let $A = (a_{ij}(x))_{1 \leq i, j \leq n}$ be an $n \times n$ matrix with complex-valued entries $a_{ij} \in L_\infty$ satisfying $\operatorname{Re} \sum_{i, j=1}^n a_{ij}(x) \zeta_i \zeta_j \geq \lambda |\zeta|^2$, for all $x \in \mathbb{R}^n$, $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{C}^n$ and some $\lambda > 0$. Consider the divergence form operator $Lf \equiv -\operatorname{div}(A \nabla f)$, which is interpreted in the usual weak sense via the appropriate sesquilinear form.

It is well known that the Gaussian bound (16) for the kernel of e^{-tL} holds when A has real-valued entries (see, for example, [1]), or when $n = 1, 2$ in the case of complex-valued entries (see [2, Chapter 1]). Therefore we can obtain similar results with Theorem 4.1.

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