

ON SOLVABILITY OF QUADRATIC HAMMERSTEIN INTEGRAL EQUATIONS IN HÖLDER SPACES

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Abstract. Using Schauder's fixed point theorem we consider the solvability of a quadratic Hammerstein integral equation in the space of functions satisfying a Hölder condition. An example is included to illustrate our results.

1. Introduction

In this paper, we investigate the existence of solutions of the following quadratic integral equation of Hammerstein type

$$x(t) = p(t) + x(t) \int_0^1 k(t, \tau) f(\tau, (\Lambda x)(\tau)) d\tau, \quad t \in [0, 1], \quad (1)$$

where Λ is a general operator.

If $f(t, y) = y$ we get an equation in [13], if $f(t, y) = y$ and $\Lambda x = \max\{|x(\tau)| : 0 \leq \tau \leq r(t)\}$, where $r : [0, 1] \rightarrow [0, 1]$ is a continuous and nondecreasing function we obtain an equation studied in [6] and if $f(t, y) = y$ and $(\Lambda x)(t) = x(r(t))$, where $r : [0, 1] \rightarrow [0, 1]$ is a measurable function, we obtain an equation studied in [5]. When $\Lambda y = y$ and $f(t, y) = -y$, (1) becomes

$$x(t) + x(t) \int_0^1 k(t, \tau) x(\tau) d\tau = p(t), \quad t \in [0, 1].$$

This equation is a generalization of a famous equation in transport theory, the so-called Chandrasekhar H -equation in which $p(t) = 1$, x must be identified with the H -function and for a nonnegative characteristic function ϕ , $k(t, \tau) = \frac{t\phi(t)}{t+\tau}$; see for example [7, 10, 12] and the references therein. Quadratic integral equations arise in the theory of radiative transfer, in the theory of neutron transport and in the theory of traffic; see [1, 4, 8, 9, 11, 14] and the references therein.

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In the space of functions satisfying a Hölder condition, Schauder's fixed point theorem and the relative compactness in these spaces are the main tools used to prove our main result.

2. Preliminaries

We denote by $C[a, b]$ the space of all continuous functions $x : [a, b] \rightarrow \mathbb{R}$ equipped with the norm $\|x\|_\infty = \sup\{|x(t)| : t \in [a, b]\}$ for $x \in C[a, b]$. Let $H_\alpha[a, b]$, $\alpha \in (0, 1]$ be the collection of all real functions x defined on $[a, b]$ which satisfies a Hölder condition

$$|x(t) - x(\tau)| \leq H_x^\alpha |t - \tau|^\alpha, \quad \forall (t, \tau) \in [a, b]^2, \quad (2)$$

where H_x^α is the least possible constant for which inequality (2) is satisfied, i.e.,

$$H_x^\alpha = \sup \left\{ \frac{|x(t) - x(\tau)|}{|t - \tau|^\alpha} : t, \tau \in [a, b], t \neq \tau \right\}.$$

The spaces $H_\alpha[a, b]$, $0 < \alpha \leq 1$, equipped with the norm $\|x\|_\alpha = \|x\|_\infty + H_x^\alpha$ are Banach spaces.

LEMMA 2.1. *The norm $\|\cdot\|_\infty$ is dominated by the norm $\|\cdot\|_\alpha$, i.e., for an arbitrarily fixed $x \in H_\alpha[a, b]$ and for an arbitrary $t \in [a, b]$, the following inequality holds $\|x\|_\infty \leq \max\{1, (b-a)^\alpha\} \|x\|_\alpha$.*

LEMMA 2.2. *For $0 < \alpha < \beta \leq 1$, we have $H_\beta[a, b] \subset H_\alpha[a, b] \subset C[a, b]$. Moreover, for $x \in H_\beta[a, b]$ the following inequality is satisfied $\|x\|_\alpha \leq \max\{1, (b-a)^{\beta-\alpha}\} \|x\|_\beta$.*

The authors in [2] established a sufficient condition for relative compactness in the spaces $H_\alpha[a, b]$, $\alpha \in (0, 1]$.

THEOREM 2.3. *Let $0 < \alpha < \beta \leq 1$ and let B be a bounded subset in $H_\beta[a, b]$ (this means that $\|x\|_\beta \leq M$ for certain constant $M > 0$, for any $x \in B$). Then B is a relatively compact subset of $H_\alpha[a, b]$.*

3. Main results

In this section we discuss the solvability of (1) in Hölder spaces.

We assume the following are satisfied.

- (a1) $p \in H_\beta[0, 1]$, $0 < \beta \leq 1$.
- (a2) The function $k : [0, 1]^2 \rightarrow \mathbb{R}$ is a continuous function and there exists a constant $\kappa_\beta > 0$ such that $|k(t, \tau) - k(s, \tau)| \leq \kappa_\beta |t - s|^\beta$ for any $t, \tau, s \in [0, 1]$.
- (a3) The function $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists a nondecreasing function $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $|f(t, x)| \leq \Psi(|x|)$, $\forall (t, x) \in ([0, 1], \mathbb{R})$.
- (a4) The operator $\Lambda : H_\beta[0, 1] \rightarrow C[0, 1]$ is continuous and there exists a nondecreasing function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for any $x \in H_\beta[0, 1]$, $\|\Lambda x\|_\infty \leq \psi(\|x\|_\beta)$.

(a5) Let r be a positive solution of the following equation $\|p\|_\beta + (K + \kappa_\beta)\Psi(\psi(r))r \leq r$, where $K = \sup \left\{ \int_0^1 |k(t, \tau)| d\tau : t \in [0, 1] \right\}$.

THEOREM 3.1. *Under assumptions (a1)–(a5), (1) has at least one solution $x \in H_\alpha[0, 1]$ (here α is an arbitrarily fixed number satisfying $0 < \alpha < \beta$).*

Proof. Consider the operator \mathfrak{T} defined on $H_\beta[0, 1]$ by

$$(\mathfrak{T}x)(t) = p(t) + x(t) \int_0^1 k(t, \tau) f(\tau, (\Lambda x)(\tau)) d\tau, \quad t \in [0, 1].$$

We claim that \mathfrak{T} maps the space $H_\beta[0, 1]$ into itself. Take $x \in H_\beta[0, 1]$ and $t, s \in [0, 1]$ with $t \neq s$. Then, by assumptions (a1) and (a2), we have

$$\begin{aligned} & \frac{|(\mathfrak{T}x)(t) - (\mathfrak{T}x)(s)|}{|t-s|^\beta} \\ &= \frac{\left| p(t) + x(t) \int_0^1 k(t, \tau) f(\tau, \Lambda x(\tau)) d\tau - p(s) - x(s) \int_0^1 k(s, \tau) f(\tau, \Lambda x(\tau)) d\tau \right|}{|t-s|^\beta} \\ &\leq \frac{|p(t) - p(s)|}{|t-s|^\beta} + \frac{\left| x(t) \int_0^1 k(t, \tau) f(\tau, \Lambda x(\tau)) d\tau - x(s) \int_0^1 k(t, \tau) f(\tau, \Lambda x(\tau)) d\tau \right|}{|t-s|^\beta} \\ &\quad + \frac{\left| x(s) \int_0^1 k(t, \tau) f(\tau, \Lambda x(\tau)) d\tau - x(s) \int_0^1 k(s, \tau) f(\tau, \Lambda x(\tau)) d\tau \right|}{|t-s|^\beta} \\ &\leq \frac{|p(t) - p(s)|}{|t-s|^\beta} + \frac{|x(t) - x(s)|}{|t-s|^\beta} \int_0^1 |k(t, \tau)| |f(\tau, \Lambda x(\tau))| d\tau \\ &\quad + \frac{|x(s)|}{|t-s|^\beta} \int_0^1 |k(t, \tau) - k(s, \tau)| |f(\tau, \Lambda x(\tau))| d\tau \\ &\leq \frac{|p(t) - p(s)|}{|t-s|^\beta} + \frac{|x(t) - x(s)|}{|t-s|^\beta} \Psi(\|\Lambda x\|_\infty) \int_0^1 |k(t, \tau)| d\tau \\ &\quad + \frac{\|x\|_\infty \Psi(\|\Lambda x\|_\infty)}{|t-s|^\beta} \int_0^1 |k(t, \tau) - k(s, \tau)| d\tau \\ &\leq \frac{|p(t) - p(s)|}{|t-s|^\beta} + K \Psi(\psi(\|x\|_\beta)) \frac{|x(t) - x(s)|}{|t-s|^\beta} + \kappa_\beta \Psi(\psi(\|x\|_\beta)) \|x\|_\infty \frac{\int_0^1 |t-s|^\beta d\tau}{|t-s|^\beta}. \end{aligned}$$

Thus $H_{\mathfrak{T}x}^\beta \leq H_p^\beta + (KH_x^\beta + \kappa_\beta \|x\|_\beta) \Psi(\psi(\|x\|_\beta))$, so

$$\begin{aligned} \|\mathfrak{T}x\|_\beta &= |(\mathfrak{T}x)(0)| + H_{\mathfrak{T}x}^\beta \\ &\leq |p(0)| + |x(0)| \int_0^1 |k(0, \tau)| |f(\tau, \Lambda x(\tau))| d\tau + H_p^\beta + (KH_x^\beta + \kappa_\beta \|x\|_\beta) \Psi(\psi(\|x\|_\beta)) \\ &\leq \|p\|_\beta + K|x(0)| \Psi(\psi(\|x\|_\beta)) + (KH_x^\beta + \kappa_\beta \|x\|_\beta) \Psi(\psi(\|x\|_\beta)) \\ &\leq \|p\|_\beta + (K + \kappa_\beta) \|x\|_\beta \Psi(\psi(\|x\|_\beta)). \end{aligned} \tag{3}$$

This proves that the operator \mathfrak{T} maps $H_\beta[0, 1]$ into itself.

Using assumption (a5) and inequality (3), we deduce that \mathfrak{T} maps the closed ball $B_{r_0}^\beta = \{x \in H_\beta[0, 1] : \|x\|_\beta \leq r_0\}$ into itself, for any r_0 satisfying $\|p\|_\beta + (K + \kappa_\beta) r_0 \Psi(\psi(r_0)) \leq r_0$. Theorem 2.3 guarantees that the set $B_{r_0}^\beta$ is relatively compact in $H_\alpha[0, 1]$ for any $0 < \alpha < \beta \leq 1$. Moreover, it is easy to see that $B_{r_0}^\beta$ is a compact subset in $H_\alpha[0, 1]$ for any $0 < \alpha < \beta \leq 1$; see the Appendix in [5].

We now prove that the operator \mathfrak{T} is continuous on $B_{r_0}^\beta$ with respect to the norm $\|\cdot\|_\alpha$, where $0 < \alpha < \beta \leq 1$. Fix $\varepsilon > 0$ and take $x, y \in B_{r_0}^\beta$ with $\|x - y\|_\alpha \leq \varepsilon$. Then, for $t \in [0, 1]$, we have

$$\begin{aligned}
& \frac{|[(\mathfrak{T}x)(t) - (\mathfrak{T}y)(t)] - [(\mathfrak{T}x)(s) - (\mathfrak{T}y)(s)]|}{|t-s|^\alpha} \\
&= \frac{1}{|t-s|^\alpha} \left| \left(x(t) \int_0^1 k(t, \tau) f(\tau, \Lambda x(\tau)) d\tau - y(t) \int_0^1 k(t, \tau) f(\tau, \Lambda y(\tau)) d\tau \right) \right. \\
&\quad \left. - \left(x(s) \int_0^1 k(s, \tau) f(\tau, \Lambda x(\tau)) d\tau - y(s) \int_0^1 k(s, \tau) f(\tau, \Lambda y(\tau)) d\tau \right) \right| \\
&= \frac{1}{|t-s|^\alpha} \left| \left(x(t) \int_0^1 k(t, \tau) f(\tau, \Lambda x(\tau)) d\tau - y(t) \int_0^1 k(t, \tau) f(\tau, \Lambda x(\tau)) d\tau \right) \right. \\
&\quad + \left(y(t) \int_0^1 k(t, \tau) f(\tau, \Lambda x(\tau)) d\tau - y(t) \int_0^1 k(t, \tau) f(\tau, \Lambda y(\tau)) d\tau \right) \\
&\quad - \left(x(s) \int_0^1 k(s, \tau) f(\tau, \Lambda x(\tau)) d\tau - y(s) \int_0^1 k(s, \tau) f(\tau, \Lambda x(\tau)) d\tau \right) \\
&\quad \left. - \left(y(s) \int_0^1 k(s, \tau) f(\tau, \Lambda x(\tau)) d\tau - y(s) \int_0^1 k(s, \tau) f(\tau, \Lambda y(\tau)) d\tau \right) \right| \\
&= \frac{1}{|t-s|^\alpha} \left| (x(t) - y(t)) \int_0^1 k(t, \tau) f(\tau, \Lambda x(\tau)) d\tau \right. \\
&\quad + y(t) \int_0^1 k(t, \tau) (f(\tau, \Lambda x(\tau)) - f(\tau, \Lambda y(\tau))) d\tau \\
&\quad - (x(s) - y(s)) \int_0^1 k(s, \tau) (f(\tau, \Lambda x(\tau)) - f(\tau, \Lambda y(\tau))) d\tau \\
&\quad \left. - y(s) \int_0^1 k(s, \tau) (f(\tau, \Lambda x(\tau)) - f(\tau, \Lambda y(\tau))) d\tau \right| \\
&= \frac{1}{|t-s|^\alpha} \left| ((x(t) - y(t)) - (x(s) - y(s))) \int_0^1 k(t, \tau) f(\tau, \Lambda x(\tau)) d\tau \right. \\
&\quad + (x(s) - y(s)) \int_0^1 (k(t, \tau) - k(s, \tau)) (f(\tau, \Lambda x(\tau)) - f(\tau, \Lambda y(\tau))) d\tau \\
&\quad + (y(t) - y(s)) \int_0^1 k(t, \tau) (f(\tau, \Lambda x(\tau)) - f(\tau, \Lambda y(\tau))) d\tau \\
&\quad \left. + y(s) \int_0^1 (k(t, \tau) - k(s, \tau)) (f(\tau, \Lambda x(\tau)) - f(\tau, \Lambda y(\tau))) d\tau \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{|(x(t)-y(t))-(x(s)-y(s))|}{|t-s|^\alpha} \int_0^1 |k(t,\tau)| |f(\tau, \Lambda x(\tau))| d\tau \\
&\quad + \frac{|x(s)-y(s)|}{|t-s|^\alpha} \int_0^1 |k(t,\tau)-k(s,\tau)| |f(\tau, \Lambda x(\tau))| d\tau \\
&\quad + \frac{|y(t)-y(s)|}{|t-s|^\alpha} \int_0^1 |k(t,\tau)| |f(\tau, \Lambda x(\tau))-f(\tau, \Lambda y(\tau))| d\tau \\
&\quad + \frac{|y(s)|}{|t-s|^\alpha} \int_0^1 |k(t,\tau)-k(s,\tau)| |f(\tau, \Lambda x(\tau))-f(\tau, \Lambda y(\tau))| d\tau \\
&\leq \frac{K|(x(t)-y(t))-(x(s)-y(s))|}{|t-s|^\alpha} \Psi(\psi(\|x\|_\beta)) + K_\beta \|x-y\|_\infty \Psi(\psi(\|x\|_\beta)) \int_0^1 |t-s|^{\beta-\alpha} d\tau \\
&\quad + \frac{K|y(t)-y(s)|}{|t-s|^\alpha} \gamma_f(\varepsilon) + K_\beta \|y\|_\infty \gamma_f(\varepsilon) \int_0^1 |t-s|^{\beta-\alpha} d\tau \\
&\leq \left(\frac{K|(x(t)-y(t))-(x(s)-y(s))|}{|t-s|^\alpha} + K_\beta \|x-y\|_\alpha \right) \Psi(\psi(\|x\|_\beta)) \\
&\quad + \left(\frac{K|y(t)-y(s)|}{|t-s|^\alpha} + K_\beta \|y\|_\alpha \right) \gamma_f(\varepsilon),
\end{aligned}$$

where, $\gamma_f(\varepsilon) = \sup\{|f(t, y_1) - f(t, y_2)| : t \in [0, 1], y_1, y_2 \in [0, \psi(r_0)], \|y_1 - y_2\| \leq \varepsilon\}$. Hence,

$$H_{\mathfrak{X}x - \mathfrak{X}y}^\alpha \leq (KH_{x-y}^\alpha + K_\beta \|x-y\|_\alpha) \Psi(\psi(\|x\|_\beta)) + (KH_y^\alpha + K_\beta \|y\|_\alpha) \gamma_f(\varepsilon). \quad (4)$$

Also, we have

$$\begin{aligned}
&|(\mathfrak{X}x)(0) - (\mathfrak{X}y)(0)| \\
&= \left| x(0) \int_0^1 k(0,\tau) f(\tau, \Lambda x(\tau)) d\tau - y(0) \int_0^1 k(0,\tau) f(\tau, \Lambda y(\tau)) d\tau \right| \\
&\leq \left| x(0) \int_0^1 k(0,\tau) f(\tau, \Lambda x(\tau)) d\tau - y(0) \int_0^1 k(0,\tau) f(\tau, \Lambda x(\tau)) d\tau \right| \\
&\quad + \left| y(0) \int_0^1 k(0,\tau) f(\tau, \Lambda x(\tau)) d\tau - y(0) \int_0^1 k(0,\tau) f(\tau, \Lambda y(\tau)) d\tau \right| \\
&\leq |x(0) - y(0)| \int_0^1 |k(0,\tau)| |f(\tau, \Lambda x(\tau))| d\tau + |y(0)| \int_0^1 |k(0,\tau)| |f(\tau, \Lambda x(\tau)) - f(\tau, \Lambda y(\tau))| d\tau \\
&\leq K|x(0) - y(0)| \Psi(\psi(\|x\|_\beta)) + K|y(0)| \gamma_f(\varepsilon). \quad (5)
\end{aligned}$$

Add (4) and (5), and we obtain

$$\begin{aligned}
\|\mathfrak{X}x - \mathfrak{X}y\|_\alpha &\leq (KH_{x-y}^\alpha + K_\beta \|x-y\|_\alpha) \Psi(\psi(\|x\|_\beta)) + (KH_y^\alpha + K_\beta \|y\|_\alpha) \gamma_f(\varepsilon) \\
&\quad + K|x(0) - y(0)| \Psi(\psi(\|x\|_\beta)) + K|y(0)| \gamma_f(\varepsilon) \\
&= (K + K_\beta) \Psi(\psi(\|x\|_\beta)) \|x-y\|_\alpha + (K + K_\beta) \|y\|_\alpha \gamma_f(\varepsilon) \\
&\leq (K + K_\beta) \Psi(\psi(r_0)) \varepsilon + (K + K_\beta) r_0 \gamma_f(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,
\end{aligned}$$

where, we used the fact that $\gamma_f(\varepsilon) \rightarrow 0$ since the function f is uniformly continuous on the set $[0, 1] \times [0, \psi(r_0)]$. Therefore, \mathfrak{X} is continuous on $B_{r_0}^\beta$.

Apply the Schauder fixed point theorem (recall $B_{r_0}^\beta$ is compact in $H_\alpha[0, 1]$) to

obtain the desired result. \square

4. Example

Here we illustrate our theory with an example.

EXAMPLE 4.1. Consider the quadratic integral equation

$$x(t) = \sqrt[8]{m \cos^2 t + n} + x(t) \int_0^1 \sqrt[6]{l \sin^2 t + \tau} \arctan \left(\frac{\tau^2 x(\tau)}{1 + \tau^2} \right)^{\frac{1}{3}} d\tau, \quad t \in [0, 1], \quad (6)$$

where, l , m and n are nonnegative constants.

Note that (6) is a special case of (1), where $p(t) = \sqrt[8]{m \cos^2 t + n}$, $k(t, \tau) = \sqrt[6]{l \sin^2 t + \tau}$, $f(\tau, y) = \arctan(\tau y)^{\frac{1}{3}}$ and $\Lambda x = \frac{\tau x}{1 + \tau^2}$.

One can easily check that:

$$\begin{aligned} |p(t) - p(s)| &= \left| \sqrt[8]{(\sqrt{m} \cos t)^2 + n} - \sqrt[8]{(\sqrt{m} \cos s)^2 + n} \right| \leq \sqrt[8]{|\sqrt{m} \cos t - \sqrt{m} \cos s|^2} \\ &= \sqrt[8]{m} |\cos t - \cos s|^{\frac{1}{4}} = \sqrt[8]{m} \sqrt[4]{|\cos t - \cos s|} = \sqrt[8]{m} |t - s|^{\frac{1}{8}}, \\ &\leq \sqrt[8]{m} |t - s|^{\frac{1}{8}} |t - s|^{\frac{1}{8}} \leq \sqrt[8]{m} |t - s|^{\frac{1}{8}}, \end{aligned}$$

for $t, s \in [0, 1]$, where we use [3, Theorem 2.1]. Thus $p \in H_{\frac{1}{8}}[0, 1]$ and $H_p^{\frac{1}{8}} = \sqrt[8]{m}$. Therefore, the assumption (a1) of Theorem 3.1 is satisfied with $0 < \alpha < \beta = \frac{1}{8}$ and $\|p\|_{\frac{1}{8}} = |p(0)| + H_p^{\frac{1}{8}} = \sqrt[8]{m+n} + \sqrt[8]{m}$. Moreover, we have

$$\begin{aligned} |k(t, \tau) - k(s, \tau)| &= \left| \sqrt[6]{l \sin^2 t + \tau} - \sqrt[6]{l \sin^2 s + \tau} \right| \leq \sqrt[6]{|l \sin^2 t - l \sin^2 s|} \leq \sqrt[6]{l} \sqrt[6]{|t^2 - s^2|} \\ &= \sqrt[6]{l} \sqrt[6]{t+s} \sqrt[6]{|t-s|} \leq \sqrt[6]{l} \sqrt[6]{2} |t-s|^{\frac{1}{6}} = \sqrt[6]{2l} |t-s|^{\frac{1}{6}} |t-s|^{\frac{1}{24}} \leq \sqrt[6]{2l} |t-s|^{\frac{1}{8}}, \end{aligned}$$

where the inequality $\left| \sqrt[6]{l \sin^2 t + \tau} - \sqrt[6]{l \sin^2 s + \tau} \right| \leq \sqrt[6]{|l \sin^2 t - l \sin^2 s|}$ follows from [3, Theorem 2.1]. Therefore, the assumption (a2) of Theorem 3.1 is satisfied with $\kappa_{\beta} = \kappa_{\frac{1}{8}} = \sqrt[6]{2l}$.

Now, since $|f(\tau, x)| = \left| \arctan(\tau x)^{\frac{1}{3}} \right| \leq |\tau x|^{\frac{1}{3}} \leq |x|^{\frac{1}{3}}$, then $f(\tau, x) = \arctan(\tau x)^{\frac{1}{3}}$, satisfies the assumption (a3) of Theorem 3.1 with a nondecreasing function $\Psi(r) = r^{\frac{1}{3}}$.

Also, we have $\|\Lambda x\|_{\infty} \leq \sup_{\tau \in [0, 1]} \frac{\tau |x(\tau)|}{1 + \tau^2} \leq \frac{1}{2} \|x\|_{\infty} \leq \frac{1}{2} \|x\|_{\beta}$, so the assumption (a4) is satisfied with $\psi(t) = \frac{1}{2}t$.

Next, we will show that the operator $\Lambda : H_{\beta}[0, 1] \rightarrow C[0, 1]$ is continuous with respect to the norm $\|\cdot\|_{\alpha}$. Take $x, y \in H_{\beta}[0, 1]$ and $\tau \in [0, 1]$, and we have

$$\left| \frac{\tau x(\tau)}{1 + \tau^2} - \frac{\tau y(\tau)}{1 + \tau^2} \right| = \frac{\tau}{1 + \tau^2} |x(\tau) - y(\tau)| \leq \frac{1}{2} |x(\tau) - y(\tau)| \leq \frac{1}{2} \|x - y\|_{\infty} \leq \frac{1}{2} \|x - y\|_{\alpha}.$$

Then $\|\Lambda x - \Lambda y\|_{\infty} \leq \frac{1}{2} \|x - y\|_{\alpha}$.

Note that the constant K satisfies

$$K = \sup \left\{ \int_0^1 |k(t, \tau)| d\tau : t \in [0, 1] \right\} = \sup \left\{ \int_0^1 \sqrt[6]{l \sin^2 t + \tau} d\tau : t \in [0, 1] \right\}$$

$$= \sup \left\{ \frac{6}{7} \left(\sqrt[6]{(l \sin t^2 + 1)^7} - \sqrt[6]{l^7 \sin^7 t^2} \right) : t \in [0, 1] \right\} \leq \frac{6}{7} \left(\sqrt[6]{(l+1)^7} - \sqrt[6]{l^7} \right),$$

so for the inequality appearing in the assumption (a5), we could consider the inequality

$$\sqrt[6]{m+n} + \sqrt[6]{m} + \left(\frac{6}{7} \left(\sqrt[6]{(l+1)^7} - \sqrt[6]{l^7} \right) + \sqrt[6]{2l} \right) \sqrt[3]{\frac{r}{2}} r \leq r. \quad (7)$$

Choosing suitable values for the constants m , n and l , one can find a positive solution of inequality (7) and then all the assumptions of Theorem 3.1 will be satisfied and (6) will have at least one solution $x \in H_\alpha[0, 1]$, where $0 < \alpha < \frac{1}{8}$.

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