

## REFINING NUMERICAL RADIUS INEQUALITIES OF HILBERT SPACE OPERATORS

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**Abstract.** Several upper estimates for the numerical radius of Hilbert space operators are given. Among many other inequalities, it is shown that

$$\omega^2(A) \leq \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\| + \frac{1}{2} \omega(A^2) - \frac{1}{2} \inf_{\substack{x \in \mathcal{H} \\ \|x\|=1}} \left( \sqrt{\langle |A|^2 x, x \rangle} - \sqrt{\langle |A^*|^2 x, x \rangle} \right)^2.$$

### 1. Introduction

Let  $(\mathcal{H}; \langle \cdot, \cdot \rangle)$  be a complex Hilbert space. The numerical range of an operator  $A$  is the subset of the complex numbers  $\mathbb{C}$  given by  $W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$ . The numerical radius  $\omega(A)$  of an operator  $A$  on  $\mathcal{H}$  is given by  $\omega(A) = \{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}$ . It is well known that  $\omega(\cdot)$  is a norm on the Banach algebra  $\mathcal{B}(\mathcal{H})$  of all bounded linear operators  $A : \mathcal{H} \rightarrow \mathcal{H}$ . This norm is equivalent with the usual operator norm  $\|A\| = \sup_{\|x\|=1, x \in \mathcal{H}} \|Ax\|$ . In fact, the following more precise result holds:  $\frac{1}{2} \|A\| \leq \omega(A) \leq \|A\|$ .

Kittaneh has shown in [7], that if  $A \in \mathcal{B}(\mathcal{H})$ ,  $\omega(A) \leq \frac{1}{2} (\|A\| + \|A^*\|)$ , where  $|A| = (A^*A)^{1/2}$ . In the same paper, and by using a refinement of triangle inequality for positive operators, namely,

$$\|A + B\| \leq \frac{1}{2} \left( \|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \left\| A^{\frac{1}{2}} B^{\frac{1}{2}} \right\|^2} \right)$$

it has been shown that

$$\omega(A) \leq \frac{1}{2} \left( \|A\| + \|A^2\|^{\frac{1}{2}} \right). \quad (1)$$

For an operator  $A$ , let  $A = U|A|$  be the polar decomposition of  $A$ , where  $U$  is a partial isometry such that  $\ker U = \ker A$ . The Aluthge transform of  $A$ , denoted by  $\tilde{A}$ ,

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is defined as  $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ . The generalized Aluthge transform, denoted by  $\tilde{A}_t$ , is defined as  $\tilde{A}_t = |A|^tU|A|^{1-t}$ ,  $0 \leq t \leq 1$ . In particular,  $\tilde{A}_0 = U^*UU|A| = U|A| = A$ ,  $\tilde{A}_1 = |A|UU^*U = |A|U$ , and  $\tilde{A}_{1/2} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}} = \tilde{A}$ . Here  $|A|^0$  is defined as  $U^*U$ .

In [12], Yamazaki proved that  $\omega(A) \leq \frac{1}{2} \left( \|A\| + \omega(\tilde{A}) \right)$ . In fact, this is a refinement of the inequality (1).

Concerning the product of two operators, Dragomir [4] (see also [8, (17)]) has shown the following estimate of  $\omega(B^*A)$ ,

$$\omega(B^*A) \leq \frac{1}{2} \left\| |A|^2 + |B|^2 \right\|. \quad (2)$$

For some recent and interesting results concerning inequalities for the numerical radius, see [6, 9, 10].

## 2. Results

We start our work with the following result.

**THEOREM 2.1.** *Let  $A \in \mathcal{B}(\mathcal{H})$  and  $0 \leq t \leq 1$ . Then*

$$\omega(A) \leq \frac{1}{4} \left\| |A|^{2t} + |A^*|^{2(1-t)} + A + A^* \right\| + \frac{1}{4} \left\| |A|^{2t} + |A^*|^{2(1-t)} - (A + A^*) \right\|.$$

*Proof.* We have

$$\begin{aligned} \omega(B^*A) &\leq \frac{1}{2} \left\| |A|^2 + |B|^2 \right\| \quad (\text{by (2)}) \\ &= \frac{1}{4} \left\| |A + B|^2 + |A - B|^2 \right\| \quad (\text{by the operator parallelogram law}) \\ &\leq \frac{1}{4} \left\| |A + B|^2 \right\| + \frac{1}{4} \left\| |A - B|^2 \right\| \quad (\text{by the triangle inequality}) \\ &= \frac{1}{4} \left\| |A|^2 + |B|^2 + A^*B + B^*A \right\| + \frac{1}{4} \left\| |A|^2 + |B|^2 - A^*B - B^*A \right\|. \end{aligned}$$

Namely,

$$\omega(B^*A) \leq \frac{1}{4} \left\| |A|^2 + |B|^2 + A^*B + B^*A \right\| + \frac{1}{4} \left\| |A|^2 + |B|^2 - (A^*B + B^*A) \right\|. \quad (3)$$

Let  $A = U|A|$  be the polar decomposition of  $A$ . By letting  $A = |A|^t$  and  $B = |A|^{1-t}U^*$ , in (3), and by taking into account that

$$|B|^2 = B^*B = U|A|^{1-t}|A|^{1-t}U^* = U|A|^{2(1-t)}U^* = |A^*|^{2(1-t)} \quad (\text{by [2, (1)]})$$

we get

$$\omega(A) \leq \frac{1}{4} \left\| |A|^{2t} + |A^*|^{2(1-t)} + A + A^* \right\| + \frac{1}{4} \left\| |A|^{2t} + |A^*|^{2(1-t)} - (A + A^*) \right\|. \quad \square$$

We continue this section by establishing another upper estimate.

**THEOREM 2.2.** *Let  $A \in \mathcal{B}(\mathcal{H})$  and  $0 \leq t \leq 1$ . Then for any mean  $\sigma$ ,*

$$\omega(A) \leq \frac{1}{2} \left( \left\| |A|^{2(1-t)} + |A^*|^{2t} \right\| \sigma \left\| |A|^{2t} + |A^*|^{2(1-t)} \right\| \right).$$

*Proof.* By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\langle Ax, x \rangle| &= |\langle U|A|x, x \rangle| = \left| \langle U|A|^{1-t}|A|^t x, x \rangle \right| \\ &= \left| \langle |A|^t x, |A|^{1-t} U^* x \rangle \right| \leq \| |A|^t x \| \| |A|^{1-t} U^* x \|, \end{aligned}$$

$$\begin{aligned} \text{and } |\langle Ax, x \rangle| &= |\langle U|A|x, x \rangle| = \left| \langle U|A|^t |A|^{1-t} x, x \rangle \right| \\ &= \left| \langle |A|^{1-t} x, |A|^t U^* x \rangle \right| \leq \| |A|^{1-t} x \| \| |A|^t U^* x \|. \end{aligned}$$

It follows from the first relation in the above that

$$\begin{aligned} |\langle Ax, x \rangle| &\leq \| |A|^t x \| \| |A|^{1-t} U^* x \| = \sqrt{\langle |A|^t x, |A|^t x \rangle \langle |A|^{1-t} U^* x, |A|^{1-t} U^* x \rangle} \\ &= \sqrt{\langle |A|^{2t} x, x \rangle \langle U|A|^{2(1-t)} U^* x, x \rangle} = \sqrt{\langle |A|^{2t} x, x \rangle \langle |A^*|^{2(1-t)} x, x \rangle} \quad (\text{by [5, (1)]}) \\ &\leq \frac{1}{2} \left( \langle |A|^{2t} x, x \rangle + \langle |A^*|^{2(1-t)} x, x \rangle \right) \quad (\text{by arithmetic-geometric mean inequality}) \\ &= \frac{1}{2} \langle (|A|^{2t} + |A^*|^{2(1-t)}) x, x \rangle = \frac{1}{2} \| |A|^{2t} + |A^*|^{2(1-t)} \|. \end{aligned}$$

Similarly, we can obtain  $|\langle Ax, x \rangle| \leq \frac{1}{2} \| |A|^{2(1-t)} + |A^*|^{2t} \|$ .

Now, by combining the inequalities 1 and 2 and employing the monotonicity property of the mean for positive numbers, we get,

$$\begin{aligned} |\langle Ax, x \rangle| &= |\langle Ax, x \rangle| \sigma |\langle Ax, x \rangle| \leq \frac{1}{2} \| |A|^{2(1-t)} + |A^*|^{2t} \| \sigma \frac{1}{2} \| |A|^{2t} + |A^*|^{2(1-t)} \| \\ &= \frac{1}{2} \left( \| |A|^{2(1-t)} + |A^*|^{2t} \| \sigma \| |A|^{2t} + |A^*|^{2(1-t)} \| \right) \quad (\text{by the homogeneity of } \sigma). \end{aligned}$$

The result follows by taking supremum over all unit vectors  $x \in \mathcal{H}$ .  $\square$

It should be mentioned here that there is no ordering between  $\| |A|^{2(1-t)} + |A^*|^{2t} \|$  and  $\| |A^*|^{2(1-t)} + |A|^{2t} \|$ , in general. The following example clarifies this statement.

EXAMPLE 2.3. Let  $A = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{bmatrix}$ .

$$\begin{aligned} \text{If } t = 0.3, \text{ then} \quad & \| |A|^{2(1-t)} + |A^*|^{2t} \| \approx 8.48, \\ & \| |A^*|^{2(1-t)} + |A|^{2t} \| \approx 8.89. \\ \text{If } t = 0.6, \text{ then} \quad & \| |A|^{2(1-t)} + |A^*|^{2t} \| \approx 7.68, \\ & \| |A^*|^{2(1-t)} + |A|^{2t} \| \approx 7.01. \end{aligned}$$

The next theorem provides an extension for the celebrated inequality  $\omega^2(A) \leq \frac{1}{2} \| |A|^2 + |A^*|^2 \|$  (see [8, Theorem 1]).

**THEOREM 2.4.** *Let  $A \in \mathcal{B}(\mathcal{H})$  and  $0 \leq t \leq 1$ . Then*

$$\omega^2(A) \leq \int_0^1 \left\| t|A|^2 + (1-t)|A^*|^2 \right\|^{1-v} \left\| (1-t)|A|^2 + t|A^*|^2 \right\|^v dv.$$

*Proof.* Applying the same procedure as in the proof of Theorem 2.2, one can write

$$\begin{aligned} |\langle Ax, x \rangle|^2 &\leq \left\| |A|^t x \right\| \left\| |A|^{1-t} U^* x \right\| \left\| |A|^{1-t} x \right\| \left\| |A|^t U^* x \right\| \\ &= \sqrt{\langle |A|^{2t} x, x \rangle \langle |A|^{2(1-t)} x, x \rangle} \sqrt{\langle U|A|^{2(1-t)} U^* x, x \rangle \langle U|A|^{2t} U^* x, x \rangle} \\ &= \sqrt{\langle |A|^{2t} x, x \rangle \langle |A|^{2(1-t)} x, x \rangle} \sqrt{\langle |A^*|^{2(1-t)} x, x \rangle \langle |A^*|^{2t} x, x \rangle} \\ &= \sqrt{\langle |A|^{2t} x, x \rangle \langle |A^*|^{2(1-t)} x, x \rangle} \sqrt{\langle |A|^{2(1-t)} x, x \rangle \langle |A^*|^{2t} x, x \rangle} \\ &\leq \int_0^1 \left( \langle |A|^{2t} x, x \rangle \langle |A^*|^{2(1-t)} x, x \rangle \right)^{1-v} \left( \langle |A|^{2(1-t)} x, x \rangle \langle |A^*|^{2t} x, x \rangle \right)^v dv \\ &\quad \text{(by logarithmic-geometric mean inequality)} \\ &\leq \int_0^1 \left( \langle |A|^2 x, x \rangle^t \langle |A^*|^2 x, x \rangle^{1-t} \right)^{1-v} \left( \langle |A|^2 x, x \rangle^{1-t} \langle |A^*|^2 x, x \rangle^t \right)^v dv \\ &\quad \text{(by the Hölder-McCarthy inequality [11, Theorem 1.4])} \\ &\leq \int_0^1 \left( t \langle |A|^2 x, x \rangle + (1-t) \langle |A^*|^2 x, x \rangle \right)^{1-v} \left( (1-t) \langle |A|^2 x, x \rangle + t \langle |A^*|^2 x, x \rangle \right)^v dv \\ &\quad \text{(by the weighted arithmetic-geometric mean inequality)} \\ &= \int_0^1 \left\langle \left( t|A|^2 + (1-t)|A^*|^2 \right) x, x \right\rangle^{1-v} \left\langle \left( (1-t)|A|^2 + t|A^*|^2 \right) x, x \right\rangle^v dv \\ &\leq \int_0^1 \left\| t|A|^2 + (1-t)|A^*|^2 \right\|^{1-v} \left\| (1-t)|A|^2 + t|A^*|^2 \right\|^v dv. \end{aligned}$$

$$\text{Thus,} \quad |\langle Ax, x \rangle|^2 \leq \int_0^1 \left\| t|A|^2 + (1-t)|A^*|^2 \right\|^{1-v} \left\| (1-t)|A|^2 + t|A^*|^2 \right\|^v dv.$$

$$\text{Whence,} \quad \omega^2(A) \leq \int_0^1 \left\| t|A|^2 + (1-t)|A^*|^2 \right\|^{1-v} \left\| (1-t)|A|^2 + t|A^*|^2 \right\|^v dv,$$

as required.  $\square$

The next result provides a refinement of the well-known inequality  $\omega^2(A) \leq \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\| + \frac{1}{2} \omega(A^2)$  (see [1, Theorem 2.4]). Notice that our method is different from [1].

**THEOREM 2.5.** *Let  $A \in \mathcal{B}(\mathcal{H})$ . Then*

$$\omega^2(A) \leq \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\| + \frac{1}{2} \omega(A^2) - \frac{1}{2} \inf_{\substack{x \in \mathcal{H} \\ \|x\|=1}} \left( \sqrt{\langle |A|^2 x, x \rangle} - \sqrt{\langle |A^*|^2 x, x \rangle} \right)^2.$$

*Proof.* Buzano's inequality [3] asserts that  $|\langle z, x \rangle| |\langle z, y \rangle| \leq \frac{\|z\|^2}{2} (|\langle x, y \rangle| + \|x\| \|y\|)$  for any  $x, y, z \in \mathcal{H}$ . Put  $x = Ax$ ,  $y = A^*x$ , and  $z = x$  with  $\|x\| = 1$ , then

$$\begin{aligned}
|\langle Ax, x \rangle|^2 &\leq \frac{1}{2} \left( \sqrt{\langle |A|^2 x, x \rangle \langle |A^*|^2 x, x \rangle} + |\langle A^2 x, x \rangle| \right) \\
&= \frac{1}{2} \left( \sqrt{\langle A^* Ax, x \rangle \langle AA^* x, x \rangle} + |\langle A^2 x, x \rangle| \right) = \frac{1}{2} \left( \sqrt{\langle Ax, Ax \rangle \langle A^*, A^* x \rangle} + |\langle A^2 x, x \rangle| \right) \\
&= \frac{1}{2} (\|Ax\| \|A^*x\| + |\langle A^2 x, x \rangle|) \\
&= \frac{1}{2} \left( \frac{1}{2} (\|Ax\|^2 + \|A^*x\|^2 - (\|Ax\| - \|A^*x\|)^2) + |\langle A^2 x, x \rangle| \right) \\
&= \frac{1}{2} \left( \frac{1}{2} \left( \langle |A|^2 x, x \rangle + \langle |A^*|^2 x, x \rangle - \left( \sqrt{\langle |A|^2 x, x \rangle} - \sqrt{\langle |A^*|^2 x, x \rangle} \right)^2 \right) + |\langle A^2 x, x \rangle| \right) \\
&= \frac{1}{2} \left( \frac{1}{2} \left( \langle (|A|^2 + |A^*|^2) x, x \rangle - \left( \sqrt{\langle |A|^2 x, x \rangle} - \sqrt{\langle |A^*|^2 x, x \rangle} \right)^2 \right) + |\langle A^2 x, x \rangle| \right) \\
&\leq \frac{1}{4} \| |A|^2 + |A^*|^2 \| + \frac{1}{2} \omega(A^2) - \frac{1}{2} \inf_{\substack{x \in \mathcal{H} \\ \|x\|=1}} \left( \sqrt{\langle |A|^2 x, x \rangle} - \sqrt{\langle |A^*|^2 x, x \rangle} \right)^2.
\end{aligned}$$

This implies,

$$\omega^2(A) \leq \frac{1}{4} \| |A|^2 + |A^*|^2 \| + \frac{1}{2} \omega(A^2) - \frac{1}{2} \inf_{\substack{x \in \mathcal{H} \\ \|x\|=1}} \left( \sqrt{\langle |A|^2 x, x \rangle} - \sqrt{\langle |A^*|^2 x, x \rangle} \right)^2. \quad \square$$

Before stating the next result, we recall the famous polarization identity, which says that

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^3 \|x + i^k y\|^2 i^k \quad (x, y \in \mathcal{H}).$$

**THEOREM 2.6.** *Let  $A \in \mathcal{B}(\mathcal{H})$  and  $0 \leq t \leq 1$ . Then,*

$$\omega(A) \leq \frac{1}{4} \sqrt{2 \| |A|^{4(1-t)} + |A^*|^{4t} \| + 4\omega^2(A) + 4 \| |A|^{2(1-t)} + |A^*|^{2t} \| \omega(A)}.$$

*Proof.* Let  $A = U|A|$  be the polar decomposition of  $A$ . We have

$$\begin{aligned}
\operatorname{Re} \langle e^{i\theta} Ax, x \rangle &= \operatorname{Re} \langle e^{i\theta} U|A|x, x \rangle = \operatorname{Re} \langle e^{i\theta} U|A|^t |A|^{1-t} x, x \rangle = \operatorname{Re} \langle e^{i\theta} |A|^{1-t} x, |A|^t U^* x \rangle \\
&= \frac{1}{4} \left\| \left( e^{i\theta} |A|^{1-t} + |A|^t U^* \right) x \right\|^2 - \frac{1}{4} \left\| \left( e^{i\theta} |A|^{1-t} - |A|^t U^* \right) x \right\|^2 \\
&\leq \frac{1}{4} \left\| \left( e^{i\theta} |A|^{1-t} + |A|^t U^* \right) x \right\|^2 \leq \frac{1}{4} \| e^{i\theta} |A|^{1-t} + |A|^t U^* \|^2 \\
&= \frac{1}{4} \left\| \left( e^{i\theta} |A|^{1-t} + |A|^t U^* \right)^* \left( e^{i\theta} |A|^{1-t} + |A|^t U^* \right) \right\|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \left\| |A|^{2(1-t)} + U|A|^{2t}U^* + 2 \operatorname{Re} (e^{i\theta}U|A|) \right\| \\
&= \frac{1}{4} \left\| |A|^{2(1-t)} + |A^*|^{2t} + 2 \operatorname{Re} (e^{i\theta}U|A|) \right\| \\
&= \frac{1}{4} \left\| |A|^{2(1-t)} + |A^*|^{2t} + 2 \operatorname{Re} (e^{i\theta}A) \right\| \\
&= \frac{1}{4} \left\| \left( (|A|^{2(1-t)} + |A^*|^{2t}) + (2 \operatorname{Re} (e^{i\theta}A)) \right)^2 \right\|^{\frac{1}{2}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\| \left( (|A|^{2(1-t)} + |A^*|^{2t}) + (2 \operatorname{Re} (e^{i\theta}A)) \right)^2 \right\| \\
&= \left\| (|A|^{2(1-t)} + |A^*|^{2t})^2 + 4(\operatorname{Re} (e^{i\theta}A))^2 \right. \\
&\quad \left. + (|A|^{2(1-t)} + |A^*|^{2t}) (2 \operatorname{Re} (e^{i\theta}A)) + (2 \operatorname{Re} (e^{i\theta}A)) (|A|^{2(1-t)} + |A^*|^{2t}) \right\| \\
&\leq \left\| (|A|^{2(1-t)} + |A^*|^{2t})^2 \right\| + 4 \left\| (\operatorname{Re} (e^{i\theta}A))^2 \right\| \\
&\quad + 2 \left\| (|A|^{2(1-t)} + |A^*|^{2t}) (\operatorname{Re} (e^{i\theta}A)) \right\| + 2 \left\| (\operatorname{Re} (e^{i\theta}A)) (|A|^{2(1-t)} + |A^*|^{2t}) \right\| \\
&\leq \left\| (|A|^{2(1-t)} + |A^*|^{2t})^2 \right\| + 4 \left\| (\operatorname{Re} (e^{i\theta}A))^2 \right\| + 4 \left\| |A|^{2(1-t)} + |A^*|^{2t} \right\| \left\| \operatorname{Re} (e^{i\theta}A) \right\| \\
&= \left\| \left( \frac{2|A|^{2(1-t)} + 2|A^*|^{2t}}{2} \right)^2 \right\| + 4 \left\| \operatorname{Re} (e^{i\theta}A) \right\|^2 + 4 \left\| |A|^{2(1-t)} + |A^*|^{2t} \right\| \left\| \operatorname{Re} (e^{i\theta}A) \right\| \\
&\leq 2 \left\| |A|^{4(1-t)} + |A^*|^{4t} \right\| + 4 \left\| \operatorname{Re} (e^{i\theta}A) \right\|^2 + 4 \left\| |A|^{2(1-t)} + |A^*|^{2t} \right\| \left\| \operatorname{Re} (e^{i\theta}A) \right\| \\
&\leq 2 \left\| |A|^{4(1-t)} + |A^*|^{4t} \right\| + 4\omega^2(A) + 4 \left\| |A|^{2(1-t)} + |A^*|^{2t} \right\| \omega(A).
\end{aligned}$$

Thus,

$$\omega(A) \leq \frac{1}{4} \sqrt{2 \left\| |A|^{4(1-t)} + |A^*|^{4t} \right\| + 4\omega^2(A) + 4 \left\| |A|^{2(1-t)} + |A^*|^{2t} \right\| \omega(A)}. \quad \square$$

REMARK 2.7. Letting  $t = \frac{1}{2}$ , we get

$$\omega(A) \leq \frac{1}{4} \sqrt{2 \left\| |A|^2 + |A^*|^2 \right\| + 4\omega^2(A) + 4 \left\| |A| + |A^*| \right\| \omega(A)}.$$

THEOREM 2.8. Let  $A$  be a non-zero operator on  $\mathcal{B}(\mathcal{H})$ . Then

$$\omega(A) \leq \frac{1}{2} \left\| \tilde{A}_t \right\| \left\| |A|^{2(1-t)} + |A|^{2(t-1)} \right\|.$$

*Proof.* Let  $x \in \mathcal{H}$  be a unit vector. Utilizing the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we can write

$$|\langle Ax, x \rangle| = |\langle U|A|x, x \rangle| = \left| \left\langle |A|^{t-1}|A|^{1-t}U|A|^t|A|^{1-t}x, x \right\rangle \right|$$

$$\begin{aligned}
&= \left| \left\langle |A|^{t-1} \tilde{A}_t |A|^{1-t} x, x \right\rangle \right| = \left| \left\langle \tilde{A}_t |A|^{1-t} x, |A|^{t-1} x \right\rangle \right| \\
&\leq \left\| \tilde{A}_t \right\| \sqrt{\left\langle |A|^{2(1-t)} x, x \right\rangle \left\langle |A|^{2(t-1)} x, x \right\rangle} \\
&\leq \frac{1}{2} \left\| \tilde{A}_t \right\| \left( \left\langle |A|^{2(1-t)} x, x \right\rangle + \left\langle |A|^{2(t-1)} x, x \right\rangle \right) \\
&= \frac{1}{2} \left\| \tilde{A}_t \right\| \left\langle \left( |A|^{2(1-t)} + |A|^{2(t-1)} \right) x, x \right\rangle \leq \frac{1}{2} \left\| \tilde{A}_t \right\| \left\| |A|^{2(1-t)} + |A|^{2(t-1)} \right\|.
\end{aligned}$$

Thus,

$$\omega(A) \leq \frac{1}{2} \left\| \tilde{A}_t \right\| \left\| |A|^{2(1-t)} + |A|^{2(t-1)} \right\|. \quad \square$$

In the next result, we give a lower bound for  $\|A + A^*\|$ . Let  $m(A)$  be the nonnegative number defined by  $m(A) = \inf_{\|x\|=1, x \in \mathcal{H}} |\langle Ax, x \rangle|$ .

**THEOREM 2.9.** *Let  $A \in \mathcal{B}(\mathcal{H})$  and  $0 \leq t \leq 1$ . Then*

$$\sqrt{\left\| |A|^2 + |A^*|^2 \right\| + m\left(A^2 + (A^*)^2\right)} \leq \|A + A^*\|.$$

*Proof.* We have

$$\begin{aligned}
\|A + A^*\| &= \sqrt{\left\| (A + A^*)^2 \right\|} = \sqrt{\left\| A^2 + |A^*|^2 + |A|^2 + (A^*)^2 \right\|} \\
&\geq \sqrt{\left| \left\langle \left( A^2 + |A^*|^2 + |A|^2 + (A^*)^2 \right) x, x \right\rangle \right|} \\
&= \sqrt{\left| \left\langle \left( |A|^2 + |A^*|^2 \right) x, x \right\rangle + \left\langle \left( A^2 + (A^*)^2 \right) x, x \right\rangle \right|} \\
&= \sqrt{\left| \left\langle \left( |A|^2 + |A^*|^2 \right) x, x \right\rangle \right| + \left| \left\langle \left( A^2 + (A^*)^2 \right) x, x \right\rangle \right|} \\
&= \sqrt{\left| \left\langle \left( |A|^2 + |A^*|^2 \right) x, x \right\rangle \right| + m\left(A^2 + (A^*)^2\right)}
\end{aligned}$$

Thus,

$$\sqrt{\left| \left\langle \left( |A|^2 + |A^*|^2 \right) x, x \right\rangle \right| + m\left(A^2 + (A^*)^2\right)} \leq \|A + A^*\|.$$

By taking supremum over all unit vectors  $x \in \mathcal{H}$ ,

$$\sqrt{\left\| |A|^2 + |A^*|^2 \right\| + m\left(A^2 + (A^*)^2\right)} \leq \|A + A^*\|. \quad \square$$

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