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THREE SOLUTIONS FOR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS WITH DIRICHLET BOUNDARY CONDITION

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Abstract. In this paper, we discuss the existence of at least three weak solutions for the following impulsive nonlinear fractional boundary value problem

$$\begin{aligned} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right) + a(t)u(t) &= \lambda f(t,u(t)), \quad t \neq t_j, \text{ a.e. } t \in [0,T], \\ \Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u\right)\right)(t_j) &= I_j(u(t_j)), \quad j = 1, \dots n, \\ u(0) = u(T) &= 0 \end{aligned}$$

where $\alpha \in (\frac{1}{2}, 1]$, $a \in C([0,T])$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function. Our technical approach is based on variational methods. An example is provided to illustrate the applicability of our results.

1. Introduction

In this paper, we consider the following impulsive nonlinear fractional boundary value problem

$$\begin{aligned} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right) + a(t)u(t) &= \lambda f(t,u(t)), \quad t \neq t_j, \text{ a.e. } t \in [0,T], \\ \Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u\right)\right)(t_j) &= I_j(u(t_j)), \quad j = 1, \dots n, \\ u(0) = u(T) &= 0 \end{aligned} \tag{1}$$

where $\alpha \in (\frac{1}{2}, 1]$, $a \in C([0,T])$ such that there are $a_0, a_1 > 0$ such that $0 < a_0 \leq a(t) \leq a_1$, $\lambda > 0$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, $\Delta({}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right))(t_j) = {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right)(t_j^+) - {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right)(t_j^-)$ and $I_j : \mathbb{R} \to \mathbb{R}$ for $j = 1, \dots, n$ are Lipschitz continuous functions with the Lipschitz constants $L_j > 0$, i.e $|I_j(x_2) - I_j(x_1)| \leq L_j|x_2 - x_1|$ for every $x_1, x_2 \in \mathbb{R}$ and $I_j(0) = 0$.

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In [17], Risken introduced an advection-dispersion equation to describe the Brownian motion of particles

$$\frac{\partial C(x,t)}{\partial t} = \left[-v\frac{\partial}{\partial x} + D\frac{\partial^2}{\partial x^2}\right] C(x,t)$$

where $C(x,t)$ is a concentration field of space variable x at time t, $D > 0$ is the diffusion coefficient and $v > 0$ is the drift coefficient. Many laboratory data [3,4] and numerical experiments [7] indicate that solutes moving through a highly heterogeneous aquifer violate the basic assumptions of the local second order theories because of the large deviations due to the stochastic process of Brownian motion.

Fractional differential equations (FDEs) have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Impulsive differential equations are used to describe various models of real-world processes that are subject to a sudden change. Due to the great development in the theory of fractional calculus and impulsive differential equations as well as having wide applications in several fields. Recently, the study of fractional differential equations with impulses has been studied by many authors using the variational methods, fixed-point theorems and critical point theory, see, for instance, [2, 8, 11, 12, 20] and the references therein for detailed discussions. For example, Anguraj and Latha Maheswari in [2] by using the fixed point theorem, established the existence of solutions for fractional impulsive neutral functional integrodifferential equations with nonlocal initial conditions and infinite delay. In [12] based on variational methods, the existence of infinitely many solutions for the perturbed impulsive fractional differential system, was studied.

Inspired by the above results, in Theorem 3.1 we obtain the existence of at least three weak solutions for the problem (1), in which one parameter is involved. In particular, we require that there is a growth of the antiderivative of f which is greater than quadratic in a suitable interval (see, for instance, condition (A_4) of Theorem 3.3), and which is less than quadratic in a following suitable interval (see, for instance, condition (A_4) of Theorem 3.3). We present Example 3.4 in which the hypotheses of Theorem 3.3 are fulfilled. As a special case of Theorem 3.3, we obtain Theorem 3.6 in the case f does not depend upon t. Theorems 3.7 and 3.8, under suitable conditions on f at zero and at infinity, ensure four distinct non-trivial solutions to the problem (1).

2. Preliminaries

Let X be a nonempty set and $\Phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r,\ r_1,\ r_2 > \inf_X \Phi,\ r_2 > r_1,\ r_3 > 0$, we define

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)) - \Psi(u)}{r - \Phi(u)},$$

$$\beta(r_1, r_2) := \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)},$$

$$\gamma(r_2, r_3) := \frac{\sup_{u\in\Phi^{-1}(-\infty, r_2+r_3)} \Psi(u)}{r_3},$$
$$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.$$

THEOREM 2.1 ([5, Theorem 3.3]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*
(a_1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
(a_2) for every $u_1, u_2 \in X$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has
$$\inf_{s\in[0,T]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that
(a_3) $\varphi(r_1) < \beta(r_1, r_2)$;   ($a_4$) $\varphi(r_2) < \beta(r_1, r_2)$;   ($a_5$) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.
Then, for each $\lambda \in \left(\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}\right)$ the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(-\infty, r_1)$, $u_2 \in \Phi^{-1}[r_1, r_2)$ and $u_3 \in \Phi^{-1}(-\infty, r_2 + r_3)$.

We refer the interested reader to the papers [1, 9, 10, 16] in which Theorem 2.1 has been successfully employed to obtain the existence of at least three solutions for boundary value problems.

In this section, we will introduce several basic definitions, notations, lemmas, and propositions used all over this paper.

DEFINITION 2.2 ([15]). For a function f defined on $[a, b]$ and $\alpha > 0$, the left and right Riemann-Liouville fractional integrals of order α for the function f are defined by

$${}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t \in [a, b],$$
$${}_tD_b^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t \in [a, b],$$

while the right-hand sides are point-wise defined on $[a, b]$, where $\Gamma(\alpha)$ is the gamma function.

DEFINITION 2.3 ([15]). Let $a, b \in \mathbb{R}$ and $\mathrm{AC}([a, b])$ be the space of absolutely continuous functions on $[a, b]$. For $0 < \alpha \leq 1$, $f \in \mathrm{AC}([a, b])$ left and right Riemann-Liouville and Caputo fractional derivatives are defined by:

$${}_aD_t^{\alpha} f(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t} {}_aD_t^{\alpha-1} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-s)^{-\alpha} f(s)\,\mathrm{d}s,$$
$${}_tD_b^{\alpha} f(t) \equiv -\frac{}{\mathrm{d}}\,\mathrm{d}t\, {}_tD_b^{\alpha-1} f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (s-t)^{-\alpha} f(s)\,\mathrm{d}s,$$
$${}_a^cD_t^{\alpha} f(t) \equiv {}^cD_{a^+}^{\alpha} f(t) := {}_aD_t^{\alpha-1} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f'(s)\,\mathrm{d}s$$

and

$$ {}_t^cD_b^\alpha f(t) \equiv {}^cD_{b-}^\alpha f(t) := -{}_tD_b^{\alpha-1} f'(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (s-t)^{-\alpha} f'(s)\,\mathrm{d}s $$

where $\Gamma(\alpha)$ is the gamma function. Note that when $\alpha = 1$, ${}_a^cD_t^1 f(t) = f'(t)$ and ${}_t^cD_b^1 f(t) = -f'(t)$.

Proposition 2.4 ([15,19]). *We have the following property of fractional integration*

$$ \int_a^b [{}_aD_t^{-\gamma} f(t)] g(t)\,\mathrm{d}t = \int_a^b [{}_tD_b^{-\gamma} g(t)] f(t)\,\mathrm{d}t, \quad \gamma > 0, $$

provided that $f \in \mathrm{L}^p([a,b],\mathbb{R}^N)$, $g \in \mathrm{L}^q([a,b],\mathbb{R}^N)$ *and* $p \geq 1$, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} \leq 1 + \gamma$ *or* $p \neq 1$, $q \neq 1$, $\frac{1}{p} + \frac{1}{q} = 1 + \gamma$.

To create suitable function spaces and apply critical point theory to explore the existence of solutions for the problem (1), we require the following essential notations and findings which will be used in establishing our main results.

Let $0 < \alpha \leq 1$, $1 < p < \infty$ and $\mathrm{E}_0^{\alpha,p}(0,T)$ be the Banach space, which is closure of $\mathrm{C}_0^\infty([0,T])$ with respect to the norm $\|u\|^p_{\mathrm{E}_0^{\alpha,p}(0,T)} = \|{}_a^cD_t^\alpha u(t)\|^p_{\mathrm{L}^p(0,T)} + \|u\|^p_{\mathrm{L}^p(0,T)}$. It is an established fact that $\mathrm{E}_0^{\alpha,p}(0,T)$ is a reflexive and separable Banach space (see [14, Proposition 3.1]). In short $\mathrm{E}_{0,T}^{\alpha,2} = \mathrm{E}^\alpha$, and by $\|.\|$ and $\|.\|_\infty$ the norms in $\mathrm{L}^2(0,T)$ and $\mathrm{C}([0,T])$:

$$ \|u\|^2 = \int_0^T |u(t)|^2\,\mathrm{d}t, \qquad u \in \mathrm{L}^2(0,T), $$

$$ \|u\|_\infty = \max_{t\in[0,T]} |u(t)|, \qquad u \in \mathrm{C}([0,T]). $$

E^α is a Hilbert space with inner product

$$ (u,v)_\alpha = \int_0^T \left({}_0^cD_t^\alpha u(t)\, {}_0^cD_t^\alpha v(t) + u(t)v(t)\right)\mathrm{d}t $$

and the norm

$$ \|u\|_\alpha^2 = \int_0^T \left(|{}_0^cD_t^\alpha u(t)|^2 + |u(t)|^2\right)\,\mathrm{d}t. $$

Pay attention that if $a \in \mathrm{C}([0,T])$ and there are two positive constants a_1 and a_2, so that $0 < a_1 \leq a(t) \leq a_2$, an equivalent norm in E^α is

$$ \|u\|_{a,\alpha}^2 = \int_0^T \left(|{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t + a(t)|u(t)|^2\right)\,\mathrm{d}t. $$

Proposition 2.5 ([14]). *Let* $0 < \alpha \leq 1$. *For* $u \in \mathrm{E}^\alpha$, *we have*

$$ \|u\| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^cD_t^\alpha u\|. \tag{2} $$

In addition, for $\frac{1}{2} < \alpha \leq 1$,

$$ \|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}} \|{}_0^cD_t^\alpha u\|. $$

By (2), we can take E^α with the norm

$$\|u\|_{0,\alpha} = \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 \,\mathrm{d}t \right)^{\frac{1}{2}} = \|{}_0^cD_t^\alpha u\|, \quad \forall u \in \mathrm{E}^\alpha.$$

By Proposition 2.5, when $\alpha > \frac{1}{2}$ for every $u \in \mathrm{E}^\alpha$, we have

$$\|u\|_\infty \leq k \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 \,\mathrm{d}t \right)^{\frac{1}{2}} = k\|u\|_{0,\alpha} < k\|u\|_{a,\alpha}, \tag{3}$$

where $$k = \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}.$$

Now, by setting $L := \sum_{i=1}^n L_j$, we put

$$C_1 := \frac{1}{2}(1 - LTk^2), \quad C_2 := \frac{1}{2}(1 + LTk^2). \tag{4}$$

We suppose that the Lipschitz constant $L > 0$ of the function h satisfies the condition $LTk^2 < 1$.

Here we give the definition of weak and classical solutions for the problem (1).

Definition 2.6. A function $u \in \mathrm{E}^\alpha$ is said to be a weak solution of the problem (1), if for every $v \in \mathrm{E}^\alpha$, we have

$$\int_0^T [({}_0^cD_t^\alpha u(t))({}_0^cD_t^\alpha v(t)) + a(t)u(t)v(t)] \,\mathrm{d}t + \sum_{j=1}^n I_j(u(t_j))v(t_j) - \lambda \int_0^T f(t,u(t))\, v(t) \,\mathrm{d}t = 0.$$

Definition 2.7. A function

$$u \in \left\{ u \in \mathrm{AC}([0,T]) \,:\, \int_{t_j}^{t_{j+1}} \left(|{}_0^cD_t^\alpha u(t)|^2 + |u(t)|^2 \right) \,\mathrm{d}t < \infty,\, j = 0, \dots n \right\}$$

is called to be a classical solution of problem (1) if

$${}_tD_T^\alpha \left({}_0^cD_t^\alpha u(t)\right) + a(t)u(t) = \lambda f(t,u(t)), \quad \text{a.e. } t \in [0,T]\backslash\{t_1, \dots, t_n\},$$

the limits ${}_tD_T^{\alpha-1}\left({}_0^cD_t^\alpha u\right)(t_j^+)$ and ${}_tD_T^{\alpha-1}\left({}_0^cD_t^\alpha u\right)(t_j^-)$ exist, $\Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^\alpha u\right)\right)(t_j) = I_j(u(t_j))$ and $u(0) = u(T) = 0$.

Lemma 2.8 ([6, Lemma 2.1]). *The function $u \in \mathrm{E}^\alpha$ is a weak solution of* (1) *if and only if u is a classical solution of* (1).

Corresponding to the functions f, h and I_j, $j = 1 \dots, n$, we introduce the functions $F : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $J_j : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, $j = 1, \dots, n$, respectively, as follows:

$$F(t,\xi) := \int_0^\xi f(t,x) \,\mathrm{d}x \quad \text{for all } \xi \in \mathbb{R}$$

and $$J_j(x) = \int_0^x I_j(\xi) \,\mathrm{d}\xi, \quad j = 1, \dots, n \quad \text{ for every } x \in \mathbb{R}.$$

In the rest of this paper we consider the following condition on impulsive terms:
$(\mathcal{H})$ $I_j \geq 0$ for all $j = 1, \dots, n$.
We also need the following proposition for establishing our main results.

PROPOSITION 2.9. *Let $S : \mathrm{E}^\alpha \longrightarrow (\mathrm{E}^\alpha)^*$ be the operator defined by*

$$S(u)(v) = \int_0^T [({}_0^cD_t^\alpha u(t))({}_0^cD_t^\alpha v(t)) + a(t)u(t)v(t)]\, \mathrm{d}t$$

for every $u, v \in \mathrm{E}^\alpha$. Then, S admits a continuous inverse on $(\mathrm{E}^\alpha)^*$.

Proof. It is obvious that

$$S(u)(u) = \int_0^T \left(|{}_0^cD_t^\alpha u(t)|^2 + a(t)|u(t)|^2\right) \mathrm{d}t \geq \|u\|^2_{a,\alpha}.$$

It follows that S is coercive. Owing to our assumptions on the data, one has

$$\langle S(u)-S(v), u-v\rangle = \int_0^T \left(|{}_0^cD_t^\alpha (u(t)-v(t))|^2 + a(t)|u(t)-v(t)|^2\right) \mathrm{d}t \geq \|u-v\|^2_{a,\alpha} > 0$$

for every $u, v \in \mathrm{E}^\alpha$, which means that S is strictly monotone. Moreover, since E^α is reflexive, for $u_n \longrightarrow u$ strongly in E^α as $n \to +\infty$, one has $S(u_n) \to S(u)$ weakly in $(E^\alpha)^*$ as $n \to \infty$. Hence, S is demicontinuous, so by [21, Theorem 26.A(d)], the inverse operator S^{-1} of S exists and it is continuous. Indeed, let e_n be a sequence in $(\mathrm{E}^\alpha)^*$ such that $e_n \to e$ strongly in $(\mathrm{E}^\alpha)^*$ as $n \to \infty$. Let $u_n, u \in \mathrm{E}^\alpha$ such that $S^{-1}(e_n) = u_n$ and $S^{-1}(e) = u$. Taking into account that S is coercive, one has that the sequence u_n is bounded in the reflexive space E^α. For a suitable subsequence, we have $u_n \to \hat{u}$ weakly in E^α as $n \to \infty$, which implies that $\langle S(u_n) - S(u), u_n - \hat{u}\rangle = \langle e_n - e, u_n - \hat{u}\rangle = 0$. Note that if $u_n \to \hat{u}$ weakly in E^α as $n \to +\infty$ and $S(u_n) \to S(\hat{u})$ strongly in $(\mathrm{E}^\alpha)^*$ as $n \to +\infty$, one has $u_n \to \hat{u}$ strongly in E^α as $n \to +\infty$, and since S is continuous, we have $u_n \to \hat{u}$ weakly in E^α as $n \to +\infty$ and $S(u_n) \to S(\hat{u}) = S(u)$ strongly in $(\mathrm{E}^\alpha)^*$ as $n \to +\infty$. Hence, taking into account that S is an injection, we have $u = \hat{u}$. □

3. Main results

Now, we present our main result.

THEOREM 3.1. *Assume that there exist positive constants γ_1, γ_2, γ_3 and σ with $\gamma_1 < \sqrt{\left(A(\alpha) + \frac{2Ta_0}{3}\right)}k\sigma$ and $\max\left\{\sigma, \sqrt{\left(A(\alpha) + \frac{2Ta_1}{3}\right)\frac{C_2}{C_1}}k\sigma\right\} < \gamma_2 < \gamma_3$ such that*

(A_1) $f(t,x) \geq 0$ for each $(t,x) \in [0, \frac{T}{4}) \cup (\frac{3T}{4}, T] \times [-\gamma_3, \gamma_3]$;

(A_2) $\max\left\{\frac{\int_0^T \sup_{|\xi|\leq\gamma_1} F(t,\xi)\mathrm{d}t}{\gamma_1^2}, \frac{\int_0^T \sup_{|\xi|\leq\gamma_2} F(t,\xi)\,\mathrm{d}t}{\gamma_2^2}, \frac{\int_0^T \sup_{|\xi|\leq\gamma_3} F(t,\xi)\,\mathrm{d}t}{\gamma_3^2-\gamma_2^2}\right\}$

$< \frac{C_1}{\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2} \frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\mathrm{d}t - \int_0^T \sup_{|\xi|\leq\gamma_1} F(t,\xi)\,\mathrm{d}t}{\sigma^2}.$

Then, for every

$$\lambda \in \left(\frac{\left(A(\alpha) + \frac{2Ta_1}{3}\right)C_2\sigma^2}{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\,\mathrm{d}t - \int_0^T \sup_{|\xi|\leq\gamma_1} F(t,\xi)\,\mathrm{d}t},\right.$$

$$\frac{C_1}{k^2}\min\left\{\frac{\gamma_1^2}{\int_0^T \sup_{|\xi|\le\gamma_1} F(t,\xi)\,\mathrm{d}t},\frac{\gamma_2^2}{\int_0^T \sup_{|\xi|\le\gamma_2} F(t,\xi)\,\mathrm{d}t},\frac{\gamma_3^2-\gamma_2^2}{\int_0^T \sup_{|\xi|\le\gamma_3} F(t,\xi)\mathrm{d}t}\right\}\Bigg)$$

the problem (1) *possesses at least three non-negative classical solutions* u_1*,* u_2*, and* u_3 *such that* $\max_{t\in[0,T]}|u_1(t)|<\gamma_1$, $\max_{t\in[0,T]}|u_2(t)|<\gamma_2$ *and* $\max_{t\in[0,T]}|u_3(t)|<\gamma_3$.

Proof. Our aim is to apply Theorem 2.1 to our problem. Let X be the Sobolev space E^α. We consider the auxiliary problem

$$\begin{aligned} {}_tD_T^\alpha\left({}_0^cD_t^\alpha u(t)\right)+a(t)u(t)&=\lambda\hat{f}(t,u(t)),\quad t\neq t_j,\ \text{a.e. } t\in[0,T],\\ \Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^\alpha u\right)\right)(t_j)&=I_j(u(t_j)),\quad j=1,\dots n,\\ u(0)=u(T)&=0 \end{aligned}\tag{5}$$

where $\hat{f}:[0,T]\times\mathbb{R}\to\mathbb{R}$ is an L^1-Carathéodory function, defined as follows

$$\hat{f}(t,\xi)=\begin{cases} f(t,0), & \text{if } \xi<-\gamma_3,\\ f(t,\xi), & \text{if } -\gamma_3\le\xi\le\gamma_3,\\ f(t,\gamma_3), & \text{if } \xi>\gamma_3. \end{cases}$$

If any solution of the problem (1) satisfies the condition $-\gamma_3\le u(t)\le\gamma_3$ for every $t\in[0,T]$, then, any classical solution of the problem (5) clearly turns to be also a classical solution of (1). Therefore, for our goal, it is enough to show that our conclusion holds for (1). Fix λ as in the conclusion. In order to apply Theorem 2.1 to our problem, let Φ,Ψ be, for every $u\in X$, defined by

$$\Phi(u):=\frac{1}{2}\|u\|_{a,\alpha}^2+\sum_{j=1}^n J_j(u(t_j)),\tag{6}$$

and

$$\Psi(u):=\int_0^T F(t,u(t))\,\mathrm{d}t\tag{7}$$

and put $I_\lambda(u)=\Phi(u)-\lambda\Psi(u)$ for every $u\in X$. Note that the classical solutions of (1) are exactly the critical points of I_λ. The functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.1. Indeed, similar arguments as in [18] show that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is given by

$$\Phi'(u)(v)=\int_0^T\left[\left({}_0^cD_t^\alpha u(t)\right)\left({}_0^cD_t^\alpha v(t)\right)+a(t)u(t)v(t)\right]\,\mathrm{d}t+\sum_{j=1}^n I_j(u(t_j))v(t_j)$$

for every $v\in X$, while Proposition 2.9 gives that Φ' admits a continuous inverse on X^*. Now from the facts $-L_j|\xi|\le I_j(\xi)\le L_j|\xi|$ for every $\xi\in\mathbb{R}$, $j=1,\dots,n$, and taking (3) and (4) into account, for every $u\in X$ we have

$$C_1\|u\|_{a,\alpha}^2\le\Phi(u)\le C_2\|u\|_{a,\alpha}^2\tag{8}$$

and thus the functional $\Phi:X\to\mathbb{R}$ is coercive. On the other hand, it is well known that Ψ is a differentiable functional whose differential at the point $u\in X$ is

$$\Psi'(u)(v)=\int_0^T f(t,u(t))v(t)\,\mathrm{d}t$$

for any $v \in X$ as well as it is sequentially weakly upper semicontinuous. Furthermore $\Psi' : X \to X^*$ is a compact operator. Put $r_1 = \frac{C_1}{k^2}\gamma_1^2$, $r_2 = \frac{C_1}{k^2}\gamma_2^2$ and $r_3 := \frac{C_1}{k^2}\left(\gamma_2^3 - \gamma_2^2\right)$. Now we define w_σ by

$$w_\sigma(t) = \begin{cases} \frac{4\sigma}{T}t, & \text{if } t \in [0, \frac{T}{4}), \\ \sigma, & \text{if } t \in [\frac{T}{4}, \frac{3T}{4}], \\ \frac{4\sigma}{T}(T-t), & \text{if } t \in (\frac{3T}{4}, T]. \end{cases}$$

Clearly, $w_\sigma \in X$. Obviously, one has

$$w'_\sigma(t) = \begin{cases} \frac{4\sigma}{T}, & \text{if } t \in (0, \frac{T}{4}), \\ 0, & \text{if } t \in (\frac{T}{4}, \frac{3T}{4}), \\ -\frac{4\sigma}{T}, & \text{if } t \in (\frac{3T}{4}, T), \end{cases}$$

and

$$|{}_0^cD_t^\alpha w_\sigma(t)| = \frac{1}{\Gamma(1-\alpha)}\left(\int_0^T (t-s)^{-\alpha} w'_\sigma(s)\,\mathrm{d}s\right)$$

$$= \frac{1}{\Gamma(1-\alpha)} \begin{cases} \frac{4\sigma}{T}\frac{t^{1-\alpha}}{1-\alpha}, & \text{if } t \in [0, \frac{T}{4}), \\ \frac{4\sigma}{T}\frac{(\frac{T}{4})^{1-\alpha}}{1-\alpha}, & \text{if } t \in [\frac{T}{4}, \frac{3T}{4}], \\ \frac{4\sigma}{T}\frac{1}{1-\alpha}[(\frac{T}{4})^{1-\alpha} - (t-(\frac{3T}{4}))^{1-\alpha}], & \text{if } t \in (\frac{3T}{4}, T], \end{cases}$$

so that

$$\left(A(\alpha) + \frac{2Ta_0}{3}\right)\sigma^2 \le \|w_\sigma\|^2_{a,\alpha} = A(\alpha)\sigma^2 + \int_0^T a(t)|w_n(t)|^2\,\mathrm{d}t \le \left(A(\alpha) + \frac{2Ta_1}{3}\right)\sigma^2,$$

and particularly, considering (8), it follows

$$\left(A(\alpha) + \frac{2Ta_0}{3}\right)C_1\sigma^2 \le \Phi(w_\sigma) \le \left(A(\alpha) + \frac{2Ta_1}{3}\right)C_2\sigma^2. \tag{9}$$

On the other hand, we observe

$$\Psi(w_\sigma) \ge \int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\,\mathrm{d}t. \tag{10}$$

From the conditions $\gamma_3 > \gamma_2$, $\gamma_1 < \sqrt{\left(A(\alpha) + \frac{2Ta_0}{3}\right)}k\sigma$ and $\sqrt{\left(A(\alpha) + \frac{2Ta_1}{3}\right)\frac{C_2}{C_1}}k\sigma < \gamma_2$, we get $r_3 > 0$ and $r_1 < \Phi(w) < r_2$.

$$\Phi^{-1}(-\infty, r_1) = \left\{u \in X; \Phi(u) < r_1\right\} \subseteq \left\{u \in X; |u| \le \gamma_1\right\} \tag{11}$$

and by the same argument as above, $\Phi^{-1}(-\infty, r_2) \subseteq \left\{u \in X; |u| \le \gamma_2\right\}$. Hence, we have

$$\sup_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u) = \sup_{u\in\Phi^{-1}(-\infty,r_1)} \int_0^T F(t,u(t))\mathrm{d}t \le \int_0^T \sup_{|\xi|\le\gamma_1} F(t,\xi)\,\mathrm{d}t.$$

In a similar way, we have

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)=\sup_{u\in\Phi^{-1}(-\infty,r_2)}\int_0^T F(t,u(t))\mathrm{d}t\le\int_0^T\sup_{|\xi|\le\gamma_2}F(t,\xi)\,\mathrm{d}t$$

$$\text{and}\quad\sup_{\Phi(u)<r_2+r_3}\Psi(u)\le\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\int_0^T F(t,u(t))\,\mathrm{d}t\le\int_0^T\sup_{|\xi|\le\gamma_3}F(t,\xi)\,\mathrm{d}t.$$

Therefore, since $0\in\Phi^{-1}(-\infty,r_1)$ and $\Phi(0)=\Psi(0)=0$, one has

$$\begin{aligned}\varphi(r_1)&=\inf_{u\in\Phi^{-1}(-\infty,r_1)}\frac{(\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u))-\Psi(u)}{r_1-\Phi(u)}\le\frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1}\\&=\frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\int_0^1F(x,u(x))dx}{r_1}\le\frac{\int_0^T\sup_{|\xi|\le\gamma_1}F(t,\xi)\,\mathrm{d}t}{\frac{C_1}{k^2}\gamma_1^2},\\ \varphi(r_2)&\le\frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2}\\&=\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\int_0^1F(x,u(x))dx}{r_2}\le\frac{\int_0^T\sup_{|\xi|\le\gamma_2}F(t,\xi)\,\mathrm{d}t}{\frac{C_1}{k^2}\gamma_2^2}\end{aligned}$$

and

$$\begin{aligned}\gamma(r_2,r_3)&\le\frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3}\\&=\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\int_0^1F(x,u(x))dx}{r_3}\le\frac{\int_0^T\sup_{|\xi|\le\gamma_2}F(t,\xi)\,\mathrm{d}t}{\frac{C_1}{k^2}(\gamma_2^3-\gamma_2^2)}.\end{aligned}$$

For each $u\in\Phi^{-1}(-\infty,r_1)$ one has

$$\beta(r_1,r_2)\ge\frac{\int_{\frac{T}{4}}^{\frac{3T}{4}}F(t,\sigma)\mathrm{d}t-\int_0^T\sup_{|\xi|\le\gamma_1}F(t,\xi)\mathrm{d}t}{\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2\sigma^2}.$$

Due to (A_2) we get $\alpha(r_1,r_2,r_3)<\beta(r_1,r_2)$. Now, we show that the functional I_λ satisfies the assumption (a_2) of Theorem 2.1. Let u_1 and u_2 be two local minima for I_λ. Then u_1 and u_2 are critical points for I_λ, and so, they are classical solutions for the problem (1). We want to prove that they are non-negative. Let u_0 be a (non-trivial) classical solution of the problem (1). Arguing by a contradiction, assume that the set $\mathcal{A}=\{t\in[0,T]:u_0(t)<0\}$ is non-empty and of positive measure. Put $\bar v(t)=\min\{0,u_0(t)\}$ for all $t\in[0,T]$. Clearly, $\bar v\in X$ and one has

$$\int_0^T[({}_0^cD_t^\alpha u_0(t))({}_0^cD_t^\alpha\bar v(t))+a(t)u_0(t)\bar v(t)]\,\mathrm{d}t+\sum_{j=1}^n I_j(u_0(t_j))\bar v(t_j)-\lambda\int_0^T f(t,u_0(t))\bar v(t)\,\mathrm{d}t=0.$$

Since we could assume that f is non-negative for fixed $\lambda>0$ and by choosing $\bar v(t)=$

$u_0(t)$ one has

$$0 \le 2C_1\|u_0\|^2_{\mathrm{E}_0^{\alpha,p}(\mathcal{A})} \le \int_{\mathcal{A}} \left[({}_0^cD_t^\alpha u_0(t))^2 + a(t)u_0^2(t)\right]\,\mathrm{d}t + \sum_{\mathcal{A}} I_j(u_0(t_j))u_0(t_j)$$
$$= \lambda \int_{\mathcal{A}} f(t,u_0(t))u_0(t)\,\mathrm{d}t \le 0,$$

that is, $\|u_0\|_{\mathrm{E}_0^{\alpha,p}(\mathcal{A})} = 0$ which is an absurd. Hence, our claim is proved. Then, we observe $u_1(t) \ge 0$ and $u_2(t) \ge 0$ for every $t \in [0,T]$. Thus, it follows that $\lambda f(t, su_1 + (1-s)u_2) \ge 0$ for all $s \in [0,T]$, and consequently, $\Psi(su_1 + (1-s)u_2) \ge 0$, for every $s \in [0,T]$. Hence, Theorem 2.1 implies that for every

$$\lambda \in \left(\frac{\left(A(\alpha) + \frac{2Ta_1}{3}\right)C_2\sigma^2}{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\,\mathrm{d}t - \int_0^T \sup_{|\xi|\le\gamma_1} F(t,\xi)\,\mathrm{d}t}, \right.$$
$$\left. \frac{C_1}{k^2}\min\left\{ \frac{\gamma_1^2}{\int_0^T \sup_{|\xi|\le\gamma_1} F(t,\xi)\,\mathrm{d}t}, \frac{\gamma_2^2}{\int_0^T \sup_{|\xi|\le\gamma_2} F(t,\xi)\,\mathrm{d}t}, \frac{\gamma_3^2-\gamma_2^2}{\int_0^T \sup_{|\xi|\le\gamma_3} F(t,\xi)\mathrm{d}t} \right\}\right)$$

and $\mu \in [0, \delta_{\lambda,g})$, the functional I_λ has three critical points $u_i,\ i = 1,2,3$, in X such that $\Phi(u_1) < r_1$, $\Phi(u_2) < r_2$ and $\Phi(u_3) < r_2 + r_3$, that is, $\max_{t\in[0,T]}|u_1(t)| < \gamma_1$, $\max_{t\in[0,T]}|u_2(t)| < \gamma_2$ and $\max_{t\in[0,T]}|u_3(t)| < \gamma_3$. Then, taking into account the fact that the weak solutions of the problem (1) are exactly critical points of the functional I_λ we have the desired conclusion. □

REMARK 3.2. We observe that, in Theorem 3.1, no asymptotic conditions on f and g are needed and only algebraic conditions on f are imposed to guarantee the existence of the classical solutions.

Now, we deduce the following straightforward consequence of Theorem 3.1.

THEOREM 3.3. *Assume that there exist positive constants* γ_1, γ_4 *and* σ *with* $\gamma_1 < \sqrt{\left(A(\alpha) + \frac{2Ta_0}{3}\right)}k\sigma$ *and* $\max\left\{\sigma, \sqrt{\left(A(\alpha) + \frac{2Ta_1}{3}\right)\frac{C_2}{C_1}}k\sigma\right\} < \gamma_4$ *such that*

(A$_3$) $f(x,t) \ge 0$ *for each* $(x,t) \in [0, \frac{T}{4}) \cup (\frac{3T}{4}, T] \times [-\gamma_4, \gamma_4]$;

(A$_4$) $\max\left\{ \frac{\int_0^T \sup_{|\xi|\le\gamma_1} F(t,\xi)\,\mathrm{d}t}{\gamma_1^2}, \frac{2\int_0^T \sup_{|\xi|\le\gamma_4} F(t,\xi)\mathrm{d}t}{\gamma_4^2} \right\}$
$< \frac{C_1}{\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2 + C_1} \frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\,\mathrm{d}t}{\sigma^2}$.

Then, for every

$$\lambda \in \left(\frac{\left(\left(A(\alpha) + \frac{2Ta_1}{3}\right)C_2 + \frac{C_1}{k^2}\right)\sigma^2}{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\,\mathrm{d}t}, \right.$$
$$\left. \frac{C_1}{k^2}\min\left\{ \frac{\gamma_1^2}{\int_0^T \sup_{|\xi|\le\gamma_1} F(t,\xi)\,\mathrm{d}t}, \frac{\gamma_4^2}{2\int_0^T \sup_{|\xi|\le\gamma_4} F(t,\xi)\,\mathrm{d}t} \right\}\right),$$

the problem (1) *possesses at least three non-negative classical solutions* u_1, u_2 *and* u_3 *such that* $\max_{t\in[0,T]}|u_1(t)| < \gamma_1$, $\max_{t\in[0,T]}|u_2(t)| < \frac{1}{\sqrt{2}}\gamma_4$ *and* $\max_{t\in[0,T]}|u_3(t)| < \gamma_4$.

Proof. Choose $\gamma_2 = \frac{1}{\sqrt{2}}\gamma_4$ and $\gamma_3 = \gamma_4$. So, from (A_4) one has

$$\begin{aligned}\frac{\int_0^T \sup_{|\xi|\le\gamma_2} F(t,\xi)\,\mathrm{d}t}{\gamma_2^2} &= \frac{2\int_0^T \sup_{|\xi|\le\frac{1}{\sqrt{2}}\gamma_4} F(t,\xi)\,\mathrm{d}t}{\gamma_4^2} \le \frac{2\int_0^T \sup_{|\xi|\le\gamma_4} F(t,\xi)\,\mathrm{d}t}{\gamma_4^2}\\ &< \frac{C_1}{\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2 + C_1}\frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\mathrm{d}t}{\sigma^2} \end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{\int_0^T \sup_{|\xi|\le\gamma_3} F(t,\xi)\mathrm{d}t}{\gamma_3^2-\gamma_2^2} &= \frac{2\int_0^T \sup_{|\xi|\le\gamma_4} F(t,\xi)\,\mathrm{d}t}{\gamma_4^2}\\ &< \frac{C_1}{\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2 + C_1}\frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\,\mathrm{d}t}{\sigma^2}. \end{aligned} \tag{13}$$

Moreover, taking into account that $\gamma_1 < \sigma$, by using (A_4) we have

$$\begin{aligned}&\frac{C_1}{\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2}\frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\mathrm{d}t - \int_0^T \sup_{|\xi|\le\gamma_1} F(t,\xi)\,\mathrm{d}t}{\sigma^2}\\ &> \frac{C_1}{\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2}\frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\mathrm{d}t}{\sigma^2} - \frac{C_1}{\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2}\frac{\int_0^T \sup_{|\xi|\le\gamma_1} F(t,\xi)\,\mathrm{d}t}{\gamma_1^2}\\ &> \frac{C_1}{\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2}\left(\frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\mathrm{d}t}{\sigma^2} - \frac{C_1}{\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2 + C_1}\frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\,\mathrm{d}t}{\sigma^2}\right)\\ &= \frac{C_1}{\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2 + C_1}\frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\mathrm{d}t}{\sigma^2}.\end{aligned}$$

Hence, from (A_4), (12) and (13), it is easy to see that the assumption (A_2) of Theorem 3.1 is satisfied, and the conclusion follows. □

We now present the following example to illustrate Theorem 3.3.

Example 3.4. We consider the problem

$$\begin{aligned} &{}_tD_T^{\frac{3}{4}}\left({}_0^cD_t^{\frac{3}{4}}u(t)\right) + u(t) = \lambda f(u), \quad t \ne \frac{1}{2}, \text{ a.e. } t\in[0,1],\\ &\Delta\left({}_tD_T^{-\frac{1}{4}}\left({}_0^cD_t^{\frac{3}{4}}u\right)\right)(\frac{1}{2}) = \frac{\Gamma^2(\frac{3}{4})}{4}\sin(u(\frac{1}{2})),\\ &u(0)=u(1)=0, \end{aligned} \tag{14}$$

where

$$f(\xi) = \begin{cases} 3\xi^2, & \text{if } \xi \le 1,\\ \frac{3}{\xi} + \sin^2(\xi-1), & \text{if } \xi > 1.\end{cases}$$

By the expression of f, we have

$$F(\xi)=\begin{cases}\xi^3, & \text{if } t\leq 1,\\ 3\ln(\xi)+\frac{1}{2}\xi-\frac{1}{4}\sin 2(\xi-1)+\frac{1}{2}, & \text{if } \xi>1.\end{cases}$$

By simple calculations, we obtain $A(\alpha)=\frac{1312}{15\Gamma^2(\frac{1}{4})}$, $k=\frac{\sqrt{2}}{\Gamma(\frac{3}{4})}$, $C_1=\frac{1}{4}$ and $C_2=\frac{3}{4}$. Taking $\gamma_1=\frac{1}{10^3}$, $\gamma_4=10^4$ and $\eta=1$, then all conditions in Theorem 3.3 are satisfied. Therefore, it follows that for each $\lambda\in\left(\frac{656}{5\Gamma^2(\frac{1}{4})}+1+\frac{\Gamma^2(\frac{3}{4})}{8},125\Gamma^2(\frac{3}{4})\right)$, the problem (14) possesses at least three non-negative classical solutions u_1, u_2 and u_3 such that $\max_{t\in[0,1]}|u_1(t)|<\frac{1}{10^3}$, $\max_{t\in[0,1]}|u_2(t)|<\frac{1}{\sqrt{2}}10^4$ and $\max_{t\in[0,1]}|u_3(t)|<10^4$.

We want to point out a simple consequence of Theorem 3.3, in which the function f has separated variables.

Theorem 3.5. *Let $f_1\in L^1([0,T])$ and $f_2\in C(\mathbb{R})$ be two functions. Put $\tilde{F}(t)=\int_0^t f_2(\xi)d\xi$ for all $t\in\mathbb{R}$ and assume that there exist positive constants γ_1, γ_4 and σ with $\gamma_1<\sqrt{\left(A(\alpha)+\frac{2Ta_0}{3}\right)}k\sigma$ and $\max\left\{\sigma,\sqrt{\left(A(\alpha)+\frac{2Ta_1}{3}\right)\frac{C_2}{C_1}}k\sigma\right\}<\gamma_4$ such that*

(A_5) $f_1(t)\geq 0$ for each $t\in[0,T]$ and $f_2(x)\geq 0$ for each $x\in[-\gamma_4,\gamma_4]$;

(A_6) $\max\left\{\frac{\sup_{|\xi|\leq\gamma_1}\tilde{F}(\xi)}{\gamma_1^2},\ \frac{2\sup_{|\xi|\leq\gamma_4}\tilde{F}(\xi)}{\gamma_4^2}\right\}<\frac{C_1}{2\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2+2C_1}\frac{\tilde{F}(\sigma)}{\sigma^2}$.

Then, for every

$$\lambda\in\left(\frac{2\left(\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2+\frac{C_1}{k^2}\right)\sigma^2}{T\tilde{F}(\sigma)},\frac{C_1}{k^2T}\min\left\{\frac{\gamma_1^2}{\sup_{|\xi|\leq\gamma_1}\tilde{F}(\xi)},\frac{\gamma_4^2}{2\sup_{|\xi|\leq\gamma_4}\tilde{F}(\xi)}\right\}\right)$$

the problem

$$\begin{aligned}&{}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right)+a(t)u(t)=\lambda f_1(t)f_2(u),\quad t\neq t_j,\ a.e.\ t\in[0,T],\\ &\Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u\right)\right)(t_j)=I_j(u(t_j)),\quad j=1,\dots n,\\ &u(0)=u(T)=0\end{aligned}$$

possesses at least three non-negative classical solutions u_1, u_2 and u_3 such that $\max_{t\in[0,T]}|u_1(t)|<\gamma_1$, $\max_{t\in[0,T]}|u_2(t)|<\frac{1}{\sqrt{2}}\gamma_4$ and $\max_{t\in[0,T]}|u_3(t)|<\gamma_4$.

Proof. Set $f(t,x)=f_1(t)f_2(x)$ for each $(t,x)\in[0,T]\times\mathbb{R}$. Since $F(t,x)=f_1(t)\tilde{F}(x)$ from (A_5) and (A_6) we obtain (A_3) and (A_4), respectively. □

Here, we present a simple consequence of Theorem 3.3 in the case when f does not depend upon t.

Theorem 3.6. *Assume that there exist positive constants γ_1, γ_4 and σ with $\gamma_1<\sqrt{\left(A(\alpha)+\frac{2Ta_0}{3}\right)}k\sigma$ and $\max\left\{\sigma,\sqrt{\left(A(\alpha)+\frac{2Ta_1}{3}\right)\frac{C_2}{C_1}}k\sigma\right\}<\gamma_4$ such that*

(A_7) $f(x)\geq 0$ for each $x\in[-\gamma_4,\gamma_4]$;

(A_8) $\max\left\{\frac{F(\gamma_1)}{\gamma_1^2},\ \frac{2F(\gamma_4)}{\gamma_4^2}\right\}<\frac{C_1}{2\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2k^2+2C_1}\frac{F(\sigma)}{\sigma^2}$.

Then, for every

$$\lambda \in \left(\frac{2\left(\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2+\frac{C_1}{k^2}\right)\sigma^2}{TF(\sigma)}, \frac{C_1}{k^2}\min\left\{\frac{\gamma_1^2}{TF(\gamma_1)}, \frac{\gamma_4^2}{2F(\gamma_4)}\right\}\right)$$

the problem

$$\begin{aligned} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right)+a(t)u(t) &= \lambda f(u(t)), \quad t\neq t_j, \ a.e.\ t\in[0,T],\\ \Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u\right)\right)(t_j) &= I_j(u(t_j)), \quad j=1,\dots n,\\ u(0)=u(T)&=0 \end{aligned}$$

possesses at least three non-negative classical solutions u_1, u_2 *and* u_3 *such that* $\max_{t\in[0,T]}|u_1(t)|<\gamma_1$, $\max_{t\in[0,T]}|u_2(t)|<\frac{1}{\sqrt{2}}\gamma_4$ *and* $\max_{t\in[0,T]}|u_3(t)|<\gamma_4$.

The following result is a consequence of Theorem 3.3.

THEOREM 3.7. *Let* $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function such that* $xf(t,x)>0$ *for all* $(t,x)\in[0,T]\times(\mathbb{R}\backslash\{0\})$. *Assume that*

(A_9*)* $\lim_{x\to 0}\frac{f(t,x)}{|x|}=\lim_{|x|\to+\infty}\frac{f(t,x)}{|x|}=0$.

Then, for every $\lambda>\overline{\lambda}$ *where*

$$\overline{\lambda}=\left(\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2+\frac{C_1}{k^2}\right)\max\left\{\inf_{\sigma>0}\frac{\sigma^2}{\int_{\frac{T}{4}}^{\frac{3T}{4}}F(t,\sigma)\mathrm{d}t}, \ \inf_{\sigma<0}\frac{(-\sigma)^2}{\int_{\frac{T}{4}}^{\frac{3T}{4}}F(t,\sigma)\mathrm{d}t}\right\},$$

the problem (1) *possesses at least four distinct non-trivial solutions.*

Proof. Set

$$f_1(t,x)=\begin{cases} f(t,x), & \text{if } (t,x)\in[0,T]\times[0,+\infty),\\ 0, & \text{otherwise,}\end{cases}$$

and $f_2(t,x)=\begin{cases} -f(t,-x), & \text{if } (t,x)\in[0,T]\times[0,+\infty),\\ 0, & \text{otherwise,}\end{cases}$

and define $F_1(t,x):=\int_0^x f_1(t,\xi)d\xi$ for every $(t,x)\in[0,T]\times\mathbb{R}$. Fix $\lambda>\lambda^*$, and let $\sigma>0$ such that $\lambda>\frac{\left(\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2+\frac{C_1}{k^2}\right)\sigma^2}{\int_{\frac{T}{4}}^{\frac{3T}{4}}F_1(t,\sigma)\,\mathrm{d}t}$. From $\lim_{x\to0}\frac{f_1(t,x)}{|x|}=\lim_{|x|\to+\infty}\frac{f_1(t,x)}{|x|}=0$, there is $\gamma_1>0$ such that

$$\gamma_1<\min\left\{\sigma, \sqrt{\left(A(\alpha)+\frac{2Ta_0}{3}\right)k\sigma}\right\} \ \text{and} \ \frac{\int_0^T\sup_{|\xi|\leq\gamma_1}F_1(t,\xi)\,\mathrm{d}t}{\gamma_1^2}<\frac{C_1}{\lambda k^2}$$

and there is $\gamma_4>0$ such that

$$\max\left\{\sigma, \sqrt{\left(A(\alpha)+\frac{2Ta_1}{3}\right)\frac{C_2}{C_1}k\sigma}\right\}<\gamma_4 \ \text{and} \ \frac{\int_0^T\sup_{|\xi|\leq\gamma_4}F_1(t,\xi)\mathrm{d}t}{\gamma_4^2}<\frac{C_1}{2\lambda k^2}.$$

Then (A_4) in Theorem 3.3 is satisfied,

$$\lambda \in \left(\frac{\left(\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2+\frac{C_1}{k^2}\right)\sigma^2}{\int_{\frac{T}{4}}^{\frac{3T}{4}} F_1(t,\sigma)\,\mathrm{d}t}, \right.$$
$$\left. \frac{C_1}{k^2}\min\left\{ \frac{\gamma_1^2}{\int_0^T \sup_{|\xi|\le\gamma_1} F_1(t,\xi)\,\mathrm{d}t}, \frac{\gamma_4^2}{2\int_0^T \sup_{|\xi|\le\gamma_4} F_1(t,\xi)\,\mathrm{d}t} \right\} \right).$$

Hence, the problem $(P_\lambda^{f_1})$ admits two positive solutions u_1, u_2, which are positive solutions of the problem (1). Next, arguing in the same way, from $\lim_{x\to 0}\frac{f_2(t,x)}{|x|} = \lim_{|x|\to+\infty}\frac{f_2(t,x)}{|x|} = 0$, we ensure the existence of two positive solutions u_3, u_4 for the problem $(P_\lambda^{f_2})$. Clearly, $-u_3$, $-u_4$ are negative solutions of the problem (1) and the conclusion is achieved. □

Remark 3.8. We explicitly observe that in Theorem 3.7 no symmetric condition on f is assumed. However, whenever f is an odd continuous non-zero function such that $f(t,x)\ge 0$ for all $(t,x)\in[0,T]\times[0,+\infty)$, ($A_9$) can be replaced by

(A_10) $\lim_{x\to 0^+}\frac{f(t,x)}{x} = \lim_{x\to+\infty}\frac{f(t,x)}{x} = 0$,

ensuring the existence of at least four distinct non-trivial solutions the problem (1) for every $\lambda > \lambda^*$ where

$$\lambda^* = \inf_{\sigma>0} \frac{\left(\left(A(\alpha)+\frac{2Ta_1}{3}\right)C_2+\frac{C_1}{k^2}\right)\sigma^2}{\displaystyle\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\sigma)\,\mathrm{d}t}.$$

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