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ON $\mathcal{I}$-STATISTICAL CONVERGENCE OF SEQUENCES IN GRADUAL NORMED LINEAR SPACES

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Abstract. In this article, we introduce the notion of $\mathcal{I}$-statistical convergence of sequences as one of the extensions of $\mathcal{I}$-convergence in the gradual normed linear spaces. We investigate some fundamental properties of the newly introduced notion and its relation with some other methods of convergence. Also we introduce and investigate the concept of $\mathcal{I}$-statistical limit points, cluster points and establish some implication relations.

1. Introduction

The idea of fuzzy sets [20] was first introduced by Zadeh in the year 1965 which was an extension of the classical set-theoretical concept. Nowadays it has wide applicability in different branches of science and engineering. The term "fuzzy number" plays a crucial role in the study of fuzzy set theory. Fuzzy numbers were basically the generalization of intervals, not numbers. In particular, fuzzy numbers do not obey some algebraic properties of the classical numbers. So the term "fuzzy number" is debatable to many authors due to its different behavior. The term "fuzzy intervals" is often used by many authors instead of fuzzy numbers. To overcome the confusion among the researchers, in 2008, Fortin et. al. [10] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual real numbers are mainly known by their respective assignment function which is defined in the interval $(0, 1]$. So in some sense, every real number can be viewed as a gradual number with a constant assignment function. The gradual real numbers also obey all the algebraic properties of the classical real numbers and can be used in computation and optimization problems.

In 2011, Sadeqi and Azari [16] first introduced the concept of gradual normed linear space. They studied various properties of the space from both algebraic and topological points of view. Further progress in this direction has occurred due to Ettefagh, Azari, and Etemad (see [7, 8]) and many others. For an extensive study on

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gradual real numbers [1, 6, 14] can be addressed, where many more references can be found.

On the other hand, the notion of statistical convergence was first introduced by Fast [9] and Steinhaus [19], independently in the year 1951. Later on, it was further investigated from the sequence space point of view by Fridy [11], Salat [17], and many others.

In 2001, the idea of $\mathcal{I}$-convergence was developed by Kostyrko et. al. [13] mainly as an extension of statistical convergence. They showed that many other known notions of convergence were a particular type of $\mathcal{I}$-convergence by considering particular ideals. Consequently, this direction gradually got more attention of the researchers and became one of the most active areas of research. Several investigations and extensions of $\mathcal{I}$-convergence can be found in the works of Kostyrko et. al. [12] and many others.

Combining the notion of statistical convergence and $\mathcal{I}$-convergence, in 2011, Savas and Das [18] introduced the notion of $\mathcal{I}$-statistical convergence. Later on, several investigations in this direction have occurred due to Mursaleen et. al. [15] and many others. For an extensive view of $\mathcal{I}$-statistical convergence, one may refer to [3–5].

Research on the convergence of sequences in gradual normed linear spaces has not yet gained much ground and it is still in its infant stage. The research carried out so far shows a strong analogy in the behavior of convergence of sequences in gradual normed linear spaces (for details one may refer to [7, 8, 16]).

The convergence of sequences in gradual normed linear spaces was introduced by Ettefagh et. al. [8]. They have investigated some properties from the topological point of view [7]. Recently, Choudhury and Debnath [2] have introduced the notion of $\mathcal{I}$-convergence of sequences in gradual normed linear spaces. From that point of view, the study of $\mathcal{I}$-statistical convergence of sequences in gradual normed linear spaces is very natural.

2. Definitions and preliminaries

Definition 2.1 ([10]). A gradual real number $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}} : (0, 1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual real number is said to be non-negative, if for every $\xi \in (0, 1]$, $A_{\tilde{r}}(\xi) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.

In [10], the gradual operations between the elements of $G(\mathbb{R})$ were defined as follows.

Definition 2.2. Let $*$ be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$ respectively. Then $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $A_{\tilde{r}_1 * \tilde{r}_2}$ given by $A_{\tilde{r}_1 * \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) * A_{\tilde{r}_2}(\xi)$, $\forall \xi \in (0, 1]$. Then the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $c\tilde{r} (c \in \mathbb{R})$ are defined by $A_{\tilde{r}_1 + \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) + A_{\tilde{r}_2}(\xi)$ and $A_{c\tilde{r}}(\xi) = cA_{\tilde{r}}(\xi)$, $\forall \xi \in (0, 1]$.

For any real number $p \in \mathbb{R}$, the constant gradual real number $\tilde{p}$ is defined by the constant assignment function $A_{\tilde{p}}(\xi) = p$ for any $\xi \in (0, 1]$. In particular, $\tilde{0}$ and $\tilde{1}$ are the constant gradual numbers defined by $A_{\tilde{0}}(\xi) = 0$ and $A_{\tilde{1}}(\xi) = 1$ respectively. One can easily verify that $G(\mathbb{R})$ with the gradual addition and multiplication forms a real vector space [10].

Definition 2.3 ([16]). Let X be a real vector space. The function $\|\cdot\|_G : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X, if for every $\xi \in (0,1]$, following three conditions are true for any $x, y \in X$:
(G1) $A_{\|x\|_G}(\xi) = A_{\tilde{0}}(\xi)$ iff $x = 0$;

(G2) (G_2) $A_{\|\lambda x\|_G}(\xi) = |\lambda| A_{\|x\|_G}(\xi)$ for any $\lambda \in \mathbb{R}$;

(G3) (G_3) $A_{\|x+y\|_G}(\xi) \leq A_{\|x\|_G}(\xi) + A_{\|y\|_G}(\xi)$.

The pair $(X, \|\cdot\|_G)$ is called a gradual normed linear space (GNLS).

Definition 2.4 ([16]). Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is said to be gradual convergent to $x \in X$, if for every $\xi \in (0,1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\xi)) \in \mathbb{N}$ such that $A_{\|x_k - x\|_G}(\xi) < \varepsilon$, $\forall k \geq N$.

Example 2.5 ([16]). Let $X = \mathbb{R}^m$ and for $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m, \xi \in (0,1]$, define $\|\cdot\|_G$ by $A_{\|x\|_G}(\xi) = e^\xi \sum_{i=1}^m |x_i|$. Then $\|\cdot\|_G$ is a gradual norm on $\mathbb{R}^m$ and $(\mathbb{R}^m, \|\cdot\|_G)$ is a GNLS.

Definition 2.6 ([11]). If K is a subset of the set of all natural numbers $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$. The natural density of K is defined by $d(K) = \lim\limits_{n\to\infty} \frac{|K_n|}{n}$.

Definition 2.7 ([11]). A real-valued sequence $x = (x_k)$ is said to be statistically convergent to l, if for every $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N} : | x_k - l | \geq \varepsilon\}$ has natural density zero. l is called the statistical limit of the sequence (x_k) and symbolically, *st-lim* $x_k = l$ or $x_k \xrightarrow{st} l$.

Definition 2.8 ([13]). Let X be a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X, if the following three conditions hold:
(i) $\emptyset \in \mathcal{I}$;

(ii) for each A, B, $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;

(iii) for each A, B, $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$.

Some standard examples of ideals are given below.
(i) The set $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$ is an admissible ideal in $\mathbb{N}$. Here $\mathbb{N}$ denotes the set of all natural numbers.

(ii) The set $\mathcal{I}_d$ of all subsets of natural numbers having natural density 0 is an admissible ideal in $\mathbb{N}$.

(iii) The set $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\}$ is an admissible ideal in $\mathbb{N}$.

(iv) Suppose $\mathbb{N} = \bigcup_{p=1}^\infty D_p$ be a decomposition of $\mathbb{N}$ (for $i \neq j$, $D_i \cap D_j = \emptyset$). Then the set $\mathcal{I}$ of all subsets of $\mathbb{N}$ which intersects finitely many D_p's forms an ideal in $\mathbb{N}$.

More important examples can be found in [12].

DEFINITION 2.9 ([13]). Let X be a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on X, if the following three conditions hold:
(i) $\emptyset \notin \mathcal{F}$;

(ii) for each A, B, $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;

(iii) for each A, B, $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. The filter $\mathcal{F}(\mathcal{I})=\{X-A : A\in\mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

DEFINITION 2.10 ([13]). Let $\mathcal{I} \subset P(\mathbb{N})$ be a non-trivial ideal on $\mathbb{N}$. A real-valued sequence (x_k) is said to be $\mathcal{I}$-convergent to l, if for each $\varepsilon > 0$, the set $C(\varepsilon) = \{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ belongs to $\mathcal{I}$. l is called the $\mathcal{I}$-limit of the sequence (x_k) and is denoted as $\mathcal{I}\text{-}\lim_k x_k = l$ or $x_k \xrightarrow{\mathcal{I}} l$.

DEFINITION 2.11 ([13]). Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A real-valued sequence (x_k) is said to be $\mathcal{I}^*$-convergent to l, if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\}$ in the associated filter $\mathcal{F}(\mathcal{I})$ such that $\lim_{k\in M} x_k = l$. Symbolically, $\mathcal{I}^*\text{-}\lim_k x_k = l$ or $x_k \xrightarrow{\mathcal{I}^*} l$.

DEFINITION 2.12 ([18]). A real-valued sequence (x_k) is said to be $\mathcal{I}$-statistically convergent to l, if for every $\varepsilon > 0, \delta > 0$,

$$\Big\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |x_k - l| \geq \varepsilon\}| \geq \delta\Big\} \in \mathcal{I}.$$

If a sequence (x_k) is $\mathcal{I}$-statistically convergent to l, then it is denoted by $\mathcal{I}\text{-}st\text{-}\lim x_k = l$ or $x_k \xrightarrow{\mathcal{I}st} l$.

DEFINITION 2.13 ([5]). A real valued sequence (x_k) is said to be $\mathcal{I}^*$-statistically convergent to l, if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, such that $st\text{-}\lim x_{m_k} = l$.

If a sequence (x_k) is $\mathcal{I}^*$-statistically convergent to l, then it is denoted by $\mathcal{I}^*\text{-}st\text{-}\lim x_k = l$ or $x_k \xrightarrow{\mathcal{I}^*st} l$.

DEFINITION 2.14 ([5]). A real number x_0 is said to be an $\mathcal{I}$-statistical limit point of a real valued sequence (x_k), if there exists $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $st\text{-}\lim x_{m_k} = x_0$.

For a sequence (x_k), the set of all $\mathcal{I}$-statistical limit points is denoted by $\mathcal{I}\text{-}S(\Lambda_{(x_k)})$.

DEFINITION 2.15 ([15]). A real number x_0 is said to be an $\mathcal{I}$-statistical cluster point of a real valued sequence (x_k), if for every $\varepsilon > 0$ and $\delta > 0$, $\{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : |x_k - x_0| \geq \varepsilon\} \mid < \delta\} \notin \mathcal{I}$.

For a sequence (x_k), the set of all $\mathcal{I}$-statistical cluster points is denoted by $\mathcal{I}$-$S(\Gamma_{(x_k)})$.

Definition 2.16. Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is said to be gradual statistically convergent to $x \in X$, if for every $\xi \in (0,1]$ and $\varepsilon > 0$, the set $B(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\xi) \geq \varepsilon\}$ has natural density zero. Symbolically we write, $x_k \xrightarrow{st\text{-}\|\cdot\|_G} x$.

Definition 2.17 ([2]). Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is said to be gradually $\mathcal{I}$-convergent (in short, $\mathcal{I}$-$\|\cdot\|_G$ convergent) to $x \in X$, if for every $\xi \in (0,1]$ and $\varepsilon > 0$, the set $B(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\xi) \geq \varepsilon\} \in \mathcal{I}$. Symbolically, $x_k \xrightarrow{\mathcal{I}\text{-}\|\cdot\|_G} x$.

Definition 2.18 ([2]). Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is said to be gradually $\mathcal{I}^*$-convergent to $x \in X$, if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_k}) is gradually convergent to x. Symbolically, $x_k \xrightarrow{\mathcal{I}^*\text{-}\|\cdot\|_G} x$.

Throughout the paper, for simplicity we use $\mathbf{0}$ to denote the m-tuple $(0, 0, \ldots .0, 0)$ and $\mathcal{I}$ stands for a non-trivial admissible ideal in $\mathbb{N}$.

3. Main results

Definition 3.1. Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is said to be gradually $\mathcal{I}$-statistical convergent (in short, $\mathcal{I}st$-$\|\cdot\|_G$ convergent) to $x \in X$, if for every $\xi \in (0,1]$ and $\varepsilon > 0, \delta > 0$, the set $\{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|x_k - x\|_G}(\xi) \geq \varepsilon\} \mid \geq \delta\} \in \mathcal{I}$. Symbolically we write, $x_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} x$.

Example 3.2. Let $X = \mathbb{R}^m$ and $\|\cdot\|_G$ be the norm defined in Example 2.5. Consider the ideal $\mathcal{I} = \mathcal{I}_d$, ideal consisting of all subsets of $\mathbb{N}$ having natural density zero. Consider the sequence (μ_n) defined by

$$\mu_n = \begin{cases} 1, & 1 \leq n \leq 10 \\ n - 10, & n \geq 10 \end{cases}$$

and let $S = \{1^2, 2^2, 3^2, \ldots\}$. Then, the sequence (x_k) defined by

$$x_k = \begin{cases} (0, 0, \ldots, 0, k), & n - [\sqrt{\mu_n}] + 1 \leq k \leq n, n \notin S \\ (0, 0, \ldots, 0, k), & n - \mu_n + 1 \leq k \leq n, n \in S \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

is $\mathcal{I}st$-$\|\cdot\|_G$ convergent to $\mathbf{0}$.

Justification. For any $\varepsilon > 0$ $(0 < \varepsilon < 1)$, since $\frac{1}{\mu_n} \mid \{n - \mu_n + 1 \leq k \leq n : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \varepsilon\} \mid = \frac{[\sqrt{\mu_n}]}{\mu_n} \to \infty$ as $n \to \infty$ and $n \notin S$, so for every $\delta > 0$,

$$\{n \in \mathbb{N} : \frac{1}{\mu_n} \mid \{n - \mu_n + 1 \leq k \leq n : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \varepsilon\} \mid \geq \delta\} \subset S \cup \{1, 2, \ldots, k_1\} \quad (1)$$

for some $k_1 \in \mathbb{N}$. Now let $\delta > 0$ be given. Then, the fact $\lim_n(1 - \frac{\mu_n}{n}) = 0$ enables us to choose $k_2 \in \mathbb{N}$ such that $1 - \frac{\mu_n}{n} < \frac{\delta}{2}$ for every $n \geq k_2$. Thus for the above $\varepsilon > 0$ we have

$$\frac{1}{n} \mid \{k \leq n : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \varepsilon\} \mid$$
$$= \frac{1}{n} \mid \{k \leq n - \mu_n : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \varepsilon\} \mid + \frac{1}{n} \mid \{n - \mu_n + 1 \leq k \leq n : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \varepsilon\} \mid$$
$$\leq 1 - \frac{\mu_n}{n} + \frac{1}{n} \mid \{n - \mu_n + 1 \leq k \leq n : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \varepsilon\} \mid$$
$$\leq \frac{\delta}{2} + \frac{1}{\mu_n} \mid \{n - \mu_n + 1 \leq k \leq n : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \varepsilon\} \mid$$

for all $n \geq k_2$. Thus,

$$\{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \varepsilon\} \mid \geq \delta\}$$
$$\subset \{n \in \mathbb{N} : \frac{1}{\mu_n} \mid \{n - \mu_n + 1 \leq k \leq n : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \varepsilon\} \mid \geq \frac{\delta}{2}\} \cup \{1, 2, 3, \ldots, k_2\}$$
$$\subset S \cup \{1, 2, 3, \ldots, k_3\}, \text{ from (1) where } k_3 = \max\{k_1, k_2\}.$$

Now, since $d(S) = 0$, from the above inclusion, we conclude that $\{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \varepsilon\} \mid \geq \delta\} \in \mathcal{I}$, proving that $x_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} \mathbf{0}$.

Theorem 3.3. *Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$ such that $x_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} x$. Then x is unique.*

Proof. If possible suppose $x_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} x$ and $x_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} y$ hold for $x, y \in X$ with $x_1 \neq x_2$. Then, for any $\varepsilon > 0, \delta > 0$ and $\xi \in (0, 1]$, we have, $B_1 = B_1(\xi, \varepsilon, \delta), B_2 = B_2(\xi, \varepsilon, \delta) \in \mathcal{F}(\mathcal{I})$, where $B_1 = \{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|x_k - x\|_G}(\xi) \geq \varepsilon\} \mid < \delta\}$ and $B_2 = \{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|x_k - y\|_G}(\xi) \geq \varepsilon\} \mid < \delta\}$. Clearly $B_1 \cap B_2 \in \mathcal{F}(\mathcal{I})$ and is non-empty. Choose $m \in B_1 \cap B_2$ and take $\varepsilon = A_{\|\frac{x-y}{3}\|_G}(\xi) > 0$. Then, $\frac{1}{m} \mid \{k \leq m : A_{\|x_k - x\|_G}(\xi) \geq \varepsilon\} \mid < \delta$ and $\frac{1}{m} \mid \{k \leq m : A_{\|x_k - y\|_G}(\xi) \geq \varepsilon\} \mid < \delta$. Now choosing δ sufficiently small, we can say most of the k's $(\leq m)$ will satisfy $A_{\|x_k - x\|_G}(\xi) < \varepsilon$ and $A_{\|x_k - y\|_G}(\xi) < \varepsilon$. Thus the set $B = \{k \leq m : A_{\|x_k - x\|_G}(\xi) < \varepsilon\} \cap \{k \leq m : A_{\|x_k - y\|_G}(\xi) < \varepsilon\} \neq \emptyset$. Choose $p \in B$.

Then, $\varepsilon = A_{\|\frac{x-y}{3}\|_G}(\xi) \leq \frac{1}{3}(A_{\|x_p - x\|_G}(\xi) + A_{\|x_p - y\|_G}(\xi)) < \frac{1}{3}(\varepsilon + \varepsilon) = \frac{2\varepsilon}{3}$, which is a contradiction. □

Theorem 3.4. *Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$ such that $x_k \xrightarrow{st\text{-}\|\cdot\|_G} x$. Then, $x_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} x$.*

Proof. $x_k \xrightarrow{st\text{-}\|\cdot\|_G} x$ implies that for every $\xi \in (0, 1]$ and $\varepsilon > 0$, the set $\{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\xi) \geq \varepsilon\}$ has natural density zero i.e., $\lim_n \frac{1}{n} \mid \{k \leq n : A_{\|x_k - x\|_G}(\xi) \geq \varepsilon\} \mid = 0$. So for every $\xi \in (0, 1]$, $\varepsilon > 0$, and $\delta > 0$, the set $\{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n :$

$A_{\|x_k-x\|_G}(\xi) \geq \varepsilon\} \,|\geq \delta\}$ is a finite set and eventually becomes a member of $\mathcal{I}$, as $\mathcal{I}$ is admissible. □

REMARK 3.5. The converse of the above theorem is not true. One can easily verify the fact by considering Example 3.2.

REMARK 3.6. For a sequence (x_k) in the GNLS $(X, \|\cdot\|_G)$, $x_k \xrightarrow{\mathcal{I}\text{-}\|\cdot\|_G} x$ implies $x_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} x$. But the converse is not true.

EXAMPLE 3.7. Let $X = \mathbb{R}^m$ and $\|\cdot\|_G$ be the norm defined in Example 2.5. Consider the ideal $\mathcal{I} = \mathcal{I}_f$, ideal consisting of all finite subsets of $\mathbb{N}$. Define the sequence (x_k) as follows:

$$x_k = \begin{cases} \mathbf{0}, & k = n^2, n \in \mathbb{N} \\ (0, 0, \ldots, 0, 1), & \text{otherwise.} \end{cases}$$

Then (x_k) is $\mathcal{I}st$-$\|\cdot\|_G$ convergent to $(0, 0, \ldots, 0, 1)$ but not $\mathcal{I}$-$\|\cdot\|_G$ convergent to $(0, 0, \ldots, 0, 1)$.

THEOREM 3.8. *Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. If each subsequence of (x_k) is $\mathcal{I}st$-$\|\cdot\|_G$ convergent to $x \in X$, then (x_k) is also gradually $\mathcal{I}$-statistical convergent to x.*

Proof. If possible suppose (x_k) is not $\mathcal{I}st$-$\|\cdot\|_G$ convergent to x in spite of having all the subsequences $\mathcal{I}st$-$\|\cdot\|_G$ converging to x. Then, by definition, there exists particular $\varepsilon > 0$ and $\delta > 0$ such that the set $B = B(\xi, \varepsilon, \delta) = \{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|x_k-x\|_G}(\xi) \geq \varepsilon\} \,|\geq \delta\} \notin \mathcal{I}$. Now admissibility of $\mathcal{I}$ ensures that B is an infinite set. Put $B = \{n_1 < n_2 < \ldots < n_j < \ldots\}$ and define $y_j = x_{k_j}, j \in \mathbb{N}$. Then, (y_j) is a subsequence of (x_k) that is not $\mathcal{I}st$-$\|\cdot\|_G$ converging to x, which is a contradiction. □

REMARK 3.9. The converse of the above theorem is not true. Example 3.7 works as a counterexample.

THEOREM 3.10. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, \|\cdot\|_G)$ such that $x_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} x$ and $y_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} y$. Then,*
(i) $x_k + y_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} x + y$ and (ii) $cx_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} cx$.

Proof. (i) From the hypothesis, we can conclude that for every $\xi \in (0, 1]$ and $\varepsilon > 0, \delta > 0$, the two sets $C_1 = C_1(\xi, \varepsilon, \delta), C_2 = C_2(\xi, \varepsilon, \delta) \in \mathcal{I}$, where $C_1 = \{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|x_k-x\|_G}(\xi) \geq \frac{\varepsilon}{2}\} \,|< \frac{\delta}{2}\}$ and $C_2 = \{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|y_k-y\|_G}(\xi) \geq \frac{\varepsilon}{2}\} \,|< \frac{\delta}{2}\}$. Then, $(\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2) \in \mathcal{F}(\mathcal{I})$ and so $(\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2) \neq \emptyset$. Choose $n \in (\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2)$. Then the following inequality

$$\begin{aligned} &\frac{1}{n} \mid \{k \leq n : A_{\|(x_k+y_k)-(x+y)\|_G}(\xi) \geq \varepsilon\} \mid \\ \leq &\frac{1}{n} \mid \{k \leq n : A_{\|x_k-x\|_G}(\xi) \geq \frac{\varepsilon}{2}\} \mid + \frac{1}{n} \mid \{k \leq n : A_{\|y_k-y\|_G}(\xi) \geq \frac{\varepsilon}{2}\} \mid, \end{aligned}$$

yields the following inclusion

$$(\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2) \subseteq \{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|(x_k+y_k)-(x+y)\|_G}(\xi) \geq \varepsilon\} \mid < \delta\}. \quad (2)$$

Now as $(\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2) \in \mathcal{F}(\mathcal{I})$, so the set in the right-hand side of (2) also belongs to $\mathcal{F}(\mathcal{I})$ which means that $x_k + y_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} x + y$.

(ii) For $c = 0$, there is nothing to prove. So let $c \neq 0$. Then, for every $\xi \in (0, 1]$ and $\varepsilon > 0$, the following inequation

$$\begin{aligned} &\frac{1}{n} \mid \{k \leq n : A_{\|cx_k - cx\|_G}(\xi) \geq \varepsilon\} \mid \\ &= \frac{1}{n} \mid \{k \leq n : |c| A_{\|x_k - x\|_G}(\xi) \geq \varepsilon\} \mid \leq \frac{1}{n} \mid \{k \leq n : A_{\|x_k - x\|_G}(\xi) \geq \frac{\varepsilon}{|c|}\} \mid \end{aligned}$$

holds good and the result follows. □

Definition 3.11. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is said to be gradually $\mathcal{I}^*st$-convergent (in short, $\mathcal{I}^*st$-$\|\cdot\|_G$ convergent) to $x \in X$, if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that $x_{m_k} \xrightarrow{st\text{-}\|\cdot\|_G} x$. Symbolically we write, $x_k \xrightarrow{\mathcal{I}^*st\text{-}\|\cdot\|_G} x$.

Theorem 3.12. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then, $x_k \xrightarrow{\mathcal{I}^*st\text{-}\|\cdot\|_G} x$ implies $x_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} x$.*

Proof. Let $x_k \xrightarrow{\mathcal{I}^*st\text{-}\|\cdot\|_G} x$. Then, by definition there exists $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that $x_{m_k} \xrightarrow{st\text{-}\|\cdot\|_G} x$. So, for any $\varepsilon, \delta > 0$ and $\xi \in (0, 1]$, the set $B = B(\xi, \varepsilon, \delta) = \{n \in \mathbb{N} : \frac{1}{n} \mid \{m_k \leq n : A_{\|x_{m_k} - x\|_G}(\xi) \geq \varepsilon\} \mid \geq \delta\}$ is a finite set and so belongs to $\mathcal{I}$, as $\mathcal{I}$ is admissible.

Now we have, $\{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|x_k - x\|_G}(\xi) \geq \varepsilon\} \mid \geq \delta\} \subseteq (\mathbb{N} \setminus M) \cup B \in \mathcal{I}$, i.e. $x_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} x$. This completes the proof. □

The converse of the above theorem is not true. The following example justifies the fact

Example 3.13. Let $X = \mathbb{R}^m$ and $\|\cdot\|_G$ be the norm defined in Example 2.5. Consider the ideal $\mathcal{I}$ consisting of all subsets of $\mathbb{N}$ which intersect finitely many D_p's where $\mathbb{N} = \bigcup_{p=1}^{\infty} D_p$ and $D_i \cap D_j = \emptyset$ for $i \neq j$. Consider the sequence (x_k) in $\mathbb{R}^m$ defined by $x_k = (0, 0, \ldots, 0, \frac{1}{p})$, *if* $k \in D_p$. Then $x_k \xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G} \mathbf{0}$ but $x_k \xrightarrow{\mathcal{I}^*st\text{-}\|\cdot\|_G} \mathbf{0}$ does not hold.

Justification. Clearly, $A_{\|x_k - \mathbf{0}\|_G}(\xi) = \frac{1}{p} e^{\xi}$ for $k \in D_p$. Let $\varepsilon > 0$ be given. Then by Archimedean property, there exists $q \in \mathbb{N}$ such that $\frac{1}{q} e^{\xi} < \varepsilon$ which in turn yields the following inclusion:

$$\{k \in \mathbb{N} : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \varepsilon\} \subseteq \{k \in \mathbb{N} : A_{\|x_k - \mathbf{0}\|_G}(\xi) \geq \frac{1}{q} e^{\xi}\} \quad (3)$$

and as $A_{\|x_k-\mathbf{0}\|_G}(\xi)=\frac{1}{p}e^{\xi}$ for $k\in D_p$, we have

$$\{k\in\mathbb{N}:A_{\|x_k-\mathbf{0}\|_G}(\xi)\geq\frac{1}{q}e^{\xi}\}=\bigcup_{p=1}^{q}D_p\in\mathcal{I}. \tag{4}$$

From (3) and (4), we obtain $\{k\in\mathbb{N}:A_{\|x_k-\mathbf{0}\|_G}(\xi)\geq\varepsilon\}\in\mathcal{I}$ i.e., $x_k\xrightarrow{\mathcal{I}\text{-}\|\cdot\|_G}x$. By Remark 3.6, we have $x_k\xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G}x$. But we claim that (x_k) is not $\mathcal{I}^*st$-$\|\cdot\|_G$ convergent to $\mathbf{0}$.

If $x_k\xrightarrow{\mathcal{I}^*st\text{-}\|\cdot\|_G}\mathbf{0}$, then there exists a set $M=\{m_1<m_2<\ldots<m_k<\ldots\}\in\mathcal{F}(\mathcal{I})$ such that

$$x_{m_k}\xrightarrow{st\text{-}\|\cdot\|_G}x. \tag{5}$$

Now for any $M\in\mathcal{F}(\mathcal{I})$, there is some $H\in\mathcal{I}$ such that $M=\mathbb{N}\backslash H$ and by the structure of $\mathcal{I}$, for that H there exists $p\in\mathbb{N}$ such that $H\subseteq\bigcup_{j=1}^{p}D_j$ and as a consequence $D_{p+1}\subseteq M$. Now for any particular $\eta\in(0,\frac{1}{p+1})$, $d(\{m_k\in D_{p+1}:A_{\|x_{m_k}-\mathbf{0}\|_G}(\xi)\geq\eta\})=\frac{1}{2^{p+1}}\neq 0$, which contradicts (5).

Definition 3.14. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and $x=(x_k)$ be a sequence in the GNLS $(X,\|\cdot\|_G)$. Then $x_0\in X$ is said to be gradual $\mathcal{I}st$-limit point of x, if there exists a set $M\subset\mathbb{N}$ with $M=\{m_1<m_2<\ldots<m_k<\ldots\}\notin\mathcal{I}$ such that $x_{m_k}\xrightarrow{st\text{-}\|\cdot\|_G}x_0$.

For any sequence (x_k), the set of all gradual $\mathcal{I}st$-limit points is denoted by $\mathcal{I}st$-$\|\cdot\|_G(\Lambda_{(x_k)})$.

Definition 3.15. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X,\|\cdot\|_G)$. Then $x_0\in X$ is said to be gradual $\mathcal{I}st$-cluster point of (x_k), if for any $\varepsilon>0,\delta>0$ and $\xi\in(0,1]$, the set $\{n\in\mathbb{N}:\frac{1}{n}\mid\{k\leq n:A_{\|x_k-x_0\|_G}(\xi)\geq\varepsilon\}\mid<\delta\}\notin\mathcal{I}$.

For any sequence (x_k), the set of all gradual $\mathcal{I}st$-cluster points is denoted by $\mathcal{I}st$-$\|\cdot\|_G(\Gamma_{(x_k)})$.

Theorem 3.16. *Let (x_k) be a sequence in the GNLS $(X,\|\cdot\|_G)$ such that $x_k\xrightarrow{\mathcal{I}st\text{-}\|\cdot\|_G}x$. Then, $\mathcal{I}st$-$\|\cdot\|_G(\Lambda_{(x_k)})=\{x\}$.*

Proof. If possible suppose $\mathcal{I}st$-$\|\cdot\|_G(\Lambda_{(x_k)})$ contains one more element y such that $y\neq x$. Then, by definition, there exists a set $M\subset\mathbb{N}$ with $M=\{m_1<m_2<\ldots<m_k<\ldots\}\notin\mathcal{I}$ such that $x_{m_k}\xrightarrow{st-\text{-}\|\cdot\|_G}x$. Let $B=B(\xi,\varepsilon,\delta)=\{n\in\mathbb{N}:\frac{1}{n}\mid\{k\leq n:A_{\|x_k-y\|_G}(\xi)\geq\varepsilon\}\mid\geq\delta\}$. Then B is a finite set, so $\mathbb{N}\setminus B\in\mathcal{F}(\mathcal{I})$. Now, since $x_k\xrightarrow{\mathcal{I}st-\|\cdot\text{-}\|_G}x$, so for any $\xi\in(0,1]$ and $\varepsilon>0,\delta>0$, the set $C=C(\xi,\varepsilon,\delta)=\{n\in\mathbb{N}:\frac{1}{n}\mid\{k\leq n:A_{\|x_k-x\|_G}(\xi)\geq\varepsilon\}\mid<\delta\}\in\mathcal{F}(\mathcal{I})$. Put $D=D(\xi,\varepsilon,\delta)=\{n\in M:\frac{1}{n}\mid\{k\leq n:A_{\|x_k-x\|_G}(\xi)\geq\varepsilon\}\mid\geq\delta\}$, then since $\mathbb{N}\setminus D\supset C$, so $\mathbb{N}\setminus D\in\mathcal{F}(\mathcal{I})$. Thus we have, $(\mathbb{N}\setminus B)\cap(\mathbb{N}\setminus D)\in\mathcal{F}(\mathcal{I})$ and eventually $(\mathbb{N}\setminus B)\cap(\mathbb{N}\setminus D)\neq\emptyset$. Let $j\in(\mathbb{N}\setminus B)\cap(\mathbb{N}\setminus D)$

and take $\varepsilon = A_{\|\frac{x-y}{2}\|_G}(\xi)$. Then we have, $\frac{1}{j} \mid \{k \leq j : A_{\|x_j - x\|_G}(\xi) \geq \varepsilon\} \mid < \delta$ and $\frac{1}{j} \mid \{k \leq j : A_{\|x_j - y\|_G}(\xi) \geq \varepsilon\} \mid < \delta$. Now choosing δ sufficiently small we can have an element say $p \in \{k \leq j : A_{\|x_j - x\|_G}(\xi) \geq \varepsilon\} \cap \{k \leq j : A_{\|x_j - y\|_G}(\xi) \geq \varepsilon\}$. But then, $\varepsilon = A_{\|\frac{x-y}{2}\|_G}(\xi) \leq \frac{1}{2}(A_{\|x_p - x\|_G}(\xi) + A_{\|x_p - y\|_G}(\xi)) < \frac{1}{2}(\varepsilon + \varepsilon) = \varepsilon$, a contradiction. □

THEOREM 3.17. *For any sequence (x_k) in the GNLS $(X, \|\cdot\|_G)$,*

$$\mathcal{I}st\text{-}\|\cdot\|_G(\Lambda_{(x_k)}) \subseteq \mathcal{I}st\text{-}\|\cdot\|_G(\Gamma_{(x_k)}).$$

Proof. Let $x_0 \in \mathcal{I}st\text{-}\|\cdot\|_G(\Lambda_{(x_k)})$. Then, there exists a set $M \subset \mathbb{N}$ with $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \notin \mathcal{I}$ such that $\lim\limits_n \frac{1}{n} \mid \{m_k \leq n : A_{\|x_{m_k} - x_0\|_G}(\xi) \geq \varepsilon\} \mid = 0$. Thus for any $\delta > 0$, there exists some $n_0 \in \mathbb{N}$ such that for any $n > n_0$, $\frac{1}{n} \mid \{m_k \leq n : A_{\|x_{m_k} - x_0\|_G}(\xi) \geq \varepsilon\} \mid < \delta$. Let $B = \{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|x_k - x_0\|_G}(\xi) \geq \varepsilon\} \mid < \delta\}$. Then, $B \supset M \setminus \{m_1, m_2, \ldots, m_{n_0}\}$. Now since $\mathcal{I}$ is admissible and $M \notin \mathcal{I}$, so $B \notin \mathcal{I}$ and the proof is complete. □

THEOREM 3.18. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, \|\cdot\|_G)$ such that $\{k \in \mathbb{N} : x_k \neq y_k\} \in \mathcal{I}$. Then,*
(i) $\mathcal{I}st\text{-}\|\cdot\|_G(\Lambda_{(x_k)}) = \mathcal{I}st\text{-}\|\cdot\|_G(\Lambda_{(y_k)})$.
(ii) $\mathcal{I}st\text{-}\|\cdot\|_G(\Gamma_{(x_k)}) = \mathcal{I}st\text{-}\|\cdot\|_G(\Gamma_{(y_k)})$.

Proof. (i) Let $x_0 \in \mathcal{I}st\text{-}\|\cdot\|_G(\Lambda_{(x_k)})$. Then, there exists a set $M \subset \mathbb{N}$ with $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \notin \mathcal{I}$ such that $x_{m_k} \xrightarrow{st\text{-}\|\cdot\|_G} x_0$. Now since the inclusion $\{k \in M : x_k \neq y_k\} \subseteq \{k \in \mathbb{N} : x_k \neq y_k\}$ holds, so $N = \{k \in M : x_k = y_k\} \notin \mathcal{I}$ and $N \subseteq M$. Therefore, $y_{m_k} \xrightarrow{st\text{-}\|\cdot\|_G} x_0$ holds and eventually we have $\mathcal{I}st\text{-}\|\cdot\|_G(\Lambda_{(x_k)}) \subseteq \mathcal{I}st\text{-}\|\cdot\|_G(\Lambda_{(y_k)})$. By symmetry, $\mathcal{I}st\text{-}\|\cdot\|_G(\Lambda_{(y_k)}) \subseteq \mathcal{I}st\text{-}\|\cdot\|_G(\Lambda_{(x_k)})$. Hence we have, $\mathcal{I}st\text{-}\|\cdot\|_G(\Lambda_{(x_k)}) = \mathcal{I}st\text{-}\|\cdot\|_G(\Lambda_{(y_k)})$.

(ii) Suppose $x_0 \in \mathcal{I}st\text{-}\|\cdot\|_G(\Gamma_{(x_k)})$. Then by definition, for any $\varepsilon > 0, \delta > 0$ and $\xi \in (0,1]$, the set $B = B(\xi, \varepsilon, \delta) = \{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|x_k - x_0\|_G}(\xi) \geq \varepsilon\} \mid < \delta\} \notin \mathcal{I}$. Let $C = C(\xi, \varepsilon, \delta) = \{n \in \mathbb{N} : \frac{1}{n} \mid \{k \leq n : A_{\|y_k - x_0\|_G}(\xi) \geq \varepsilon\} \mid < \delta\}$. We claim that $C \notin \mathcal{I}$. Because if $C \in \mathcal{I}$, then $\mathbb{N} \setminus C \in \mathcal{F}(\mathcal{I})$ and then by the hypothesis we obtain, $(\mathbb{N} \setminus C) \cap \{k \in \mathbb{N} : x_k = y_k\} \in \mathcal{F}(\mathcal{I})$. Consequently, the inclusion $(\mathbb{N} \setminus B) \supset (\mathbb{N} \setminus C) \cap \{k \in \mathbb{N} : x_k = y_k\}$ leads us to the contradiction that $\mathbb{N} \setminus B \in \mathcal{F}(\mathcal{I})$. Therefore, we must have, $C \notin \mathcal{I}$ i.e., $x_0 \in \mathcal{I}st\text{-}\|\cdot\|_G(\Gamma_{(y_k)})$. Thus, $\mathcal{I}st\text{-}\|\cdot\|_G(\Gamma_{(x_k)}) \subseteq \mathcal{I}st\text{-}\|\cdot\|_G(\Gamma_{(y_k)})$. By symmetry, $\mathcal{I}st\text{-}\|\cdot\|_G(\Gamma_{(y_k)}) \subseteq \mathcal{I}st\text{-}\|\cdot\|_G(\Gamma_{(x_k)})$. Hence we have, $\mathcal{I}st\text{-}\|\cdot\|_G(\Gamma_{(x_k)}) = \mathcal{I}st\text{-}\|\cdot\|_G(\Gamma_{(y_k)})$. □

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