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MORE ON THE GENERALIZED PASCAL TRIANGLES

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Abstract. The aim of this article is to obtain a new factorization of a generalized Pascal triangle. This factorization particularly emphasizes that there is a close relation between generalized Pascal triangles and Toeplitz matrices. However, we will show that in general there is no such relation between generalized Pascal triangles and Hankel matrices.

1. Introduction

Let $S(\infty)$ be the infinite symmetric matrix with entries $S_{i,j} = \binom{i+j}{i}$ for $i, j \geqslant 0$. Indeed, this matrix has the following form:

$$S(\infty) = [S_{i,j}]_{i,j\geqslant 0} = \left[\binom{i+j}{j}\right]_{i,j\geqslant 0} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 3 & 4 & \dots \\ 1 & 3 & 6 & 10 & \dots \\ 1 & 4 & 10 & 20 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

One can easily check that (see also [1, 3]): $S(\infty) = L(\infty)U(\infty)$, where $L(\infty)$ is the infinite left-lower-triangular Pascal's matrix:

$$L(\infty) = [L_{i,j}]_{i,j\geqslant 0} = \left[\binom{i}{j}\right]_{i,j\geqslant 0} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and $U(\infty) = L(\infty)^t$, the superscript t denotes transpose. We remark that the entries of matrix $S(\infty)$ satisfy the following recurrence relation:

$$S_{i,0} = S_{0,j} = 1 \quad (i, j \geqslant 0), \qquad S_{i,j} = S_{i-1,j} + S_{i,j-1} \quad (i, j \geqslant 1),$$

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while the entries of matrix $L(\infty)$ satisfy the following recurrence relation:

$$L_{i,0} = 1,\ L_{0,j} = 0 \quad (i \geqslant 0,\ j \geqslant 1), \qquad L_{i,j} = L_{i-1,j-1} + L_{i-1,j} \quad (i,j \geqslant 1).$$

In [1], Bacher introduced the generalized Pascal triangles (which can be regarded as a generalization of the infinite symmetric matrix $S(\infty)$) as follows. Let $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be two sequences starting with a common first term $\alpha_0 = \beta_0$. Then, the *generalized Pascal triangle associated with* α *and* β, is the infinite matrix $P_{\alpha,\beta}(\infty) = [P_{i,j}]_{i,j\geqslant 0}$ with entries $P_{i,0} = \alpha_i$, $P_{0,j} = \beta_j$ $(i,j \geqslant 0)$ and $P_{i,j} = P_{i,j-1} + P_{i-1,j}$, for $i,j \geqslant 1$. In a more general case, we allow the matrix entries to be defined recursively by $P_{i,j} = xP_{i,j-1} + yP_{i-1,j-1} + zP_{i-1,j}$, where x, y and z are constant coefficients. It is worth noting that these matrices and similar matrices with recursive entries have been studied in scattered articles (see e.g. [2, 4–10]). Some of these articles [2, 7, 9] focused in particular on identifying some factorizations and determinants of these matrices, while a comprehensive collection of results can be found in [8].

There is a close relationship between generalized Pascal triangles and Toeplitz matrices, which we will discuss later. Recall that a matrix $T(\infty) = [T_{i,j}]_{i,j\geqslant 0}$ is *Toeplitz* if $T_{i,j} = T_{k,l}$ whenever $i - j = k - l$. Obviously, a Toeplitz matrix is determined by its first row and its first column, so in what follows we use $T_{\alpha,\beta}(\infty)$ to describe an infinite Toeplitz matrix $T(\infty)$, where $\alpha = (T_{i,0})_{i\geqslant 0}$ and $\beta = (T_{0,j})_{j\geqslant 0}$.

We denote by $L(n)$ (resp. $T_{\alpha,\beta}(n)$, $P_{\alpha,\beta}(n)$) the finite submatrix of $L(\infty)$ (resp. $T_{\alpha,\beta}(\infty)$, $P_{\alpha,\beta}(\infty)$) consisting of the entries in its first n rows and columns.

Given an arbitrary sequence $\alpha = (\alpha_i)_{i\geqslant 0}$, the *binomial transform* of α is the sequence $\check{\alpha} = (\check{\alpha}_i)_{i\geqslant 0}$ defined by $\check{\alpha}_i = \sum_{k=0}^{i} \binom{i}{k} \alpha_k$. If we consider the sequence α to be the (infinite) column matrix $[\alpha_0, \alpha_1, \ldots]^t$, then we obtain the binomial transform $\check{\alpha} = (\check{\alpha}_i)_{i\geqslant 0}$ by multiplying this column matrix by the left-lower-triangular Pascal's matrix $L(\infty)$, that is

$$\begin{bmatrix} \check{\alpha}_0 \\ \check{\alpha}_1 \\ \check{\alpha}_2 \\ \check{\alpha}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{bmatrix}.$$

This transformation is invertible, and we have

$$L^{-1}(\infty) = \left[(-1)^{i-k} \binom{i}{j} \right]_{i,j\geqslant 0} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & & \\ -1 & 3 & -3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Analogously, we define the *inverse binomial transform* $\hat{\alpha} = (\hat{\alpha}_i)_{i\geqslant 0}$ of α as $\hat{\alpha}_i =$

$\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\alpha_k$, or equivalently,

$$\begin{bmatrix}\hat{\alpha_0}\\ \hat{\alpha_1}\\ \hat{\alpha_2}\\ \hat{\alpha_3}\\ \vdots\end{bmatrix}=\begin{bmatrix}1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & & \\ -1 & 3 & -3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \vdots\end{bmatrix}.$$

It is clear that (see also [7, Lemma 2.1]):

$$\hat{\check{\alpha}} = \check{\hat{\alpha}} = \alpha. \tag{1}$$

In [7, Theorem 3.1] (see also [2,9]), we obtained a *factorization* of the generalized Pascal triangle $P_{\alpha,\beta}(n)$ associated with the arbitrary sequences α and β (with $\alpha_0 = \beta_0$), as a product of the left-lower-triangular Pascal's matrix $L(n)$, the Toeplitz matrix $T_{\hat{\alpha},\hat{\beta}}(n)$ and the right-upper-triangular Pascal's matrix $U(n) = L(n)^t$, that is

$$P_{\alpha,\beta}(n) = L(n)T_{\hat{\alpha},\hat{\beta}}(n)U(n). \tag{2}$$

Similarly, we showed that $T_{\alpha,\beta}(n) = L(n)^{-1}P_{\check{\alpha},\check{\beta}}(n)U(n)^{-1}$. In fact, we obtained a *connection* between generalized Pascal triangles and Toeplitz matrices. In particular, it follows easily that $\det(P_{\alpha,\beta}(n)) = \det(T_{\hat{\alpha},\hat{\beta}}(n))$.

Similarly, a matrix $H(\infty) = [H_{i,j}]_{i,j\geqslant 0}$ is *Hankel* if $H_{i,j} = H_{k,l}$ whenever $i + j = k + l$. Thus, a Hankel matrix is characterized by the property that the (i, j)-th entry depends only on the sum $i + j$. Again, it is easy to see that an $n \times n$ Hankel matrix $H(n) = [H_{i,j}]_{0\leqslant i,j<n}$ is fully determined by its first row and its last column, henceforth, we will use $H_{\alpha,\beta}(n)$ to describe an $n \times n$ Hankel matrix $H(n)$, where $\alpha = (\alpha_i)_{0\leqslant i<n} = (H_{0,n-1-i})_{0\leqslant i<n}$ and $\beta = (\beta_i)_{0\leqslant i<n} = (H_{i,n-1})_{0\leqslant i<n}$. Note that $\alpha_0 = \beta_0$. There exists a special relation between Toeplitz matrices and the Hankel matrices. Indeed, if $J(n)$ is the $n \times n$ *reflection* (*exchange*) matrix, as

$$J(n) = H_{(1,0,\dots,0),(1,0,\dots,0)}(n) = \begin{bmatrix}0 & 0 & \dots & 0 & 1\\ 0 & 0 & \dots & 1 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 1 & \dots & 0 & 0\\ 1 & 0 & \dots & 0 & 0\end{bmatrix},$$

then we have $J(n)H_{\alpha,\beta}(n) = T_{\alpha,\beta}(n)$, $H_{\alpha,\beta}(n)J(n) = T_{\beta,\alpha}(n)$, or equivalently $H_{\alpha,\beta}(n) = J(n)T_{\alpha,\beta}(n)$, $H_{\alpha,\beta}(n) = T_{\beta,\alpha}(n)J(n)$.

In the present article we obtain a new factorization of generalized Pascal triangles (Theorem 2.2). Then we show that if we replace "Toeplitz matrices" by "Hankel matrices" in (2) or (3) in Theorem 2.2, the multiplication of the factors leads again to a Hankel matrix. Of course, we also note that there are certain families of Hankel and Toeplitz matrices which are also generalized Pascal triangles.

We conclude the introduction with some notations and terms that are used throughout the article. Given a matrix A, we denote by $\mathrm{R}_i(A)$ and $\mathrm{C}_j(A)$ the row i and the column j of A, respectively. The notation A^t denotes the transpose of A. In this article, all matrices are indexed starting with the $(0, 0)$-th element.

2. Main results

We begin with the following simple observation.

LEMMA 2.1. *Let i, j be positive integers and $i \geqslant j$. Then, we have*

$$\sum_{t=0}^{i-j}(-1)^t\binom{i}{i-t}\binom{i-t}{j} = \begin{cases} 0 & \text{if } \; i \neq j, \\ 1 & \text{if } \; i = j. \end{cases}$$

The *proof* follows from the simple identity $\binom{i}{i-t}\binom{i-t}{j} = \binom{i}{j}\binom{i-j}{t}$.

Let $\mathbb{M}(n) := L(n)J(n)$. We call $\mathbb{M}(n)$ the $n \times n$ *right-lower-triangular Pascal's matrix*, whose (i,j)-th entery is equal to

$$\mathbb{M}_{i,j} = \begin{cases} 0 & \text{if} \quad i < n-j-1 \\ \binom{i}{n-j-1} & \text{if} \quad i \geqslant n-j-1, \end{cases}$$

and so

$$\mathbb{M}(n) = \left[\binom{i}{n-j-1}\right]_{0 \leqslant i,j<n} = \begin{bmatrix} & & & & 1 \\ & & & 1 & 1 \\ & & 1 & 2 & 1 \\ & 1 & 3 & 3 & 1 \\ \iddots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-1}{n-1} \; \cdots & \binom{n-1}{3} & \binom{n-1}{2} & \binom{n-1}{1} & \binom{n-1}{0} \end{bmatrix}.$$

Let $\mathbb{N}(n) = \mathbb{M}(n)^t$. We preserve this notation throughout the paper.

THEOREM 2.2. *Let $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be two sequences starting with a common first term $\alpha_0 = \beta_0 = \gamma$. Then, we have*

$$P_{\alpha,\beta}(n) = \mathbb{M}(n)T_{\hat{\beta},\hat{\alpha}}(n)\mathbb{N}(n), \tag{3}$$

and

$$T_{\alpha,\beta}(n) = \mathbb{M}(n)^{-1}P_{\check{\beta},\check{\alpha}}(n)\mathbb{N}(n)^{-1}. \tag{4}$$

Proof. First, we claim that $P_{\alpha,\beta}(n) = \mathbb{M}(n)Q(n)$, where $\mathbb{M}(n) = [\mathbb{M}_{i,j}]_{0\leqslant i,j<n}$ is the right-lower-triangular Pascal's matrix with

$$\mathbb{M}_{i,j} = \binom{i}{n-j-1},$$

and $Q(n) = (Q_{i,j})_{0\leqslant i,j<n}$, with $Q_{i,0} = \hat{\alpha}_{n-i-1}$, $Q_{n-1,j} = \beta_j$ and

$$Q_{i,j} = Q_{i,j-1} + Q_{i+1,j-1}, \qquad 0 \leqslant i < n-1, \qquad 0 < j \leqslant n-1. \tag{5}$$

For instance, when $n = 4$ the matrices $\mathbb{M}(4)$ and $Q(4)$ are given by:

$$\mathbb{M}(4) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix},$$

and $$Q(4)=\begin{bmatrix}\alpha_3-3\alpha_2+3\alpha_1-\gamma & \alpha_3-2\alpha_2+\alpha_3 & \alpha_3-\alpha_2 & \alpha_3\\ \alpha_2-2\alpha_1+\gamma & \alpha_2-\alpha_1 & \alpha_2 & \alpha_2+\alpha_1+\beta_1\\ \alpha_1-\gamma & \alpha_1 & \alpha_1+\beta_1 & \alpha_1+\beta_2+\beta_1\\ \gamma & \beta_1 & \beta_2 & \beta_3\end{bmatrix}.$$

Note that the entries of $\mathbb{M}(n)$ satisfy the following recurrence

$$\mathbb{M}_{i,j}=\mathbb{M}_{i-1,j}+\mathbb{M}_{i-1,j+1},\quad 0<i\leqslant n-1,\quad 0\leqslant j<n-1. \tag{6}$$

In order to have a convenient notation, let us write $\mathbb{M}$, $\mathbb{N}$, Q, $P_{\alpha,\beta}$ and $T_{\alpha,\beta}$ for $\mathbb{M}(n)$, $\mathbb{N}(n)$, $Q(n)$, $P_{\alpha,\beta}(n)$ and $T_{\alpha,\beta}(n)$, respectively. For the proof of the claimed factorization we compute the (i,j)-th entry of $\mathbb{M}Q$, that is $(\mathbb{M}Q)_{i,j}=\sum_{k=0}^{n-1}\mathbb{M}_{i,k}Q_{k,j}$. In fact, it suffices to show that $\mathrm{R}_0(\mathbb{M}Q)=\mathrm{R}_0(P_{\alpha,\beta})$, $\mathrm{C}_0(\mathbb{M}Q)=\mathrm{C}_0(P_{\alpha,\beta})$ and

$$(\mathbb{M}Q)_{i,j}=(\mathbb{M}Q)_{i,j-1}+(\mathbb{M}Q)_{i-1,j},\ \text{for } 1\leqslant i,j<n. \tag{7}$$

First, suppose that $i=0$. Then, we obtain

$$(\mathbb{M}Q)_{0,j}=\sum_{k=0}^{n-1}\mathbb{M}_{0,k}Q_{k,j}=\mathbb{M}_{0,n-1}Q_{n-1,j}=1\beta_j=\beta_j,$$

and so $\mathrm{R}_0(LQ)=\mathrm{R}_0(P_{\alpha,\beta})=(\beta_0,\ \beta_1,\ \ldots,\ \beta_{n-1})$.

Next, suppose that $i\geqslant 1$ and $j=0$. In this case, we have

$$\begin{aligned}(\mathbb{M}Q)_{i,0}&=\sum_{k=0}^{n-1}\mathbb{M}_{i,k}Q_{k,0}=\sum_{k=n-i-1}^{n-1}\left\{\binom{i}{n-k-1}\hat{\alpha}_{n-k-1}\right\}\\ &=\sum_{k=n-i-1}^{n-1}\left\{\binom{i}{n-k-1}\sum_{l=0}^{n-k-1}(-1)^{n-k-1+l}\binom{n-k-1}{l}\alpha_l\right\}\\ &=\sum_{k=n-i-1}^{n-1}\left\{\sum_{l=0}^{n-k-1}(-1)^{n-k-1+l}\binom{i}{n-k-1}\binom{n-k-1}{l}\alpha_l\right\}\\ &=\sum_{l=0}^{i}\alpha_l\sum_{t=0}^{i-l}\left\{(-1)^{i+l-t}\binom{i}{i-t}\binom{i-t}{l}\right\}\\ &=\sum_{l=0}^{i}\alpha_l(-1)^{i+l}\sum_{t=0}^{i-l}\left\{(-1)^{t}\binom{i}{i-t}\binom{i-t}{l}\right\}=\alpha_i,\qquad\text{(by Lemma 2.1)}\end{aligned}$$

and so $\mathrm{C}_0(\mathbb{M}Q)=\mathrm{C}_0(P_{\alpha,\beta})=(\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{n-1})^t$.

Finally, we must establish (7). We therefore assume that $1\leqslant i,j<n$. In this case, using (5) and (6), we obtain

$$\begin{aligned}(\mathbb{M}Q)_{i,j}&=\sum_{k=0}^{n-1}\mathbb{M}_{i,k}Q_{k,j}=\mathbb{M}_{i,n-1}Q_{n-1,j}+\sum_{k=0}^{n-2}\mathbb{M}_{i,k}(Q_{k,j-1}+Q_{k+1,j-1})\\ &=\beta_j+\sum_{k=0}^{n-2}\mathbb{M}_{i,k}Q_{k,j-1}+\sum_{k=0}^{n-2}\mathbb{M}_{i,k}Q_{k+1,j-1}\end{aligned}$$

$$
\begin{aligned}
=&\beta_j + \sum_{k=0}^{n-1} \mathbb{M}_{i,k} Q_{k,j-1} - \mathbb{M}_{i,n-1} Q_{n-1,j-1} + \sum_{k=0}^{n-2} (\mathbb{M}_{i-1,k} + \mathbb{M}_{i-1,k+1}) Q_{k+1,j-1} \\
=&\beta_j + (\mathbb{M}Q)_{i,j-1} - \beta_{j-1} + \sum_{k=0}^{n-2} \mathbb{M}_{i-1,k} Q_{k+1,j-1} + \sum_{k=0}^{n-2} \mathbb{M}_{i-1,k+1} Q_{k+1,j-1} \\
=&\beta_j + (\mathbb{M}Q)_{i,j-1} - \beta_{j-1} + \sum_{k=0}^{n-2} \mathbb{M}_{i-1,k} (Q_{k,j} - Q_{k,j-1}) + \sum_{k=1}^{n-1} \mathbb{M}_{i-1,k} Q_{k,j-1} \\
=&\beta_j + (\mathbb{M}Q)_{i,j-1} - \beta_{j-1} + \sum_{k=0}^{n-2} \mathbb{M}_{i-1,k} Q_{k,j} - \sum_{k=0}^{n-2} \mathbb{M}_{i-1,k} Q_{k,j-1} + \sum_{k=1}^{n-1} \mathbb{M}_{i-1,k} Q_{k,j-1} \\
=&\beta_j + (\mathbb{M}Q)_{i,j-1} - \beta_{j-1} - \mathbb{M}_{i-1,n-1} Q_{n-1,j} \\
&+ \sum_{k=0}^{n-1} \mathbb{M}_{i-1,k} Q_{k,j} - \mathbb{M}_{i-1,0} Q_{0,j-1} + \mathbb{M}_{i-1,n-1} Q_{n-1,j-1} \\
=&\beta_j + (\mathbb{M}Q)_{i,j-1} - \beta_{j-1} - \beta_j + (\mathbb{M}Q)_{i-1,j} - 0 + \beta_{j-1} = (\mathbb{M}Q)_{i-1,j} + (\mathbb{M}Q)_{i,j-1},
\end{aligned}
$$

which is precisely (7).

Next, we claim that $Q = T_{\hat\beta,\hat\alpha}\mathbb{N}$. Note that we have

$$
\mathbb{N}_{i,j} = \begin{cases} 0 & \text{if} \quad j < n-i-1, \\ \binom{j}{n-i-1} & \text{if} \quad j \geqslant n-i-1, \end{cases}
$$

and it is also easy to see that $\mathbb{N}_{i,j} = \mathbb{N}_{i,j-1} + \mathbb{N}_{i+1,j-1}$, $0 \leqslant i < n-1$, $1 \leqslant j < n$.

For instance, if $n = 4$, then the matrices $T_{\hat\beta,\hat\alpha}$ and $\mathbb{N}$ are given by:

$$
T_{\hat\beta,\hat\alpha} = \begin{bmatrix} \gamma & \alpha_1-\gamma & \alpha_2-2\alpha_1+\gamma & \alpha_3-3\alpha_2+3\alpha_1-\gamma \\ \beta_1-\gamma & \gamma & \alpha_1-\gamma & \alpha_2-2\alpha_1+\gamma \\ \beta_2-2\beta_1+\gamma & \beta_1-\gamma & \gamma & \alpha_1-\gamma \\ \beta_3-3\beta_2+3\beta_1-\gamma & \beta_2-2\beta_1+\gamma & \beta_1-\gamma & \gamma \end{bmatrix},
$$

and $$\mathbb{N} = \begin{bmatrix} 0&0&0&1 \\ 0&0&1&3 \\ 0&1&2&3 \\ 1&1&1&1 \end{bmatrix}.$$

Using similar arguments as previously, it suffices to show that

$$
\begin{aligned}
\mathrm{C}_0(T_{\hat\beta,\hat\alpha}\mathbb{N}) =& \mathrm{C}_0(Q), \quad \mathrm{R}_{n-1}(T_{\hat\beta,\hat\alpha}\mathbb{N}) = \mathrm{R}_{n-1}(Q), \\
\text{and} \qquad (T_{\hat\beta,\hat\alpha}\mathbb{N})_{i,j} =& (T_{\hat\beta,\hat\alpha}\mathbb{N})_{i,j-1} + (T_{\hat\beta,\hat\alpha}\mathbb{N})_{i+1,j-1}, \qquad (8)
\end{aligned}
$$

for $0 \leqslant i < n-1$, $0 < j \leqslant n-1$.

As before, the proof of our claim requires some calculations. On the one hand, we have $(T_{\hat\beta,\hat\alpha}\mathbb{N})_{i,0} = \sum_{k=0}^{n-1} T_{i,k}\mathbb{N}_{k,0} = T_{i,n-1}\mathbb{N}_{n-1,0} = \hat\alpha_{n-i-1} \cdot 1 = \hat\alpha_{n-i-1} = Q_{i,0}$, which implies that $\mathrm{C}_0(T_{\hat\beta,\hat\alpha}\mathbb{N}) = \mathrm{C}_0(Q)$. On the other hand, we must evaluate the following sum:

$$\begin{aligned}(T_{\hat{\beta},\hat{\alpha}}\mathbb{N})_{n-1,j} &= \sum_{k=0}^{n-1} T_{n-1,k}\mathbb{N}_{k,j} = \sum_{k=0}^{n-1} \hat{\beta}_{n-1-k}\binom{j}{n-k-1} \\ &= \sum_{k=0}^{n-1}\left\{\left(\sum_{l=0}^{n-1-k}(-1)^{l+n-1-k}\binom{n-1-k}{l}\beta_l\right)\binom{j}{n-k-1}\right\} \\ &= \sum_{k=0}^{n-1}\left(\sum_{l=0}^{n-1-k}(-1)^{l+n-1-k}\binom{n-1-k}{l}\binom{j}{n-k-1}\beta_l\right) \\ &= \sum_{i=0}^{n-1}\beta_i\left\{\sum_{t=0}^{n-1-i}(-1)^{i+n-1-t}\binom{n-1-t}{i}\binom{j}{n-1-t}\right\}. \qquad (9)\end{aligned}$$

Since $\binom{j}{n-1-t} = 0$ for $n-1-t > j$, we may restrict the second sum on the right-hand side of (9) to $t \geqslant n-1-j$, and so we obtain

$$(T_{\hat{\beta},\hat{\alpha}}\mathbb{N})_{n-1,j} = \sum_{i=0}^{n-1}\beta_i\left\{\sum_{t=n-1-j}^{n-1-i}(-1)^{i+n-1-t}\binom{n-1-t}{i}\binom{j}{n-1-t}\right\}. \qquad (10)$$

Moreover, we have $\binom{n-1-t}{i}\binom{j}{n-1-t} = \binom{j}{i}\binom{j-i}{n-1-t-i}$. If this is substituted in (10), then we obtain

$$(T_{\hat{\beta},\hat{\alpha}}\mathbb{N})_{n-1,j} = \sum_{i=0}^{n-1}\beta_i\left\{\sum_{t=n-1-j}^{n-1-i}(-1)^{i+n-1-t}\binom{j}{i}\binom{j-i}{n-1-t-i}\right\},$$

and after some simplification this leads to

$$(T_{\hat{\beta},\hat{\alpha}}\mathbb{N})_{n-1,j} = \sum_{i=0}^{n-1}\beta_i\left\{\binom{j}{i}\sum_{t=0}^{j-i}(-1)^{j+i-t}\binom{j-i}{j-i-t}\right\},$$

or equivalently,

$$(T_{\hat{\beta},\hat{\alpha}}\mathbb{N})_{n-1,j} = \sum_{i=0}^{n-1}\beta_i\left\{(-1)^{j+i}\binom{j}{i}\sum_{t=0}^{j-i}(-1)^{t}\binom{j-i}{t}\right\}.$$

Therefore, using the fact that

$$\sum_{t=0}^{j-i}(-1)^t\binom{j-i}{t} = \begin{cases}0 & \text{if} \quad i \neq j,\\ 1 & \text{if} \quad i = j,\end{cases}$$

it follows that $(T_{\hat{\beta},\hat{\alpha}}\mathbb{N})_{n-1,j} = \beta_j = Q_{n-1,j}$, and so $\mathrm{R}_{n-1}(T_{\hat{\beta},\hat{\alpha}}\mathbb{N}) = \mathrm{R}_{n-1}(Q)$, as desired.

Finally, we assume that $0 \leqslant i < n-1$, $0 < j \leqslant n-1$ and establish (8). Indeed, using (11), we observe that

$$(T_{\hat{\beta},\hat{\alpha}}\mathbb{N})_{i,j} = \sum_{k=0}^{n-1} T_{i,k}\mathbb{N}_{k,j} = \sum_{k=0}^{n-2} T_{i,k}\mathbb{N}_{k,j} + T_{i,n-1}\mathbb{N}_{n-1,j}$$

$$
\begin{aligned}
&=\sum_{k=0}^{n-2} T_{i,k}(\mathbb{N}_{k,j-1}+\mathbb{N}_{k+1,j-1})+T_{i,n-1} \quad (\text{note that } \mathbb{N}_{n-1,j}=1)\\
&=\sum_{k=0}^{n-2} T_{i,k}\mathbb{N}_{k,j-1}+\sum_{k=0}^{n-2} T_{i,k}\mathbb{N}_{k+1,j-1}+T_{i,n-1}\\
&=\sum_{k=0}^{n-1} T_{i,k}\mathbb{N}_{k,j-1}-T_{i,n-1}\mathbb{N}_{n-1,j-1}+\sum_{k=1}^{n-1} T_{i,k-1}\mathbb{N}_{k,j-1}+T_{i,n-1}\\
&=(T_{\hat\beta,\hat\alpha}\mathbb{N})_{i,j-1}-T_{i,n-1}+\sum_{k=0}^{n-1} T_{i+1,k}\mathbb{N}_{k,j-1}-T_{i+1,0}\mathbb{N}_{0,j-1}+T_{i,n-1}\\
&\quad(\text{note that } T_{i,k-1}=T_{i+1,k} \text{ and } \mathbb{N}_{n-1,j-1}=1)\\
&=(T_{\hat\beta,\hat\alpha}\mathbb{N})_{i,j-1}+(T_{\hat\beta,\hat\alpha}\mathbb{N})_{i+1,j-1}, \quad (\text{ note that } \mathbb{N}_{0,j-1}=0),
\end{aligned}
$$

which is (8). The proof of (3) is now complete.

To prove (4), we use the fact that $\mathbb{M}$ is invertible. Observe that

$$\mathbb{M}^{-1}P_{\check\alpha,\check\beta}\mathbb{N}^{-1} \overset{(3)}{=} \mathbb{M}^{-1}(\mathbb{M}T_{\hat{\check\alpha},\hat{\check\beta}}\mathbb{N})\mathbb{N}^{-1}=T_{\hat{\check\alpha},\hat{\check\beta}} \overset{(1)}{=} T_{\alpha,\beta}.$$

This completes the proof of the theorem. □

Remark 2.3. Actually, Theorem 2.2 is just a reformulation of [7, Theorem 3.1]. It is worth pointing out that in (2) $L(n)$ is a lower triangular matrix, while $U(n)$ is an upper triangular matrix, but here neither of the matrices $\mathbb{M}$ and $\mathbb{N}$ are lower or upper triangular matrices, and they have the same structure. Note also that we have proved Theorem 2.2 without using [7, Theorem 3.1]. Nevertheless, let us give a specific and at the same time very simple proof of (3) *with* [7, Theorem 3.1]. For simplicity, we use the notation $\mathbb{M}$, J, $P_{\alpha,\beta}$ and $T_{\alpha,\beta}$ for $\mathbb{M}(n)$, $J(n)$, $P_{\alpha,\beta}(n)$ and $T_{\alpha,\beta}(n)$. Using the fact that $J^t=J$ and that $JT_{\hat\beta,\hat\alpha}J=(JT_{\hat\beta,\hat\alpha})J=H_{\hat\beta,\hat\alpha}J=T_{\hat\alpha,\hat\beta}$, (i.e. Toeplitz matrices $T_{\alpha,\beta}$ are *persymmetric*), it follows that

$$\mathbb{M}T_{\hat\beta,\hat\alpha}\mathbb{M}^t=(LJ)T_{\hat\beta,\hat\alpha}(LJ)^t=L(JT_{\hat\beta,\hat\alpha}J^t)L^t=L(JT_{\hat\beta,\hat\alpha}J)U=LT_{\hat\alpha,\hat\beta}U=P_{\alpha,\beta}.$$

The second main theorem shows that the multiplication of a Hankel matrix by the matrices $L(n)$ and $U(n)=L(n)^t$ (and also $\mathbb{M}(n)$ and $\mathbb{M}(n)^t$) leads again to a Hankel matrix.

Theorem 2.4. *Let $\alpha=(\alpha_i)_{i\geqslant 0}$ and $\beta=(\beta_i)_{i\geqslant 0}$ be two sequences starting with a common first term $\alpha_0=\beta_0=\gamma$. Let n be a fixed positive integer, and let*

$$\omega=(\omega_i)_{0\leqslant i\leqslant 2n-2}=(\alpha_{n-1},\alpha_{n-2},\dots,\alpha_1,\gamma,\beta_1,\dots,\beta_{n-2},\beta_{n-1}).$$

Then there holds

$$L(n)H_{\alpha,\beta}(n)U(n)=H_{\lambda,\mu}(n), \tag{11}$$

where $\lambda=(\breve\omega_{n-1},\breve\omega_{n-2},\dots,\breve\omega_0)$ and $\mu=(\breve\omega_{n-1},\breve\omega_n,\dots,\breve\omega_{2n-2})$. Also, we have

$$\mathbb{M}(n)H_{\alpha,\beta}(n)\mathbb{M}(n)^t=H_{\varphi,\psi}(n), \tag{12}$$

where $\varphi=(\breve\omega_{n-1},\breve\omega_{n-2},\dots,\breve\omega_{2n-2})$ and $\psi=(\breve\omega_{n-1},\breve\omega_{n-2},\dots,\breve\omega_0)$.

Proof. Again, for the sake of simplicity we use the notation L, U, $\mathbb{M}$, $H_{\alpha,\beta}$, $H_{\lambda,\mu}$ and $H_{\varphi,\psi}$ for $L(n)$, $U(n)$, $\mathbb{M}(n)$, $H_{\alpha,\beta}(n)$, $H_{\lambda,\mu}(n)$ and $H_{\varphi,\psi}(n)$. To prove (11), we show that $\mathrm{R}_0(LH_{\alpha,\beta}U) = \mathrm{R}_0(H_{\lambda,\mu})$, $\mathrm{C}_{n-1}(LH_{\alpha,\beta}U) = \mathrm{C}_{n-1}(H_{\lambda,\mu})$, and $(LH_{\alpha,\beta}U)_{i,j} = (LH_{\alpha,\beta}U)_{k,l}$, whenever $i+j=k+l$.

First, suppose that $0 \leqslant j \leqslant n-1$. Then, we obtain

$$\begin{aligned} LH_{\alpha,\beta}U)_{0,j} =& \mathrm{R}_0(LH_{\alpha,\beta})\mathrm{C}_j(U) \\ =& \mathrm{R}_0(H_{\alpha,\beta})\mathrm{R}_j(L) \qquad \text{(note that } \mathrm{R}_0(L) = (1,0,\ldots,0) \text{ and } U = L^t) \\ =& \sum_{k=0}^{j} \binom{j}{k} \alpha_{n-1-k} \qquad \text{(note that } \mathrm{R}_0(H_{\alpha,\beta}) = (\alpha_{n-1},\alpha_{n-2},\ldots,\alpha_1,\gamma)) \\ =& \check{\omega}_j = (H_{\lambda,\mu})_{0,j}, \end{aligned}$$

and so $\mathrm{R}_0(LH_{\alpha,\beta}U) = \mathrm{R}_0(H_{\lambda,\mu})$.

Next, suppose that $0 \leqslant i \leqslant n-1$. Then, we obtain

$$\begin{aligned} (LH_{\alpha,\beta}U)_{i,n-1} =& \mathrm{R}_i(LH_{\alpha,\beta})\mathrm{C}_{n-1}(U) = \mathrm{R}_i(LH_{\alpha,\beta})\mathrm{R}_{n-1}(L) \quad (\text{since } U = L^t) \\ =& \sum_{k=0}^{n-1} (LH_{\alpha,\beta})_{i,k} L_{n-1,k} = \sum_{k=0}^{n-1} \left\{ \sum_{l=0}^{n-1} L_{i,l}(H_{\alpha,\beta})_{l,k} \right\} \binom{n-1}{k} \\ =& \sum_{k=0}^{n-1} \left\{ \sum_{l=0}^{n-1} \binom{i}{l} (H_{\alpha,\beta})_{l,k} \right\} \binom{n-1}{k} = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \binom{i}{l}\binom{n-1}{k} (H_{\alpha,\beta})_{l,k} \\ =& \sum_{0 \leqslant l+k \leqslant 2n-2} \binom{i}{l}\binom{n-1}{k} (H_{\alpha,\beta})_{l,k} \\ & \text{(note that the } (i,j)\text{-th entry depends only on } i+j). \end{aligned}$$

Let $d := l+k$ be fixed with $0 \leqslant d \leqslant 2n-2$. Now we use the fact that the coefficient $(H_{\alpha,\beta})_{l,k}$ is equal to

$$\binom{i}{0}\binom{n-1}{d} + \binom{i}{1}\binom{n-1}{d-1} + \cdots + \binom{i}{d}\binom{n-1}{0},$$

which is exactly the coefficient of x^d in $(1+x)^i(1+x)^{n-1} = (1+x)^{n+i-1}$, and so this coefficient is equal to $\binom{n+i-1}{d}$. Thus we obtain

$$\begin{aligned} (LH_{\alpha,\beta}U)_{i,n-1} =& \sum_{d=0}^{n-1} \binom{n+i-1}{d} (H_{\alpha,\beta})_{0,d} + \sum_{d=n}^{2n-2} \binom{n+i-1}{d} (H_{\alpha,\beta})_{d,n-1} \\ =& \sum_{d=0}^{n-1} \binom{n+i-1}{d} \alpha_{n-1-d} + \sum_{d=n}^{2n-2} \binom{n+i-1}{d} \beta_{d-n+1} \\ =& \sum_{d=0}^{2n-2} \binom{n+i-1}{d} \omega_d = \check{\omega}_{n-i+1} = (H_{\lambda,\mu})_{i,n-1}, \end{aligned}$$

and so $\mathrm{C}_{n-1}(LH_{\alpha,\beta}U) = \mathrm{C}_{n-1}(H_{\lambda,\mu})$.

Finally, suppose that $i+j=k+l$. If one argues exactly as above, one comes to the conclusion that

$$\begin{aligned}(LH_{\alpha,\beta}U)_{i,j} =& \mathrm{R}_i(LH_{\alpha,\beta})\mathrm{C}_j(U) = \mathrm{R}_i(LH_{\alpha,\beta})\mathrm{R}_j(L) \quad (\text{since } U = L^t \\ =& \sum_{t=0}^{n-1}(LH_{\alpha,\beta})_{i,t}L_{j,t} = \sum_{t=0}^{n-1}\left\{\sum_{s=0}^{n-1} L_{i,s}(H_{\alpha,\beta})_{s,t}\right\}\binom{j}{t} \\ =& \sum_{t=0}^{n-1}\left\{\sum_{s=0}^{n-1}\binom{i}{s}(H_{\alpha,\beta})_{s,t}\right\}\binom{j}{t} = \sum_{t=0}^{n-1}\sum_{s=0}^{n-1}\binom{i}{s}\binom{j}{t}(H_{\alpha,\beta})_{s,t} \\ =& \sum_{0\leqslant s+t\leqslant 2n-2}\binom{i}{s}\binom{j}{t}(H_{\alpha,\beta})_{s,t} \qquad (13)\end{aligned}$$

(note that the (s,t)-th entry depends only on $s+t$).

Again it follows that if $d = s+t$, then the coefficient $(H_{\alpha,\beta})_{s,t}$ is equal to

$$\binom{i}{0}\binom{j}{d} + \binom{i}{1}\binom{j}{d-1} + \cdots + \binom{i}{d}\binom{j}{0} = \binom{i+j}{d}.$$

Since $k+l = i+j$, we can write

$$\binom{i+j}{d} = \binom{k+l}{d} = \binom{k}{0}\binom{l}{d} + \binom{k}{1}\binom{l}{d-1} + \cdots + \binom{k}{d}\binom{l}{0}.$$

If this is substituted in (13) and the sums are put together, then we obtain

$$(LH_{\alpha,\beta}U)_{i,j} = \sum_{0\leqslant s+t\leqslant 2n-2}\binom{k}{s}\binom{l}{t}(H_{\alpha,\beta})_{s,t} = (LH_{\alpha,\beta}U)_{k,l}.$$

This completes the proof of (11).

To prove (12), observe that

$$\mathbb{M}H_{\alpha,\beta}\mathbb{M}^t = (LJ)H_{\alpha,\beta}(LJ)^t = L(JH_{\alpha,\beta}J^t)L^t = L(H_{\beta,\alpha})L^t = H_{\varphi,\psi},$$

where $\varphi = (\check{\omega}_{n-1}, \check{\omega}_{n-2}, \ldots, \check{\omega}_{2n-2})$ and $\mu = (\check{\omega}_{n-1}, \check{\omega}_{n-2}, \ldots, \check{\omega}_0)$. Note that in the last equality we have applied (11). □

Remark 2.5. We note here that there are some families of Hankel and Toeplitz matrices, which are also generalized Pascal triangles. It is routine to check that for all $m, n \in \mathbb{Z}$, the Hankel matrices $H_m(n) = 2^m H_{\alpha,\beta}(n)$, where $\alpha = (2^{n-1}, 2^{n-2}, \ldots, 2, 1)$ and $\beta = (2^{n-1}, 2^n, \ldots, 2^{2n-2})$, are the only Hankel matrices of this type. Similarly, the Toeplitz matrices $T_{a,b}(n) = T_{\varphi,\psi}(n)$, where $\varphi = (\varphi_i)_{\geqslant 0}$ and $\psi = (\psi_i)_{\geqslant 0}$ are two sequences satisfying linear recursions:

$$\begin{cases}\varphi_0 = a+b,\ \varphi_1 = a, \\ \varphi_i = \varphi_{i-1} - \varphi_{i-2}, & i \geqslant 2,\end{cases} \quad \text{and} \quad \begin{cases}\psi_0 = a+b,\ \psi_1 = b, \\ \psi_i = \psi_{i-1} - \psi_{i-2}, & i \geqslant 2,\end{cases}$$

are the only Toeplitz matrices of this type. Note that the sequences φ, ψ are also 6-periodic sequences:

$$\begin{aligned}\varphi =& (\varphi_i)_{i\geqslant 0} = (a+b, a, -b, -b-a, -a, b, a+b, a, \ldots) \qquad (\text{6-periodic}), \\ \psi =& (\psi_i)_{i\geqslant 0} = (a+b, b, -a, -a-b, -b, a, a+b, b, \ldots) \qquad (\text{6-periodic}).\end{aligned}$$

For instance, when $m = 0$, $a = 1$, $b = 2$ and $n = 5$, we have:

$$H_0(5) = \begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 2 & 4 & 8 & 16 & 32 \\ 4 & 8 & 16 & 32 & 64 \\ 8 & 16 & 32 & 64 & 128 \\ 16 & 32 & 64 & 128 & 256 \end{bmatrix} = P_{(1,2,4,8,16),(1,2,4,8,16)}(5),$$

and $$T_{1,2}(5) = \begin{bmatrix} 3 & 2 & -1 & -3 & -2 \\ 1 & 3 & 2 & -1 & -3 \\ -2 & 1 & 3 & 2 & -1 \\ -3 & -2 & 1 & 3 & 2 \\ -1 & -3 & -2 & 1 & 3 \end{bmatrix} = P_{(3,1,-2,-3,-1),(3,2,-1,-3,-2)}(5).$$

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