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CLOSURE OPERATIONS AND TERNARY RELATIONS

Chandan Chattopadhyay

Abstract. In this paper, the concept of a ternary relation (named as C-relation) is introduced. It is observed that every closure operator can be used to define a C-relation and conversely, any C-relation induces a closure operator. Thus, topological concepts can be studied in terms of relations.

1. Introduction

It is well known that binary relations play an important role in the study of uniformity [1,2,6] and proximity [4,5]. A uniformity on X is a family of binary relations on X. A proximity on X is a binary relation on $P(X)$, where $P(X)$ denotes the power set of X. In the study of proximity spaces, we have seen that a topology can be generated by considering binary relations that satisfy certain axioms.

It should be noted that:

(i) A function $\mathrm{cl} : P(X) \to P(X)$ is called a closure operator [3] if

$$\mathrm{cl}(\emptyset) = \emptyset, \qquad A \subseteq B \Rightarrow \mathrm{cl}\, A \subseteq \mathrm{cl}\, B, \text{ for all } A \subseteq X, B \subseteq X,$$
$$A \subseteq \mathrm{cl}\, A, \text{ for all } A \subseteq X, \qquad \mathrm{cl}(A \cup B) \subseteq \mathrm{cl}\, A \cup \mathrm{cl}\, B, \text{ for all } A \subseteq X, B \subseteq X,$$
$$\mathrm{cl}(\mathrm{cl}\, A) = \mathrm{cl}\, A, \text{ for all } A \subseteq X.$$

(ii) A closure operator $\mathrm{cl} : P(X) \to P(X)$ generates a topology on X and vice versa, for a topology on X there is a closure operator $\mathrm{cl} : P(X) \to P(X)$ that generates the given topology.

Section 2 examines a ternary relation (called C-relation) that satisfies certain axioms. The third section examines the following:

(i) conditions imposed on the C-relation for respective topological structures,

(ii) the concept of C-continuous function and its relation to continuous functions and

(iii) a new characterization of compact spaces.

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2. Concept of C-relation

Let Y be a non-empty set. Consider a subset ρ of the Cartesian product $P(Y) \times Y \times P(Y)$ satisfying the following axioms.
C(i) : $(A, t, B) \in \rho \Rightarrow A \cap B \neq \emptyset$ for all non-empty subsets A and B of Y and $t \in Y$.

C(ii) : $t \in A \cap B \Rightarrow (A, t, B) \in \rho$ for all $A, B \subseteq Y$ and $t \in Y$.

C(iii) : $(A, t, B) \in \rho$ and $A \subseteq D$, $B \subseteq F \Rightarrow (D, t, F) \in \rho$ for all $A, B, D, F \subseteq Y$ and $t \in Y$.

C(iv) : $(A \cup B, t, D) \in \rho \Rightarrow (A, t, D) \in \rho$ or $(B, t, D) \in \rho$ for all $A, B, D \subseteq Y$ and $t \in Y$.

C(v) : Let $(A, t, B) \in \rho$ and $D \subseteq A \cap B$. If $(D, y, D) \in \rho \, \forall y \in A \cap B$ then $(D, t, D) \in \rho$, for all $A, B \subseteq Y$ and $t \in Y$.

C(vi) : $(A, t, B) \in \rho \Leftrightarrow (B, t, A) \in \rho$ for all $A, B \subseteq Y$ and $t \in Y$.

The subset ρ of $P(Y) \times Y \times P(Y)$ satisfying the above axioms is said to be a C-relation on Y. From C(v), C(ii) and C(i) we observe that if $(A, t, B) \in \rho$ then $(A \cap B, t, A \cap B) \in \rho$.

Example 2.1. Consider $Y = \{a, b\}$. Let $\rho = \{(\{a\}, a, \{a\}), (\{a\}, b, \{a\}), (\{b\}, b, \{b\}), (\{a\}, a, Y), (\{a\}, b, Y), (\{b\}, b, Y), (Y, a, \{a\}), (Y, b, \{a\}), (Y, b, \{b\}), (Y, a, Y), (Y, b, Y)\}$. Then ρ is a C-relation on Y.

Theorem 2.2. *A closure operator* $\mathrm{cl} : P(Y) \to P(Y)$ *generates a C-relation* ρ *on* Y *and this* ρ *induces the same closure operator.*

Proof. Let τ be the topology on Y corresponding to the given closure operator. Define ρ by the rule: $(A, t, B) \in \rho$ iff $t \in \mathrm{cl}(A \cap B)$. Obviously, C(i) holds.

For C(ii), let $t \in A \cap B$. Then let $t \in \mathrm{cl}(A \cap B) \Rightarrow (A, t, B) \in \rho$.

For C(iii) let $(A, t, B) \in \rho$ and $A \subseteq D$, $B \subseteq F$. Then $t \in \mathrm{cl}(A \cap B)$. Now $(A \cap B) \subseteq D \cap F \Rightarrow \mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(D \cap F)$. So $t \in \mathrm{cl}(D \cap F) \Rightarrow (D, t, F) \in \rho$.

For C(iv) let $(A \cup B,\ t, D) \in \rho$. Then $t \in \mathrm{cl}[(A \cup B) \cap D] \Rightarrow t \in \mathrm{cl}(A \cap D) \cup \mathrm{cl}(B \cap D) \Rightarrow t \in \mathrm{cl}(A \cap D)$ or $t \in \mathrm{cl}(B \cap D) \Rightarrow (A, t, D) \in \rho$ or $(B, t, D) \in \rho$.

For C(v) let $(A, t, B) \in \rho$ and $D \subseteq A \cap B$. Furthermore, let $(D, y, D) \in \rho \ \forall y \in A \cap B$. Now let $t \in \mathrm{cl}(A \cap B)$ and $\forall y \in A \cap B$, $y \in \mathrm{cl}(D \cap D) = clD$. Then $A \cap B \subseteq clD \Rightarrow cl(A \cap B) \subseteq \mathrm{cl}\, D$. So $t \in \mathrm{cl}\, D = cl(D \cap D) \Rightarrow (D, t, D) \in \rho$.

For C(vi), $(A, t, B) \in \rho \Leftrightarrow t \in \mathrm{cl}(A \cap B) \Leftrightarrow t \in \mathrm{cl}(B \cap A) \Leftrightarrow (B, t, A) \in \rho$. Therefore, ρ is a C relation on Y.

Now let ρ be a C-relation on Y. We will first show that ρ induces a closure operator $\mathrm{cl} : P(Y) \to P(Y)$. Let $A \subseteq Y$. We define $\mathrm{cl}\, A$ as follows: $t \in \mathrm{cl}\, A$ iff $(A, t, A) \in \rho$. We will show that 'cl' is a closure operator. Note that $\mathrm{cl}\, \emptyset = \emptyset$. Now if $t \in A$, then $t \in A \cap A \Rightarrow (A, t, A) \in \rho$ (by C(ii)). Hence $t \in \mathrm{cl}\, A$. Thus $A \subseteq \mathrm{cl}\, A$, for all $A \subseteq Y$.

Then let $A \subseteq B \subseteq Y$ and let $t \in \mathrm{cl}\, A$. Then $(A, t, A) \in \rho$, and since $A \subseteq B$, by C(iii), $(B, t, B) \in \rho$. Consequently, $t \in \mathrm{cl}\, B$, i.e. $\mathrm{cl}\, A \subseteq \mathrm{cl}\, B$.

Now let $t \in \mathrm{cl}(A \cup B)$. Then $(A \cup B, t, A \cup B) \in \rho$. According to C(iv), $(A, t, A \cup B) \in \rho$ or $(B, t, A \cup B) \in \rho$. So by C(vi), $(A \cup B, t, A) \in \rho$ or $(A \cup B, t, B) \in \rho$. Through C(iv), $(A, t, A) \in \rho$ or $(B, t, A) \in \rho$ or $(A, t, B) \in \rho$ or $(B, t, B) \in \rho$. With C(vi), $(A, t, A) \in \rho$ or $(A, t, B) \in \rho$ or $(B, t, B) \in \rho$. Therefore, $t \in \mathrm{cl}\, A$ or $(A, t, B) \in \rho$ or $t \in \mathrm{cl}\, B$. If $(A, t, B) \in \rho$, then $(A \cap B, t, A \cap B) \in \rho$ according to C(v). This implies $t \in \mathrm{cl}(A \cap B)$. Since $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}\, A$ and $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}\, B$ (as shown above), it follows that $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}\, A \cup clB$.

Now prove that for every $A \subseteq Y$, $\mathrm{cl}(clA) = \mathrm{cl}\, A$. Obviously, $\mathrm{cl}\, A \subseteq \mathrm{cl}(\mathrm{cl}\, A)$. Let $t \in \mathrm{cl}(\mathrm{cl}\, A)$. Then $(\mathrm{cl}\, A, t, \mathrm{cl}\, A) \in \rho$. Now let $y \in \mathrm{cl}\, A$. Then $(A, y, A) \in \rho$. Thus $A \subseteq \mathrm{cl}\, A \cap \mathrm{cl}\, A$ and therefore for all $y \in \mathrm{cl}\, A \cap \mathrm{cl}\, A$, we have $(A, y, A) \in \rho$. It then follows from C(v) that $(A, t, A) \in \rho$. So $t \in \mathrm{cl}\, A$. Consequently, $\mathrm{cl}(\mathrm{cl}\, A) \subseteq \mathrm{cl}\, A$. So $\mathrm{cl}(\mathrm{cl}\, A) = \mathrm{cl}\, A$. Thus 'cl' is a closure operator on Y. Let σ be the corresponding topology on Y generated by this closure operator 'cl'.

As defined above, we now write $t \in \mathrm{cl}_\sigma(A)$ iff $(A, t, A) \in \rho$. We will now show that $\sigma = \tau$. If we can show that $\mathrm{cl}_\sigma A = \mathrm{cl}_\tau A$ for all $A \subseteq Y$, then our proof is complete.

Let $A \subseteq Y$ and $t \in \mathrm{cl}_\sigma A$. Then by definition $(A, t, A) \in \rho \Rightarrow t \in \mathrm{cl}_\tau A$. Thus $\mathrm{cl}_\sigma A \subseteq \mathrm{cl}_\tau A$. Now let $t \in \mathrm{cl}_\tau A$. Then $t \in \mathrm{cl}_\tau(A \cap A) \Rightarrow (A, t, A) \in \rho \Rightarrow t \in \mathrm{cl}_\sigma A$. Therefore $\mathrm{cl}_\tau A \subseteq \mathrm{cl}_\sigma A$. Thus, $\mathrm{cl}_\sigma A = \mathrm{cl}_\tau A$. □

3. Topological concepts and nature of C-relations

For a closure operator 'cl', the generated topology τ on Y and a C-relation ρ on Y is called a C-joint on Y iff the following holds: '$(A, x, B) \in \rho$ iff $x \in \mathrm{cl}_\tau(A \cap B)$', for any two subsets A and B of Y. If τ and ρ on $Y$are C-joint on Y, then (Y, τ, ρ) is called a TR-space.

Consider any TR-space (Y, τ, ρ). The following results in Theorem 3.1 can be easily derived.

Theorem 3.1. *(i) A is a closed subset of Y with respect to τ iff $(A, x, A) \notin \rho$ for all $x \in Y - A$.*

(ii) τ is indiscrete iff for any non-empty subset A of Y, $(A, x, A) \in \rho$ for all $x \in Y$.

(iii) τ is discrete iff for any $x \in Y$, $(Y - \{x\}, x, Y - \{x\}) \notin \rho$.

(iv) τ is separable iff there exists a countable subset A of Y such that $(A, x, A) \in \rho$ for all $x \in Y$.

(v) τ is disconnected iff there exists a non-empty proper subset A of Y such that $(A, x, A) \notin \rho$ for all $x \in Y - A$ and $(Y - A, x, Y - A) \notin \rho$ for all $x \in A$.

Proof. (i) Let A be a closed subset of Y with respect to τ. Let $x \in Y - A$. Then $x \notin A$. Therefore, $x \notin \mathrm{cl}_\tau(A \cap A)$ (since $A = \mathrm{cl}_\tau A$). But we have $(A, x, A) \in \rho$ iff $x \in \mathrm{cl}_\tau(A \cap A)$. Hence it follows that $(A, x, A) \notin \rho$. The 'only if' part follows easily.

(ii) Let τ be indiscrete. Let $A \subseteq Y$ and let A be nonempty. Let $x \in Y$. Since $\mathrm{cl}_\tau A = Y$, we have $x \in \mathrm{cl}_\tau(A \cap A)$. Since τ and ρ are C-joint, it follows that

$(A, x, A) \in \rho$. Conversely, suppose that for any non-empty subset A of Y, $(A, x, A) \in \rho$ for all $x \in Y$. Now τ and ρ are C-joint. Hence, $x \in \mathrm{cl}_\tau(A \cap A)$ for all $x \in Y$, i.e. $\mathrm{cl}_\tau A = Y$. So, Y is the only non-empty closed set in τ. Therefore, τ is indiscrete.

(iii) Let τ be discrete. Let $x \in Y$. Now $Y - \{x\}$ is closed in τ. So $x \notin \mathrm{cl}_\tau(Y - \{x\})$, i.e. $x \notin \mathrm{cl}_\tau((Y - \{x\}) \cap (Y - \{x\}))$. Since τ and ρ are C-joint, it follows that $(Y - \{x\}, x, Y - \{x\}) \notin \rho$. Conversely, let $(Y - \{x\}, x, Y - \{x\}) \notin \rho$ for every $x \in Y$. Since τ and ρ are C-joint, $x \notin \mathrm{cl}_\tau((Y - \{x\}) \cap (Y - \{x\}))$ for every $x \in Y$. So, $x \notin \mathrm{cl}_\tau(Y - \{x\})$ for every $x \in Y$. Hence for each $x \in Y$, $Y - \{x\}$ is closed in τ, i.e. $\{x\}$ is open in τ for every $x \in Y$. Therefore τ is discrete.

(iv) Let τ be separable. Then there exists a countable subset say, A of Y which is dense in τ. Then for each $x \in Y$, $x \in \mathrm{cl}_\tau A$. So, for each $x \in Y$, $x \in \mathrm{cl}_\tau(A \cap A)$. Since τ and ρ are C-joint, it follows that $(A, x, A) \in \rho$ for all $x \in Y$. Conversely, Let there exist a countable subset A of Y for which $(A, x, A) \in \rho$ for all $x \in Y$. Clearly $x \in \mathrm{cl}_\tau(A \cap A)$ for all $x \in Y$. Thus A is dense in τ. Hence τ is separable.

(v) Let τ be disconnected. Then there exists a proper subset A of Y which is both open and closed in τ. Now if $x \in Y - A$, then $x \notin \mathrm{cl}_\tau A$, i.e. $x \notin \mathrm{cl}_\tau(A \cap A)$, i.e. $(A, x, A) \notin \rho$. similarly, $(Y - A, x, Y - A) \notin \rho$ for all $x \in A$. Conversely, let there exist a non-empty proper subset A of Y such that $(A, x, A) \notin \rho$ for all $x \in Y - A$ and $(Y - A, x, Y - A) \notin \rho$ for all $x \in A$. It follows easily that A and $Y - A$ are both closed in τ. Thus A is both open and closed in τ. Hence τ is disconnected. □

DEFINITION 3.2. Let (X, τ, ρ_1) and (Y, σ, ρ_2) be two TR-spaces. A function $f : (X, \tau, \rho_1) \to (Y, \sigma, \rho_2)$ is called C-continuous if $(A, x, B) \in \rho_1 \Rightarrow (f(A), f(x), f(B)) \in \rho_2$.

THEOREM 3.3. *$f : (X, \tau) \to (Y, \sigma)$ is continuous iff $f : (X, \tau, \rho_1) \to (Y, \sigma, \rho_2)$ is C-continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be continuous. Let $(A, x, B) \in \rho_1$. We will show that $(f(A), f(x), f(B)) \in \rho_2$. It suffices to show that $f(x) \in \mathrm{cl}_\sigma(f(A) \cap f(B))$. Let H be any open set in σ that contains $f(x)$. Since f is continuous, $f^{-1}(H) \in \tau$.Now $(A, x, B) \in \rho_1 \Rightarrow x \in \mathrm{cl}_\tau(A \cap B)$. Also $x \in f^{-1}(H)$ and therefore $f^{-1}(H) \cap (A \cap B) \neq \emptyset$. Let $z \in f^{-1}(H) \cap (A \cap B)$. Then $f(z) \in H$ and $f(z) \in f(A \cap B)$. So $f(z) \in f(A) \cap f(B)$. Consequently $H \cap f(A) \cap f(B) \neq \emptyset$. Since H is arbitrarily taken from σ, which contains $f(x)$, it follows that $f(x) \in \mathrm{cl}_\sigma(f(A) \cap f(B))$. This part is therefore proven.

Conversely, let $f : (X, \tau, \rho_1) \to (Y, \sigma, \rho_2)$ be C-continuous. We will show that $f : (X, \tau) \to (Y, \sigma)$ is continuous. It suffices to show that for any subset A of $Y, f(cl_\tau(A)) \subset \mathrm{cl}_\sigma f(A)$. Let $A \subseteq Y$. Let $x \in f(\mathrm{cl}_\tau(A))$. Then there exists $y \in \mathrm{cl}_\tau(A)$ such that $f(y) = x$. Now $y \in \mathrm{cl}_\tau(A) \Rightarrow (A, y, A) \in \rho_1$. Now, since f is C-continuous, $(f(A), f(y), f(A)) \in \rho_2$. Therefore, $f(y) \in \mathrm{cl}_\sigma(f(A) \cap f(A)) = \mathrm{cl}_\sigma f(A)$ i.e. $x \in \mathrm{cl}_\sigma f(A)$. It follows that $f(\mathrm{cl}_\tau(A)) \subseteq \mathrm{cl}_\sigma f(A)$. This completes the proof. □

DEFINITION 3.4. Let (Y, τ, ρ) be a TR-space. A net $\{x_n : n \in D\}$ in Y is said to have a c-cluster point x in Y if for any $A \subseteq Y$ and for any $n \in D$, $(A, x, A) \notin \rho \Rightarrow$ there exists $m \in D$ such that $m \geq n$ and $(A, x_m, A) \notin \rho$.

THEOREM 3.5. *Let (Y, τ, ρ) be a TR-space.Then (Y, τ) is compact iff every net in Y has a c-cluster point in Y.*

Proof. Let (Y, τ) be compact. If possible, let $\{x_n : n \in D\}$ be a net in Y that has no c-cluster point in Y. Then for each $x \in Y$ there exists $A_x \subseteq Y$ and $n_x \in D$ such that $(A_x, x, A_x) \notin \rho$ but for all $m \in D$ with $m \geq n_x$ we have $(A_x, x_m, A_x) \in \rho$.

Now $(A_x, x, A_x) \notin \rho \Rightarrow x \notin cl(A_x)$ for each $x \in Y$ whereas

$$(A_x, x_m, A_x) \in \rho_1 \Rightarrow x_m \in cl(A_x) \quad \text{for all} \quad m \in D \quad \text{with} \quad m \geq n_x. \tag{1}$$

For each $x \in X$ therefore there exists an open set G_x (which contains x) such that

$$G_x \cap A_x = \emptyset. \tag{2}$$

Let us now consider the collection $\{G_x : x \in Y\}$. Since (Y, τ) is compact, there exists a finite subcollection say, $G_{x_1}, G_{x_2}, \ldots, G_{x_k}$ from $\{G_x : x \in Y\}$ such that

$$Y = \bigcup_{i=1}^{k} G_{x_i}. \tag{3}$$

Consider the corresponding n_{x_i} for each $i = 1, 2, \ldots, k$. Since D is a directed set, there exists $n \geq n_{x_i}$ for all $i = 1, 2, \ldots, k$. By (3) $x_n \in G_{x_j}$ for some $j = 1, 2 \ldots, k$. Also by (1), $x_n \in clA_{x_i}$ for all $i = 1, 2, \ldots, k$, so that

$$G_{x_j} \cap A_{x_i} \neq \emptyset \quad \text{for all} \quad i = 1, 2, \ldots, k. \tag{4}$$

But by (2), $G_{x_1} \cap A_{x_1} = \emptyset$, $G_{x_2} \cap A_{x_2} = \emptyset$, …, $G_{x_k} \cap A_{x_k} = \emptyset$. This contradicts with (4). It follows that if (Y, τ) is compact, then every net in Y has a c-cluster point in Y.

Conversely, let (Y, τ) not be compact. Then there exists a net $\{x_n : n \in D\}$ in Y that has no cluster point in Y. We claim that $\{x_n : n \in D\}$ has no c-cluster point in Y. If possible, let $\{x_n : n \in D\}$ have a c-cluster point, say x, in Y. We now claim that x is a cluster point of $\{x_n : n \in D\}$. Let G_x be any open set in τ that contains x. Let $n \in D$. Take $A = X - G_x$. Then A is closed in τ. Since $x \notin A = \mathrm{cl}\, A$, we have $(A, x, A) \notin \rho$. Since x is a c-cluster point of $\{x_n : n \in D\}$, there is $m \in D$ such that $m \geq n$ and $(A, x_m, A) \notin \rho$. So $x_m \notin \mathrm{cl}(A) = A = X - G_x \Rightarrow x_m \in G_x$. So for $n \in D$ there is $m \in D$ such that $m \geq n$ and $x_m \in G_x$. Therefore, x is a cluster point of $\{x_n : n \in D\}$. This is a contradiction. This completes the proof. □

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Department of Mathematics, Narasinha Dutt College, Howrah, West Bengal-711101, India
E-mail: chandanmath2011@gmail.com
ORCID iD: https://orcid.org/0009-0008-0394-9146