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# CHARACTERIZATION OF GENERALIZED HAUSDORFF OPERATOR ON VARIOUS INTEGRABLE SPACES

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**Abstract**. We characterize those $\alpha \in \mathbb{R}$ and $\mu$ positive Borel measure on $(0,1]$ for which generalized Hausdorff operator acts on Hardy spaces of the unit disk. Further, certain conditions on $\mu$, we prove the operator is bounded linear on $H^p(\mathbb{D})$, for different cases of $p$. For $\alpha = 0$, we determine the characterization of the operator on weighted spaces of integrable functions.

## 1. Introduction

For any sequence $\{c_n\}_{n=0}^{\infty}, c_n \in \mathbb{C}$ and $A = (a_{n\ k})_{n,k\in\mathbb{Z}^+}$, which is an infinite matrix, define $\sigma_n := \sum\limits_{k=0}^{\infty} a_{n\ k} c_k$, provided the series converges for each $n \in \mathbb{Z}^+$. In [1, 6] the authors studied the properties of the above sequence for Cesàro, Euler, Taylor and Hausdorff means on various sequence spaces. These matrices were examined in connection with the summability of the series and assorted function spaces, see e.g. [7, 9, 10].

For the Cesàro matrix, Hardy [4] proved that $H : \{c_n\}_{n=0}^{\infty} \to \{\sigma_n\}_{n=0}^{\infty}$ is a bounded linear operator on $l_p$,$1 < p < \infty$ with $\|H\|_p = \frac{p}{p-1}$. Later Hardy [5] obtained the sufficient condition $\int\limits_0^1 t^{-\frac{1}{p}} d\mu(t) < \infty$, for $H$ to be a bounded linear operator on $l_p$, $1 < p < \infty$, for the Hausdorff means $A$ of the matrix.

In another direction, the Hardy result in [8, 12] has been generalized for various function spaces such as real Hardy spaces, the Hardy spaces on the unit disk, Bergmann spaces, etc. Here we will mention some results that were inspiring for this work.

For an analytic function $f$ on the unit disk $\mathbb{D}$, consider the power series

$$H(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_{n\ k} c_k \right) z^n$$

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where $f(z) = \sum_{n=0}^{\infty} c_n z^n$.

It is natural to ask the following questions.

(a) Does the power series converge?

(b) Is a transformation $f \to H(f)$, bounded and linear?

In this paper, we consider $(a^{\alpha}_{n\ k})_{n,k \in \mathbb{Z}^+}$ as a generalized Hausdorff matrix, since it plays an important role in summability theory for various cases of $\mu$ and $\alpha$. It leads to well-known matrices such as Cesàro, Euler, Riesz, Taylor, Holder and so on.

**Generalized Hausdorff matrix:** Let $\alpha \in \mathbb{R}$ and $\mu$ be of bounded variation on $(0, 1]$. For $n, k \in \mathbb{Z}^+$, we define

$$a^{\alpha}_{n\ k} := \begin{cases} \binom{n+\alpha}{n-k} \int\limits_0^1 t^{k+\alpha}(1-t)^{n-k} d\mu(t) & \text{for } 0 \leq k \leq n, \\ 0 & \text{otherwise} \end{cases}$$

For a Cesàro matrix $(a_{n\ k})_{n,k \in \mathbb{Z}^+}$, Siskakis [12] gave an elegant proof that $H$ is a bounded linear operator on $H^p(\mathbb{D})$, $1 \leq p < \infty$. In 2001, P. Galanopoulos and A. G. Siskakis [2] studied the boundedness of the operator for the Hausdorff matrix on $H^p(\mathbb{D})$, $1 < p < \infty$ and obtained the sufficient condition on measure $\mu$.

Finally, we mention that Galanopoulos and Papadimitrakis [3] Hausdorff and quasi-Hausdorff matrices as operators on Hardy and Bergmann spaces and gave a sufficient condition for them to act as bounded linear operators on the corresponding spaces. In this paper, we will study the generalized Hausdorff operator on $H^p(\mathbb{D})$ and find the sufficient condition on $\mu$ to confirm the boundedness of the operator.

The boundedness and weak compactness of the generalized Hardy-Cesàro operator acting on spaces with weighted integrable spaces was studied by Pedersen [11] was investigated. The aim of this work is to characterize the generalized Hausdorff operator on various spaces.

In this paper we take $\alpha > -1$. In Section 2 we study the convergence of the power series $H^{\alpha}(f)(z)$ and establish the relation between the series and the composition operator for a generalized Hausdorff matrix. In Section 3 we show that $H^{\alpha}$ on $H^p(\mathbb{D})$ under certain conditions on $\mu$. In Section 4 we consider the case $\alpha = 0$. The aim of this section is to investigate the action of the Hausdorff operator on weighted spaces of integrable functions and to determine the boundedness of the operator under certain conditions on weighted functions $u$ and $v$ on $\mathbb{R}$.

## 2. A characterization of the generalized Hausdorff operator

THEOREM 2.1. *Let $(a^{\alpha}_{n\ k})_{n,k \in \mathbb{Z}^+}$ be an generalized Hausdorff matrix and $f \in H^p(\mathbb{D})$ for $1 \leq p < \infty$. Then $H^{\alpha}(f)(z)$ represents an analytic function on $\mathbb{D}$.*

*Proof.* Let $f(z) = \sum_{k \in \mathbb{Z}^+} c_k z^k, z \in \mathbb{D}$ and $A_n = \sum_{k=0}^{n} a^{\alpha}_{n\ k} c_k$. Since Taylor coefficients of an analytic function $f \in H^p(\mathbb{D})$ are bounded by $M > 0$.Thus

$$|A_n| = \left|\sum_{k=0}^{n} a_{n\ k}^{\alpha} c_k\right| \leq M \left|\sum_{k=0}^{n} a_{n\ k}^{\alpha}\right| \leq M\mu(0,1] \quad \text{as} \quad \sum_{k=0}^{n} \binom{n+\alpha}{n-k} t^{k+\alpha}(1-t)^{n-k} \leq 1$$

which yields the radius of convergence for $H^{\alpha}(f)(z)$ is atleast 1. Hence $H^{\alpha}(f)(z)$ is an analytic on the unit disk $\mathbb{D}$. □

For $t \in (0,1]$, define $\upsilon_t(z) = \dfrac{tz}{1-z(1-t)}, z \in \mathbb{D}$.

THEOREM 2.2. *Let $f \in H^p(\mathbb{D}), 1 \leq p < \infty$.*
*(a) If $\alpha \geq 0$, then integral representation of $H^{\alpha}(f)$ is given by*

$$H^{\alpha}(f)(z) = \int_0^1 \frac{\upsilon_t(z)^{\alpha+1}}{tz^{\alpha+1}} f\left(\frac{tz}{1-z(1-t)}\right) d\mu(t), z \in \mathbb{D}. \tag{1}$$

*(b) If $\alpha < 0$ and $\int_0^1 t^{\alpha} d\mu(t) < \infty$, then (1) is true for each $z \in \mathbb{D}$.*

*Proof.* (a) For $\alpha \geq 0$, we have $\sup\limits_{0<t\leq 1} \left|\dfrac{t^{\alpha}}{(1-z(t-1))^{\alpha+1}}\right| \leq (1-|z|)^{-\alpha-1}$. Consider the function $\upsilon_t(z)$, as $\upsilon_t(z)$ satisfies the hypothesis of Schwartz's lemma on $\mathbb{D}$ and hence

$$\sup_{0<t\leq 1} |f(\upsilon_t(z))| \leq \frac{C_p\|f\|_p}{(1-|z|)^{\frac{1}{p}}}. \tag{2}$$

Thus integral representation of (1) is finite for each $z \in \mathbb{D}$. The series expansion of

$$H^{\alpha}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_{n\ k}^{\alpha} c_k\right) z^n = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \left(\int_0^1 \binom{n+\alpha}{n-k}(1-t)^{n-k} z^{n-k}\right) c_k z^k t^{k+\alpha} d\mu(t).$$

For each $t \in [0,1]$, the series $\sum_{n=k}^{\infty} \binom{n+\alpha}{n-k}(1-t)^{n-k} z^{n-k}$ converges uniformly to $[1-z(1-t)]^{-(k+\alpha+1)}$. Hence, by interchanging the sum and integral in the above equation,

$$H^{\alpha}(f)(z) = \sum_{k=0}^{\infty} \int_0^1 \frac{1}{[1-z(1-t)]^{k+\alpha+1}} c_k z^k t^{k+\alpha} d\mu(t)$$

$$= \int_0^1 \frac{t^{\alpha}}{[1-z(1-t)]^{\alpha+1}} f\left(\frac{tz}{1-z(1-t)}\right) d\mu(t) = \int_0^1 \frac{\upsilon_t(z)^{\alpha+1}}{tz^{\alpha+1}} f\left(\frac{tz}{1-z(1-t)}\right) d\mu(t).$$

(b) For $\alpha \leq 0$, we have $\left|\dfrac{t^{\alpha}}{(1-z(1-t))^{\alpha+1}}\right| \leq \dfrac{t^{\alpha}}{1-|z|}$. Assuming $f \in H^p(\mathbb{D})$ and by (2),

$$\int_0^1 \frac{t^{\alpha}}{|1-z(1-t)|^{\alpha+1}} |f(\upsilon_t(z))| d\mu(t) \leq \int_0^1 \frac{t^{\alpha}}{1-|z|} \frac{c_p\|f\|_p}{(1-|z|)^{\alpha+1}} d\mu(t) \leq \frac{C_p\|f\|_p}{(1-|z|)^{\alpha+2}} \int_0^1 t^{\alpha} d\mu(t).$$

Thus the integral expression of (1) is finite for each $z \in \mathbb{D}$ if $\int_0^1 t^{\alpha} d\mu(t) < \infty$. □

Theorem 2.3. *For $t \in (0,1]$, define* $T_t(f)(z) = \dfrac{\upsilon_t(z)^{\alpha+1}}{tz^{\alpha+1}} f\left(\dfrac{tz}{(t-1)z+1}\right)$ *for $z$ in $\mathbb{D}$ and $\upsilon_t$ as in Theorem 2.2. Then*

*(i) $T_t$ is bounded linear operator on $H^p(\mathbb{D})$ and $\|T_t\| \le t^{-1+\frac{1}{p}}$, when $p \ge \dfrac{2}{\alpha+1}$.*

*(ii) For $1 < p < \frac{2}{\alpha+1}$, $T_t$ is bounded linear operator on $H^p(\mathbb{D})$ and $\|T_t\| \le t^{\alpha+1-\frac{1}{p}}$.*

*Proof.* (i) For $1 \le p < \infty, H^p(\mathbb{P})$ be the Hardy space of the right half plane $\mathbb{P} = \{z \in \mathbb{C} | \Re(z) > 0\}$. Note that there is a isometric between $H^p(\mathbb{D})$ and $H^p(\mathbb{P})$ through the linear map $V_p : H^p(\mathbb{P}) \to H^p(\mathbb{D})$ defined by

$$V_p(f)(z) = \frac{(4\pi)^{\frac{1}{p}}}{(1-z)^{\frac{2}{p}}} f\left(\frac{1+z}{1-z}\right).$$

Now define the operator on $H^P(\mathbb{P})$ by

$$\widetilde{T_t} = {V_p}^{-1} T_t V_p. \tag{3}$$

Then

$$\begin{aligned}
\widetilde{T_t}(f)(z) &= {V_p}^{-1} T_t[g(z)] \text{ where } g(z) = V_p(f(z)) \\
&= {V_p}^{-1}\left(\frac{t^\alpha}{[(t-1)z+1]^{\alpha+1}} \frac{(4\pi)^{\frac{1}{p}}}{\left[1-\dfrac{tz}{(t-1)z+1}\right]^{\frac{2}{p}}} f(\mu(\upsilon_t(z)))\right), \\
\mu(z) &= \frac{1+z}{1-z} \\
&= \frac{1}{\pi^{\frac{1}{p}}(1+z)^{\frac{2}{p}}}\left(\frac{t^\alpha}{[(t-1)\mu^{-1}(z)+1]^{\alpha+1}} \frac{(4\pi)^{\frac{1}{p}}[(t-1)\mu^{-1}(z)+1]^{\frac{2}{p}}}{[1-\mu^{-1}(z)]^{\frac{2}{p}}} f(\mu(\upsilon_t(\mu^{-1}(z))))\right) \\
&= \frac{4^{\frac{1}{p}}}{(1+z)^{\frac{2}{p}}}\left(\frac{t^\alpha(-z-1)^{\alpha+1}}{(-t(z-1)-2)^{\alpha+1}} \frac{(-tz+t-2)^{\frac{2}{p}}}{(-2)^{\frac{2}{p}}}\right) f(t(z-1)+1) \\
&= t^\alpha \left(\frac{1+z}{-t(z+1)+2}\right)^{\alpha+1-\frac{2}{p}} f(t(z-1)+1)
\end{aligned}$$

For each $t \in (0,1], z \in \mathbb{P}$ and $\frac{2}{\alpha+1} \le p < \infty$, it is easy to check,

$$\left|\frac{z+1}{tz+2-t}\right|^{\alpha+1-\frac{2}{p}} \le t^{\frac{2}{p}-1-\alpha}. \tag{4}$$

Then

$$\|\widetilde{T_t}(f)\|_{H^p(\mathbb{P})} = \sup_{0<x<\infty}\left(\int_{-\infty}^{\infty} |\widetilde{T_t}(f)(x+iy)|^p dy\right)^{\frac{1}{p}}$$

$$= \sup_{0<x<\infty} \left( \int_{-\infty}^{\infty} \left| t^{\alpha} \left( \frac{1+z}{t(z-1)+2} \right)^{\alpha+1-\frac{2}{p}} f(tx+ity-t+1) \right|^{p} dy \right)^{\frac{1}{p}}$$

$$\overset{(4)}{\leq} \sup_{0<x<\infty} \left( \int_{-\infty}^{\infty} \left| t^{\alpha} t^{\frac{2}{p}-1-\alpha} f(tx+ity-t+1) \right|^{p} dy \right)^{\frac{1}{p}}.$$

Replace $y$ by $\frac{y}{t}$, we have

$$\|\widetilde{T}_t(f)\|_{H^p(\mathbb{P})} \leq t^{-1+\frac{1}{p}} \|f\|_{H^p(\mathbb{P})}. \tag{5}$$

Since $V_p$ and its inverse are isometrics maps and by (3), we have $\|\widetilde{T}_t\| = \|T_t\|$. Hence for $\frac{2}{\alpha+1} \leq p < \infty$, we have a desired conclusion.

(ii) Let $f \in H^p(\mathbb{D})$. Using Cauchy integral representation of $f$, we have

$$T_t(f)(z) = \frac{\upsilon_t(z)^{\alpha+1}}{tz^{\alpha+1}} f\left( \frac{tz}{(t-1)z+1} \right) = \frac{t^{\alpha}}{[(t-1)z+1]^{\alpha}} \frac{1}{2\pi} \int_{0}^{2\pi} \frac{f(e^{i\theta})}{1-(e^{-i\theta}t-t+1)z} d\theta.$$

Note that for $z \in \mathbb{D}$,

$$\begin{aligned} &[1-(t-1)z]^{-\alpha}[1-(e^{-i\theta}t-t+1)z]^{-1} \\ &= \sum_{n=0}^{\infty} \binom{\alpha+n-1}{n} (t-1)^n z^n \sum_{n=0}^{\infty} (e^{-i\theta}t-t+1)^n z^n = \sum_{n=0}^{\infty} b_n z^n, \end{aligned}$$

where $b_n = \sum\limits_{k=0}^{n} \binom{\alpha+k-1}{k} (t-1)^k (e^{-i\theta}t-t+1)^{n-k}$. Thus

$$T_t(f)(z) = t^{\alpha} \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{0}^{2\pi} b_n f(e^{i\theta}) d\theta z^n,$$

hence $\displaystyle\int_{0}^{2\pi} e^{-in\mu} T_t(f)(e^{i\mu}) d\mu(\mathbf{t}) = t^{\alpha} \int_{0}^{2\pi} b_n f(e^{i\theta}) d\theta \langle T_t(f), e^{in\mu} \rangle = \langle f, t^{\alpha} \overline{b_n} \rangle \quad n \in \mathbb{Z}^+.$

We know that $H^q$ is spanned by $\{e^{in\theta}\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ is also linear combination of $\{(e^{i\theta}t+1-t)^n\}_{n=0}^{\infty}$, hence $\{b_n\}_{n=0}^{\infty}$ also forms spanning set for $H^q$. Thus for $h \in H^q$, we have

$$\langle T_t(f), h \rangle = \langle f, t^{\alpha} h(\chi_t) \rangle, \quad \chi_t(z) = tz+1-t. \tag{6}$$

For $1 < p < \dfrac{2}{\alpha+1}$,

$$\begin{aligned} \|T_t(f)\|_p &= \sup\{|<T_t(f), h>| : h \in H^q, \| \wedge_h \| \leq 1\} \\ &= \sup\{|<f, t^{\alpha} h(\chi_t)>| : h \in H^q, \| \wedge_h \| \leq 1\} \text{ by (6)} \\ &\leq \sup\{\|f\|_p t^{\alpha} \|h(\chi_t)\|_q : h \in H^q, \| \wedge_h \| \leq 1\} \leq \|f\|_p t^{\alpha} t^{\frac{1}{q}}. \end{aligned}$$

Conclusion of theorem follows from the above inequality. ∎

## 3. Generalized Hausdorff matrix on $H^p(\mathbb{D})$

THEOREM 3.1. *Let $(a^{\alpha}_{n\ k})_{n,k\in\mathbb{Z}^+}$ be a generalized Hausdorff matrix.*

*(a) Suppose $\frac{2}{\alpha+1} \leq p < \infty$ and $\int_0^1 t^{-1+\frac{1}{p}} d\mu(t)$ is finite. Then $H^{\alpha}$ is bounded linear operator on $H^p(\mathbb{D})$ with $\|H^{\alpha}\| \leq C \int_0^1 t^{-1+\frac{1}{p}} d\mu(t)$ for some $C > 0$.*

*(b) For $1 < p < \frac{2}{\alpha+1}$, the corresponding operator $H^{\alpha}$ is bounded on $H^p(D)$ if $\int_0^1 t^{\alpha+1-\frac{1}{p}} d\mu(t) < \infty$ with $\|H^{\alpha}\| \leq \int_0^1 t^{\alpha+1-\frac{1}{p}} d\mu(t)$.*

*Proof.* (a) For $f \in H^p(\mathbb{D})$, we have

$$\|H^{\alpha}(f)\|_{H^p(\mathbb{D})} = \sup_{0<r<1} \left( \int_0^{2\pi} |H^{\alpha}(f)(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} = \sup_{0<r<1} \left( \int_0^{2\pi} \left| \int_0^1 T_t(f)(re^{i\theta}) d\mu(t) \right|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}}.$$

Using Minkowski's integral inequality in the above equation, we have

$$\|H^{\alpha}(f)\|_{H^p(\mathbb{D})} \leq \int_0^1 \|T_t(f)\|_{H^p(\mathbb{D})} d\mu(t) \leq C\|f\|_{H^p(\mathbb{D})} \int_0^1 t^{-1+\frac{1}{p}} d\mu(t) \tag{7}$$

for some $C > 0$. The above inequality obtained by Theorem 2.3 (i).

(b) By (7),we have

$$\|H^{\alpha}(f)\|_{H^p(\mathbb{D})} = \int_0^1 \|T_t(f)\|_{H^p(\mathbb{D})} d\mu(t) \leq \int_0^1 t^{\alpha+1-\frac{1}{p}} d\mu(t).$$

The above inequality obtained by Theorem 2.3 (ii). □

## 4. Hausdorff operator on weighted spaces of integrable functions

In this section, we consider the case $\alpha = 0$. As in Theorem 2.2, the integrable representation of $H(f)(x) = \int_0^1 \frac{\upsilon_t(x)}{tx} f(\upsilon_t(x)) dt$. We will study the action of Generalized Hausdorff matrix on weighted spaces of integrable functions via $H(f)$ on $\mathbb{R}^+$. Let $u$ be a non-negative continuous function on $\mathbb{R}^+$ and $L_1(u)$ be a space of measurable functions $f$ on $\mathbb{R}^+$ with

$$\|f\|_{L_1(u)} = \int_0^1 |f(t)|u(t)dt < \infty.$$

Denote $L_\infty\left(\frac{1}{u}\right)$ be a space of measurable functions $f$ on $\mathbb{R}^+$ with

$$\|f\|_{L_\infty\left(\frac{1}{u}\right)} = ess \sup_{t\in\mathbb{R}^+} \frac{|f(t)|}{u(t)} < \infty.$$

Let $\mathfrak{M}(u)$ be a space of locally finite, complex Borel measures $\mu$ on $\mathbb{R}^+$ with

$$\|\mu\|_{\mathfrak{M}(u)} = \int_0^\infty u(t)d|\mu|(t).$$

In the following theorem, we find the sufficient condition on weighted functions $u$ and $v$ for the boundedness of the operator $H$ from $L_1(u)$ to $L_1(v)$.

Theorem 4.1. *Let $u$ and $v$ be the non-negative weighted functions on $\mathbb{R}^+$. The operator $H$ is bounded linear operator from $L_1(u)$ to $L_1(v)$ iff*

$$\int_x^\infty \frac{v(z)}{z}dz \leq C|1-x|u(x) \quad a.e. \text{ on } \mathbb{R}^+ \text{ for some } C > 0. \tag{8}$$

*Proof.* Let $f \in L_1(u)$ and assume the weighted functions $u$ and $v$ satisfies the inequality (8). Then

$$\int_0^\infty\int_x^\infty \frac{1}{z|1-x|}|f(x)|v(z)dzdx = \int_0^\infty |f(x)|\frac{1}{|1-x|}\int_x^\infty \frac{v(z)}{z}dzdx \leq C\int_0^\infty |f(x)|u(x)dx \tag{9}$$

by the assumptions on $u$ and $v$ in (8). From Fubini's theorem, it follows that

$$\begin{aligned}\int_0^\infty \left|\int_0^z \frac{1}{z}\frac{f(x)}{1-x}dx\right| v(z)dz &\leq \int_0^\infty\int_0^z \frac{1}{z}\frac{|f(x)|}{|1-x|}dxv(z)dz \\ &\leq \int_0^\infty\int_x^\infty \frac{1}{z|1-x|}|f(x)|v(z)dzdx \leq C\|f\|_{L_1(u)},\end{aligned}$$

the last inequality obtained by (9). Note that

$$\begin{aligned}\int_0^\infty |Hf(z)|v(z)dz &= \int_0^\infty \left|\int_0^1 \frac{1}{(t-1)z+1}f\left(\frac{tz}{(t-1)z+1}\right)dt\right| v(z)dz \\ &= \int_0^\infty \left|\int_0^z \frac{1}{z(1-x)}f(x)dx\right| v(z)dz \leq C\|f\|_{L_1(u)}.\end{aligned}$$

Thus $H(f) \in L_1(v)$ and $\|H(f)\|_{L_1(v)} \leq C\|f\|_{L_1(u)}$ for some $C > 0$. Conversely, assume $H$ is bounded linear operator from $L_1(u)$ to $L_1(v)$. Let $h \in C_0\left(\frac{1}{v}\right)$, then by usual inner product with $H(f)$, we obtain

$$\begin{aligned}\langle h, H(f)\rangle &= \int_0^\infty h(z)H(f)(z)dz = \int_0^\infty h(z)\frac{1}{z}\int_0^z \frac{f(x)}{1-x}dxdz \\ &= \int_0^\infty\int_x^\infty \frac{h(z)}{z}\frac{f(x)}{1-x}dzdx = \int_0^\infty \frac{1}{u(x)(1-x)}\int_x^\infty \frac{h(z)}{z}dzf(x)u(x)dx. \qquad (10)\end{aligned}$$

Since $L_1(v) \subset \mathfrak{M}(v)$ which is closed and identify $\mathfrak{M}(v)$ is the dual of $C_0\left(\frac{1}{v}\right)$, then there exists a mapping $\omega : \mathbb{R}^+ \to \mathfrak{M}(v)$ such that $< h, \omega(x) >$ is measurable and bounded on $\mathbb{R}^+$, for every $h \in C_0\left(\frac{1}{v}\right)$ with $\|H\| = ess \sup_{x\in\mathbb{R}^+} \|\omega(x)\|_{\mathfrak{M}(v)}$. Thus

$$\langle h, H(f)\rangle = \int_0^\infty \langle h, \omega(x)\rangle f(x)u(x)dx = \int_0^\infty\int_0^\infty h(z)d\omega(x)(z)f(x)u(x)dx. \qquad (11)$$

Equating (10) and (11), we obtain

$$\int_0^\infty h(z)d\omega(x)(z) = \frac{1}{u(x)(1-x)}\int_x^\infty \frac{h(z)}{z}dz \quad \text{a.e. on} \quad \mathbb{R}^+,$$

for every $h \in C_0(\frac{1}{v})$. As $\omega(x)$ is an element of $\mathfrak{M}(v)$, we get

$$d\omega(x)(z) = \frac{1}{zu(x)(1-x)}\chi_{z\geq x}dz \quad \text{a.e. on} \quad \mathbb{R}^+,$$

where $\chi$ is the characteristic function on the irrespective domain. Then

$$\|\omega(x)\|_{L_1(v)} = \int_0^\infty v(z)d\omega(x)(z) = \int_0^\infty v(z)\frac{1}{zu(x)|1-x|}\chi_{z\geq x}dz = \frac{1}{u(x)|1-x|}\int_x^\infty \frac{v(z)}{z}dz.$$

By the last equality, we conclude the inequality (8) holds a.e. on $\mathbb{R}^+$, if the operator is bounded on weighted spaces of the integrable functions from $L_1(u)$ to $L_1(v)$. □

THEOREM 4.2. *Let $H$ be a bounded linear operator from $L_1(u)$ to $L_1(v)$ such that the condition (8) holds a.e. on $\mathbb{R}^+$. Then there exists an adjoint operator $H^* : L_\infty(\frac{1}{v}) \to L_\infty(\frac{1}{u})$ such that $H^*h(x) = \frac{1}{1-x}\int_x^\infty \frac{h(z)}{z}dz.$*

*Proof.* For $f \in L_1(u)$ and $h \in L_1(v)$, we have

$$\begin{aligned}\langle H(f), h\rangle &= \int_0^\infty H(f)(z)h(z)dz = \int_0^\infty\int_0^1 \frac{1}{(t-1)z+1}f\left(\frac{tz}{(t-1)z+1}\right)dth(z)dz \\ &= \int_0^\infty\int_0^z \frac{1}{z(1-x)}f(x)dxh(z)dz = \int_0^\infty\int_x^\infty \frac{1}{z(1-x)}f(x)h(z)dzdx\end{aligned}$$

$$= \int_0^\infty f(x) \int_x^\infty \frac{1}{z(1-x)} h(z)dz = \langle f, H^* h\rangle.$$

Thus $H^*$ is an adjoint operator of $H$ on $L_1(u)$. Also, for $h \in L_\infty\left(\frac{1}{v}\right)$, we have

$$\left| \frac{1}{1-x} \int_x^\infty \frac{h(z)}{z} dz \right| \leq \|h\|_{L_\infty\left(\frac{1}{v}\right)} \frac{1}{|1-x|} \int_x^\infty \frac{v(z)}{z} dz \leq \|h\|_{L_\infty\left(\frac{1}{v}\right)} u(x) \text{ a.e. on } \mathbb{R}^+.$$

The last inequality is obtained by the boundedness of $H$. □

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