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A NOTE ON PO-EQUIVALENT TOPOLOGIES

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Abstract. Two topologies on a set X are called PO-equivalent if their families of preopen sets concide. Let $P(\mathcal{T})$ stand for the class of all topologies on X which are PO-equivalent to $\mathcal{T}$ and denote by $\mathcal{T}_M$ the topology on X having for a base $\mathcal{T}_\alpha \cup \{\{x\} \mid \{x\}$ is closed-and-open in $\mathcal{T}_\gamma\}$. It was proved in [Andrijević, M. Ganster, *On PO-equivalent topologies*, Suppl. Rend. Circ. Mat. Palermo, **24** (1990), 251–256] that the class $P(\mathcal{T})$ does not have the largest member in general. Precisely, if $P(\mathcal{T})$ has the largest member, say $\mathcal{U}$, then $\mathcal{U} = \mathcal{T}_M$. On the other hand, it was shown that $\mathcal{T}_M$ does not necessarily belong to $P(\mathcal{T})$. In this paper we are going to show that the topology $\mathcal{T}_M$ is actually the least upper bound of the class $P(\mathcal{T})$.

1. Introduction

Let A be a subset of a topological space $(X, \mathcal{T})$. We denote the closure and the interior of A in $(X, \mathcal{T})$ by clA and $intA$ respectively. The class of closed sets (resp. closed-and-open sets) in $(X, \mathcal{T})$ is denoted by $C(\mathcal{T})$ (resp. $CO(\mathcal{T})$). The class of nowhere dense sets in $(X, \mathcal{T})$ is denoted by $N(\mathcal{T})$.

Definition 1.1. A subset A of a space X is called:
(i) an α-set if $A \subset int(cl(intA))$ ([6]), (ii) semi-open if $A \subset cl(intA)$ ([4]),
(iii) preopen if $A \subset int(clA)$ ([5]).

We denote the classes of these sets in $(X, \mathcal{T})$ by $\mathcal{T}_\alpha$, $SO(\mathcal{T})$ and $PO(\mathcal{T})$ respectively. They are all larger than $\mathcal{T}$ and closed under forming arbitrary unions. It was shown in [6] that $\mathcal{T}_\alpha$ is a topology on X. The closure and interior of A in $(X, \mathcal{T}_\alpha)$ are denoted by $cl_\alpha A$ and $int_\alpha A$. The complement of a semi-open set (resp. preopen set) is called *semi-closed* (resp. *preclosed*). We denote these classes by $SC(\mathcal{T})$ and $PC(\mathcal{T})$. For a subset A of X, the *semi-closure* (resp. *preclosure*) of A, denoted by $sclA$ (resp. $pclA$), is the intersection of all semi-closed (resp. preclosed) subsets of X

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that contain A. The *semi-interior* (resp. *preinterior*) of A, denoted by $sintA$ (resp. $pintA$), is the union of all semi-open (resp. preopen) subsets of X contained in A.

Although the classes $SO(\mathcal{T})$ and $PO(\mathcal{T})$ are not topologies on X in general, they generate a topology in a natural way. Let $\mathcal{T}(\mathcal{A}) = \{G \in \mathcal{A} \mid G \cap A \in \mathcal{A}$ whenever $A \in \mathcal{A}\}$ where $\mathcal{A}$ stands for $SO(\mathcal{T})$ and $PO(\mathcal{T})$. It is clear that $\mathcal{T}(\mathcal{A})$ is a topology on X that is larger than $\mathcal{T}$ and $\{x\} \in \mathcal{T}(\mathcal{A})$ if $\{x\} \in \mathcal{A}$. It was shown in [6] that $\mathcal{T}(\mathcal{A}) = \mathcal{T}_\alpha$ for $\mathcal{A} = SO(\mathcal{T})$. The topology generated in this way by $PO(\mathcal{T})$ was studied in [1] and denoted by $\mathcal{T}_\gamma$. The closure and the interior of a set A in $(X, \mathcal{T}_\gamma)$ are denoted by $cl_\gamma A$ and $int_\gamma A$. Further details on $\mathcal{T}(\mathcal{A})$ can be found in [2].

Definition 1.2 ([3]). Two topologies $\mathcal{T}$ and $\mathcal{U}$ on a set X are called PO-equivalent if $PO(\mathcal{T}) = PO(\mathcal{U})$.

The class of all topologies on X that are PO-equivalent to $\mathcal{T}$ is denoted by $P(\mathcal{T})$. For a space $(X, \mathcal{T})$ let $M = M(\mathcal{T}) = \{x \in X \mid \{x\} \in CO(\mathcal{T}_\gamma)\}$ and let $\mathcal{T}_M$ be the topology on X that has for a base $\mathcal{T}_\alpha \cup \{\{x\} \mid x \in M\}$, i.e. $V \in \mathcal{T}_M$ if and only if $V = G \cup K$ with $G \in \mathcal{T}_\alpha$ and $K \subset M$ [3]. The question arose as to whether the class $P(\mathcal{T})$ has the largest member and in [3] it was answered in the negative. It was proved [3, Theorem 2.9] that if $P(\mathcal{T})$ has the largest member, say $\mathcal{U}$, then $\mathcal{U} = \mathcal{T}_M$. The next example [3] shows that $\mathcal{T}_M$ does not necessarily belong to $P(\mathcal{T})$. Let $X = \mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Set $\mathcal{A} = \{A \subset X \mid z \in A \text{ iff } -z \in A\}$ and let $\mathcal{T} = \{\emptyset, X\} \cup \{G \in \mathcal{A} \mid 0 \notin G$ or $X \setminus G$ is finite$\}$. Then:

(i) $\mathcal{T}$ is a topology on X,

(ii) $PO(\mathcal{T}) = \{\emptyset, X\} \cup \{A \subset X \mid 0 \notin A$ or clA is open$\}$,

(iii) $\mathcal{T}_\gamma = \{\emptyset, X\} \cup \{A \subset X \mid 0 \notin A$ or $X \setminus A$ is finite$\}$,

(iv) $PO(\mathcal{T}_\gamma) = \mathcal{T}_\gamma$.

If now $S = \{0, 1, 2, \ldots\}$, then $S \in PO(\mathcal{T}) \setminus PO(\mathcal{T}_\gamma)$, so $\mathcal{T}$ and $\mathcal{T}_\gamma$ are not PO-equivalent. On the other hand, since $z \in CO(\mathcal{T}_\gamma)$ for every $z \neq 0$we have $\mathcal{T}_M = \mathcal{T}_\gamma$. In our case, $\mathcal{T}_M$ does not belong to $P(\mathcal{T})$, i.e. $P(\mathcal{T})$ does not have the largest member.

In this paper we will show that the topology $\mathcal{T}_M$ is indeed the smallest upper bound or supremum of the class $P(\mathcal{T})$.

Now we recall some results that we will need in the sequel.

Proposition 1.3 ([2]). *Let A be a subset of a space X. Then:*
(i) $cl_\alpha A = A \cup cl(int(clA))$, $int_\alpha A = A \cap int(cl(intA))$,

(ii) $sclA = A \cup int(clA)$, $sintA = A \cap cl(intA)$,

(iii) $pclA = A \cup cl(intA)$, $pintA = A \cap int(clA)$.

Proposition 1.4 ([2]). *Let A be a subset of a space X. Then* $int_\alpha cl_\alpha A = int(clA)$.

Proposition 1.5 ([6]). *Let* $(X, \mathcal{T})$ *be a space. Then* $\mathcal{T}_\alpha = \{U \setminus A \mid U \in \mathcal{T}, A \in N(\mathcal{T})\}$.

Proposition 1.6 ([1]). *Let A be a subset of a space X. Then:*
(i) $cl_\gamma intA = cl(intA)$, $int_\gamma clA = int(clA)$, *(ii)* $pint(cl_\gamma A) = cl_\gamma A \cap int(clA)$.

Proposition 1.7 ([1]). *] Let A be a subset of a space X. Then:*
(i) $cl_\alpha A = cl_\gamma A \cup int(clA)$, *(ii)* $int_\alpha A = int_\gamma A \cap cl(intA)$.

Proposition 1.8 ([1]). *Let* $(X, \mathcal{T})$ *be a space and* $A \in \mathcal{T}_\gamma$. *Then* $sintA = int_\alpha A$.

Proposition 1.9 ([2]). *Let* $G \in \mathcal{T}_\gamma$ $x \in G \setminus cl(intG)$. *Then* $\{x\} \in PO(\mathcal{T}) \setminus \mathcal{T}$.

Proposition 1.10 ([2]). *Let A be a subset of a space* $(X, \mathcal{T})$ *and* $x \in int(clA) \setminus cl_\gamma A$. *Then* $\{x\} \in PO(\mathcal{T}) \setminus \mathcal{T}$.

Proposition 1.11 ([2]). *Let A be a subset of a space* $(X, \mathcal{T})$. *Then* $A \in \mathcal{T}_\gamma$ *if and only if* $A = G \cup H$ *with* $G \in \mathcal{T}_\alpha$ *and* $\{h\} \in PO(\mathcal{T}) \setminus \mathcal{T}$ *for every* $h \in H$.

Proposition 1.12 ([3]). *Let* $(X, \mathcal{T})$ *be a space,* $A \in CO(\mathcal{T}_\gamma)$ *and* $\mathcal{U}$ *the topology on X having* $\mathcal{T} \cup \{A, X \setminus A\}$ *as a subbase. Then* $PO(\mathcal{U}) = PO(\mathcal{T})$.

2. Topological space $(X, \mathcal{T}_M)$

We have already mentioned that our main goal is to show that $\mathcal{T}_M$ is the smallest upper bound of $P(\mathcal{T})$. First, we establish a few lemmas. The operators on a set A in $(X,\mathcal{U})$ with $\mathcal{U} \in P(\mathcal{T})$ are denoted by $cl_\mathcal{U} A$, $int_\mathcal{U} A$, $pcl_\mathcal{U} A$, etc.

Lemma 2.1. *Let* $\mathcal{U} \in P(\mathcal{T})$ *and* $A \in \mathcal{U}$. *Then* $cl_\gamma A = pclA$.

Proof. Since $\mathcal{U} \in P(\mathcal{T})$, we have that $P(\mathcal{U}) = P(\mathcal{T})$ and so $\mathcal{U} \subset \mathcal{U}_\gamma = \mathcal{T}_\gamma$. Thus by Proposition 1.3(iii) we have $cl_\gamma A \subset cl_\mathcal{U} A = A \cup cl_\mathcal{U} int_\mathcal{U} A = pcl_\mathcal{U} A = pclA \subset cl_\gamma A$. □

Lemma 2.2. *Let* $\mathcal{U} \in P(\mathcal{T})$ *and* $A \in \mathcal{U}$. *Then* $\{x\} \in PO(\mathcal{T}) \setminus \mathcal{T}$ *for every* $x \in int(clA) \setminus cl(intA)$.

Proof. By Lemma 2.1 we have that $cl_\gamma A = A \cup cl(intA)$ and thus $int(clA) \setminus cl(intA) = (int(clA) \setminus cl_\gamma A)) \cup (A \setminus cl(intA))$. Now the statement follows from Propositions 1.9 and 1.10. □

Lemma 2.3. *Let* $A \in PO(\mathcal{T}) \cap C(\mathcal{T}_\gamma)$. *Then* $clA \in \mathcal{T}$.

Proof. Since $cl_\alpha A = clA$ for $A \in PO(\mathcal{T})$, applying Proposition 1.7(i) we have that $clA = cl_\gamma A \cup int(clA) = A \cup int(clA) = int(clA)$ that is $clA \in \mathcal{T}$. □

The next lemma follows immediately from Proposition 1.10.

Lemma 2.4. *Let* $\{x\} \in PO(\mathcal{T})$ *and* $y \in int(cl\{x\}) \setminus cl_\gamma\{x\}$. *Then* $cl\{y\} = cl\{x\}$.

Lemma 2.5. *Let* $\mathcal{U} \in P(\mathcal{T})$ *and* $\{x\} \in PO(\mathcal{T}) \setminus C(\mathcal{T}_\gamma)$ *such that* $int(cl\{x\}) \cap cl_\gamma\{x\} = \{x\}$. *Then* $int_\mathcal{U} cl_\mathcal{U}\{x\} = int(cl\{x\})$.

Proof. First, we note that $cl\{x\} = cl_\alpha\{x\} = cl_\gamma\{x\} \cup int(cl\{x\})$ by Proposition 1.7. Since $PO(\mathcal{T}) = PO(\mathcal{U})$ implies$\mathcal{T}_\gamma = \mathcal{U}_\gamma$, we have by Proposition 1.6(ii) that $int_\mathcal{U} cl_\mathcal{U}\{x\} \cap cl_\gamma\{x\} = pint_\mathcal{U} cl_\gamma\{x\} = pint(cl_\gamma\{x\}) = cl_\gamma\{x\} \cap int(cl\{x\}) = \{x\}$. Now set $U =$

$int_{\mathcal{U}} int(cl\{x\})$. Since $int(cl\{x\}) \notin C(\mathcal{T})$, we have that $int(cl\{x\}) \notin PC(\mathcal{T}) = PC(\mathcal{U})$ and therefore $U \neq \emptyset$. On the other hand, since by Lemma 2.4 $clU = cl\{x\} \notin \mathcal{T}$, it follows from Lemma 2.3 that $U \notin C(\mathcal{T}_\gamma)$ and therefore $U \notin C(\mathcal{U})$. Consequently, $U \notin PC(\mathcal{U}) = PC(\mathcal{T})$ and therefore $intU \neq \emptyset$. Therefore, we have by Lemma 2.4 that $intU = int(cl\{x\})$ and thus $U = int(cl\{x\})$, that is $int(cl\{x\}) \in \mathcal{U}$.

In a similar way, we prove that $int_{\mathcal{U}} cl_{\mathcal{U}}\{x\} \in \mathcal{T}$ and thus $int(cl\{x\}) \cap int_{\mathcal{U}} cl_{\mathcal{U}}\{x\}$ is open in both $\mathcal{T}$ and $\mathcal{U}$. Therefore, $int_{\mathcal{U}} cl_{\mathcal{U}}\{x\} = int(cl(\{x\}))$ is again given by Lemma 2.4. $\square$

PROPOSITION 2.6. *Let $\mathcal{U} \in P(\mathcal{T})$ and $A \in \mathcal{U}$. Then $A \setminus cl(intA) \subset M$.*

Proof. Let $A \in \mathcal{U}$ and $x \in A \setminus cl(intA)$. Then $\{x\} \in \mathcal{T}_\gamma$ by Proposition 1.9 and assume that $\{x\} \notin C(\mathcal{T}_\gamma)$. Since $int(cl\{x\}) \subset int(clA) \setminus cl(intA)$, we have by Lemma 2.2 that $int(cl\{x\}) \cap cl_\gamma\{x\} = \{x\}$. Now it follows from Lemma 2.5, Proposition 1.6(i) and Lemma 2.1 that $int(cl\{x\}) = int_{\mathcal{U}} cl_{\mathcal{U}}\{x\} \subset cl_{\mathcal{U}} A \setminus cl(intA) = cl_{\mathcal{U}_\gamma} A \setminus cl(intA) = cl_\gamma A \setminus cl(intA) = (A \cup cl(intA)) \setminus cl(intA) = A \setminus cl(intA)$, a contradiction. Therefore $\{x\} \in C(\mathcal{T}_\gamma)$ and thus $A \setminus cl(intA) \subset M$. $\square$

Now we are in a position to prove [3, Theorem 2.8] without the condition $\mathcal{T} \subset \mathcal{U}$.

PROPOSITION 2.7. *Let $(X, \mathcal{T})$ be a space and $\mathcal{U} \in P(\mathcal{T})$. Then $\mathcal{U} \subset \mathcal{T}_M$.*

Proof. Let $A \in \mathcal{U}$. Since $A = sintA \cup (A \setminus cl(intA))$, the statement follows from Propositions 1.8 and 2.6. $\square$

COROLLARY 2.8. *Let $(X, \mathcal{T})$ be a space. Then $\mathcal{T}_M$ is the least upper bound of the class $P(\mathcal{T})$.*

Proof. Let $\mathcal{V}$ be an upper bound of the class $P(\mathcal{T})$ and suppose that $\{x\} \in CO(\mathcal{T}_\gamma)$. Then $\{x\} \in \mathcal{V}$ by Proposition 1.12. On the other hand, $\mathcal{T}_\alpha \subset \mathcal{V}$ follows from Proposition 1.4 and hence $\mathcal{T}_M \subset \mathcal{V}$. $\square$

We conclude our investigation with some further results on $\mathcal{T}_M$. The closure and the interior of a set A in $(X, \mathcal{T}_M)$ are denoted by $cl_M A$ and $int_M A$.

LEMMA 2.9. *Let $(X, \mathcal{T})$ be a space, and $\{x\} \in CO(\mathcal{T}_\gamma)$. Then $\{y\} \in CO(\mathcal{T}_\gamma)$ for every $y \in cl\{x\}$.*

Proof. By Lemma 2.3 we have that $cl\{x\} \in \mathcal{T}$ and Proposition 1.10 implies that all singletons in $cl\{x\}$ are preopen in $(X, \mathcal{T})$. Therefore all of them are closed-and-open in $(X, \mathcal{T}_\gamma)$. $\square$

PROPOSITION 2.10. *Let $(X, \mathcal{T})$ be a space. Then $N(\mathcal{T}_M) = N(\mathcal{T})$.*

Proof. By Proposition 1.6(i) we have that $int_M cl_M A \subset int_\gamma clA = int(clA)$ and thus $N(\mathcal{T}) \subset N(\mathcal{T}_M)$. To prove the reverse inclusion, assume that $int_M cl_M A = \emptyset$ and $int(clA) \neq \emptyset$. Let $x \in U = int(clA) \setminus cl_M A$. Then $U \in \mathcal{T}_M$, $intU = \emptyset$ and thus $\{x\} \in CO(\mathcal{T}_\gamma)$. Then by Lemma 2.3 $cl\{x\} \in \mathcal{T}$ and thus $cl\{x\} \cap A \neq \emptyset$. So by Lemma 2.9 $int_M A \neq \emptyset$, a contradiction. Therefore, $int(clA) = \emptyset$, i.e. $N(\mathcal{T}_M) \subset N(\mathcal{T})$. $\square$

Corollary 2.11. *Let $(X, \mathcal{T})$ be a space and $x \in X$. Then $\{x\} \in PO(\mathcal{T})$ if and only if $\{x\} \in PO(\mathcal{T}_M)$.*

Proposition 2.12. *Let $(X, \mathcal{T})$ be a space. Then:*
(i) $\mathcal{T}_{M\alpha} = \mathcal{T}_M$, (ii) $\mathcal{T}_{M\gamma} = \mathcal{T}_\gamma$, (iii) $\mathcal{T}_{MM} = \mathcal{T}_M$.

Proof. (i) By Proposition 1.5 it suffices to show that every nowhere dense set in $(X, \mathcal{T}_M)$ is closed in $(X, \mathcal{T}_M)$ and from Proposition 2.10 we have that $N(\mathcal{T}_M) = N(\mathcal{T}) \subset C(\mathcal{T}_\alpha) \subset C(\mathcal{T}_M)$.

(ii) Suppose that $A \in \mathcal{T}_{M\gamma}$. Then by Proposition 1.9 we have that $A = G \cup H$ with $G \in \mathcal{T}_{M\alpha}$ and $\{h\} \in PO(\mathcal{T}_M) \setminus \mathcal{T}_M$ for every $h \in H$. Hence $A \in \mathcal{T}_\gamma$ by (i) and Corollary 2.11. To prove the reverse inclusion suppose that $A \in \mathcal{T}_\gamma$. Then by Proposition 1.11 we have that $A = G \cup H$ with $G \in \mathcal{T}_\alpha$ and $\{h\} \in PO(\mathcal{T}) \setminus \mathcal{T}$ for every $h \in H$. By Corollary 2.11 we have that $\{h\} \in PO(\mathcal{T}_M)$ that is $\{h\} \in \mathcal{T}_{M\gamma}$ and so $A \in \mathcal{T}_{M\gamma}$.

(iii) Let $A \in \mathcal{T}_{MM}$. Then $A = G \cup H$ with $G \in \mathcal{T}_{M\alpha}$ and $\{h\} \in CO(\mathcal{T}_{M\gamma})$ for every $h \in H$. Now the statement follows from (i) and (ii). □

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