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ON ERDŐS-LAX AND BERNSTEIN INEQUALITIES FOR GENERATING OPERATORS

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Abstract. In this paper, some generalizations and improved versions of classical Erdős-Lax inequality and Bernstein inequality are stated and proved.

1. Introduction

Let $P(z) = c_0 + \sum_{v=1}^{n} c_v z^v$ be a polynomial of degree n and $P'(z)$ its derivative. The following inequality, which concerns the norm of the derivative and the polynomial itself on the unit disc, is classical and known as Bernstein's inequality:

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

In the literature there are various inequalities, or rather systems of inequalities, that revolve around the Bernstein inequality. For further details, we refer to [3, 7, 8].

Furthermore, some additional information about $P(z)$ and its zeros, i.e., for the class of polynomials $P(z)$ that do not vanish in the interior of the unit disc, the inequality (1) was replaced by:

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|, \tag{2}$$

and is sharp with equality for the polynomials that have all their zeros on the unit disc. As is well known, the inequality (2) was conjectured by Erdős and later proved by Lax [5]. These inequalities are known for various regions of the complex plane and for various norms, such as weighted L_p-norms, and for many classes of functions, such as polynomials with various constraints. Here we investigate some new operators and thereby establish some operator-preserving inequalities such as (1) and (2). In addition, we will see that the inequalities (1) and (2) preserved by ordinary derivatives are actually special cases of them.

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Let us now take the set $S_n = \{z_1, z_2, \ldots, z_n\}$ of n complex numbers that are not necessarily distinct in the complex plane, let $\{P_k(z)|1 \leq k \leq n\}$ denote the sequence of n polynomials each of degree $n-1$ given by:

$$P_k(z) = \prod_{j=1, j\neq k}^{n} (z - z_j).$$

This term was given by Barrero and Egozcue [2] and called a sequence of "*incomplete polynomials*". Now we say that the essence of the ordinary derivative of $P(z)$ normalised to a monic polynomial $P(z)$ of degree n whose zeros are $z_1, z_2, \ldots, z_n$ is a convex linear combination of members of $\{P_k(z)\}$ in the sense that:

$$P'(z) = \sum_{k=1}^{n} \prod_{j=1, j\neq k}^{n} (z - z_j).$$

Equivalently, the derivative of the monic polynomial $P(z)$ can be expressed as:

$$\frac{1}{n}P'(z) = \sum_{k=1}^{n} \frac{1}{n} P_k(z) \tag{3}$$

where all coefficients of the convex linear combination are $\frac{1}{n}$, and $\sum_{k=1}^{n} \frac{1}{n} = 1$.

Definition 1.1. Let $P(z) = \prod_{j=1}^{n}(z - z_j)$ be a polynomial of degree n and if $\alpha_k > 0$, $1 \leq k \leq n$ with $\sum_{k=1}^{n} \alpha_k = 1$, then $\mathcal{D}_{(\alpha_1, \alpha_2, \ldots, \alpha_n)} P(z) = \sum_{k=1}^{n} \alpha_k P_k(z)$.

We call this operator "convex generalized derivative or convex derivative of $P(z)$".

Note that $\mathcal{D}_{(\alpha_1, \alpha_2, \ldots, \alpha_n)} P(z)$ is a polynomial of degree $n-1$ in its general form and ordinary derivative $P'(z)$ is one such representation, i.e. if $\alpha_k = \frac{1}{n}$, $1 \leq k \leq n$, then $\mathcal{D}_{(\alpha_1, \alpha_2, \ldots, \alpha_n)} P(z) = \frac{1}{n} P'(z)$.

Now we state the following recently proved result due to Kumar and Dhankar [4] concerning $\mathcal{D}_{(\alpha_1, \alpha_2, \ldots, \alpha_n)} P(z)$ which is the extension of the Theorem of Laguerre.

Theorem 1.2 ([4]). *Let $P(z) = \prod_{j=1}^{n}(z - z_j)$ be a polynomial of degree n having no zeros in the disc $|z| < 1$. Then the polynomial $cP(z) + (\alpha - z)\mathcal{D}_{(\alpha_1, \alpha_2, \ldots, \alpha_n)} P(z)$ has no zeros in the disc $|z| < 1$, for all α with $|\alpha| < 1$ and $c \geq 1$.*

In the same paper they also established the following result by making the choice of $\alpha_k = \frac{1}{n}$, $1 \leq k \leq n$, in Theorem 1.2.

Theorem 1.3 ([4]). *Let $P(z) = \prod_{j=1}^{n}(z - z_j)$ be a polynomial of degree n having no zeros in the disc $|z| < 1$. Then the polynomial $cP(z) + (\alpha - z)P'(z)$ has no zeros in the disc $|z| < 1$, for all α with $|\alpha| < 1$ and $c \geq n$.*

Definition 1.4. If $P(z)$ is a polynomial of degree n, and α is any complex number, then

$$D_\alpha P(z) = -\left[\frac{P(z)}{(z - \alpha)^n}\right]' (z - \alpha)^{n+1} = nP(z) + (\alpha - z)P'(z),$$

is called the *polar derivative* of $P(z)$. Note that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and it generalizes the concept of "ordinary derivative" is evident and convincing from the fact that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

DEFINITION 1.5. Given a polynomial $P(z) = \sum_{v=0}^{n} c_v z^v$ and $Q(z) = z^n \, \overline{P\left(\frac{1}{\bar{z}}\right)} = \sum_{v=0}^{n} \overline{c_{n-v}} z^v$. If $P(z) = \zeta Q(z)$, where $|\zeta| = 1$, then the polynomial $P(z)$ is said to be self-inverse.

2. Main results

As we know that $P'(z)$ is one of the representation of $\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)$. In this connection, we obtain the following result in terms of convex derivative.

THEOREM 2.1. *Let $P(z) = \prod_{j=1}^{n}(z - z_j)$ be a polynomial of degree n having no zeros in the disc $|z| < 1$. Then for $c \geq 1$ and $|z| = 1$:*

$$\max_{|z|=1} |\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| \leq \frac{c}{2} \max_{|z|=1} |P(z)|. \tag{4}$$

According to Theorem 1.3 and the choice of $\alpha_k = \frac{1}{n}$, $1 \leq k \leq n$, Theorem 2.1 reduces to an interesting result that generalizes inequality (2) as given below.

COROLLARY 2.2. *Let $P(z) = \prod_{j=1}^{n}(z - z_j)$ be a polynomial of degree n having no zeros in the disc $|z| < 1$. Then for $c \geq n$ and $|z| = 1$:*

$$\max_{|z|=1} |P'(z)| \leq \frac{c}{2} \max_{|z|=1} |P(z)|. \tag{5}$$

REMARK 2.3. If $c = n$, Corollary 2.2 reduces to inequality (2).

THEOREM 2.4. *If $P(z)$ is a polynomial of degree n and $M = \max_{|z|=1} |P(z)|$, then*

$$\max_{|z|=1} |\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| \leq M \sum_{k=1}^{n} \alpha_k. \tag{6}$$

Equality in (6) holds for $P(z) = \sigma z^n$, σ being a non zero complex number.

REMARK 2.5. Take $\alpha_k = \frac{1}{n}$, $1 \leq k \leq n$ in Theorem 2.4 and note (3), we get (1) the so called Bernstein inequality.

Until now we studied the ordinary derivative as the convex linear combination of members of $\{P_k(z)\}$ and there by discussed inequalities between polynomials preserved by convex derivatives. Now instead of convex linear combination, we take simply a linear combination of members of $\{P_k(z)\}$. If we choose γ_k's such that $\sum_{k=1}^{n} \gamma_k = \Lambda$, then for $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{R}_+^n$, we take $P^\gamma(z) = \sum_{k=1}^{n} \gamma_k P_k(z)$, the generalized derivative of $P(z)$. Note that $P^\gamma(z)$ is a polynomial of degree $n - 1$ and for $\gamma = (1, 1, \ldots, 1)$: $P^\gamma(z) = P'(z)$.

Definition 2.6. If $P(z)$ is a polynomial of degree n, and α is any complex number, then $D_\alpha^\gamma P(z) = \Lambda P(z) + (\alpha - z)P^\gamma(z)$ is called the *generalized polar derivative* of $P(z)$ (see [9]). Noting that, for $\gamma = (1, 1, \dots, 1)$: $D_\alpha^\gamma P(z) = D_\alpha P(z)$. In this paper, we also add that

$$\lim_{\alpha\to\infty} \frac{D_\alpha^\gamma P(z)}{\alpha} = P^\gamma(z)$$

uniformly with respect to z for $|z| \le R$, $R > 0$.

Next we prove another generalization of Bernstein's inequality in terms of $P^\gamma(z)$. In fact, we prove the following.

Theorem 2.7. *If $P(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1} |P^\gamma(z)| \le \Lambda \max_{|z|=1} |P(z)|. \tag{7}$$

Equality in (7) holds for $P(z) = \sigma z^n$, σ being a non zero complex number.

Now we prove the next theorem.

Theorem 2.8. *If $P(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| \ge 1$:*

$$\max_{|z|=1} |D_\alpha^\gamma P(z)| \le \Lambda|\alpha| \max_{|z|=1} |P(z)|. \tag{8}$$

The result is best possible and equality in (8) holds for $P(z) = Cz^n$, $C \ne 0$.

Remark 2.9. Dividing both sides of inequality (8) by $|\alpha|$ and letting $|\alpha| \to \infty$, we obtain Theorem 2.7.

Theorem 2.10. *If $P(z)$ is a polynomial of degree n such that $\max_{|z|=1} |P(z)| = 1$ and $P(z)$ has no zeros in $|z| < 1$ then for every real or complex number α with $|\alpha| \ge 1$:*

$$|D_\alpha^\gamma P(z)| \le \frac{\Lambda}{2}\{|\alpha z^{n-1}| + 1\}, \quad \text{for} \quad |z| \ge 1. \tag{9}$$

The result is sharp and equality holds in (9) for the polynomial $P(z) = az^n + b$, $|a| = |b| = \frac{1}{2}$.

Remark 2.11. For the n-tuple $\gamma = (1, 1, \dots, 1)$, we obtain the following inequality from Theorem 2.10 under the same hypothesis as $|D_\alpha P(z)| \le \frac{n}{2}\{|\alpha z^{n-1}| + 1\}$ and this inequality is ascribed to Aziz [1].

Remark 2.12. Dividing both sides of inequality (9) by $|\alpha|$ and letting $|\alpha| \to \infty$, we under the same hypothesis obtain the following inequality $|P^\gamma(z)| \le \frac{\Lambda}{2}$, for $|z| = 1$, and for n-tuple $\gamma = (1, 1, \dots, 1)$, we obtain inequality (2) in case $\max_{|z|=1} |P(z)| = 1$.

Theorem 2.13. *If $P(z)$ is a self-inverse polynomial of degree n such that $\max_{|z|=1} |P(z)| = 1$ then for every real or complex number α with $|\alpha| \ge 1$:*

$$|D_\alpha^\gamma P(z)| \le \frac{\Lambda}{2}\{|\alpha z^{n-1}| + 1\}, \quad \text{for} \quad |z| \ge 1. \tag{10}$$

Inequality (10) also holds for $|z| \le 1$ and $|\alpha| \le 1$. Equality in (10) holds for $P(z) = \frac{z^n+1}{2}$.

THEOREM 2.14. *Let $F(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P(z)$ be a polynomial of degree not exceeding that of $F(z)$. If $|P(z)| \leq |F(z)|$ for $|z| = 1$, then for any $\beta \in \mathbb{C}$ with $|\beta| \leq 1$:*

$$\left|\frac{zP^{\gamma}(z)}{\Lambda} + \beta\frac{P(z)}{2}\right| \leq \left|\frac{zF^{\gamma}(z)}{\Lambda} + \beta\frac{F(z)}{2}\right|, \quad \text{for} \quad |z| = 1. \tag{11}$$

One should note that Theorem 2.14 presents a generalization of a result due to Malik and Vong [6]. For an appropriate choice of the argument of β in (11) and letting $\beta \to 1$, we obtain the following corollary.

COROLLARY 2.15. *Under the hypothesis of Theorem 2.14:*

$$\left|\frac{P^{\gamma}(z)}{\Lambda}\right| + \left|\frac{F(z)}{2}\right| \leq \left|\frac{F^{\gamma}(z)}{\Lambda}\right| + \left|\frac{P(z)}{2}\right|, \quad \text{for} \quad |z| = 1. \tag{12}$$

For $F(z) = Mz^n$, where $M = \max_{|z|=1} |P(z)|$ and $\beta = -1$ in (11), we have the following.

COROLLARY 2.16. *If $P(z)$ is a polynomial of degree at most n, then*

$$\left|\frac{P^{\gamma}(z)}{\Lambda} - \frac{P(z)}{2}\right| \leq \frac{1}{2}\max_{|z|=1}|P(z)|, \quad \text{for} \quad |z| = 1, \tag{13}$$

and equality in (13) holds for $P(z) = \alpha z^n + \beta$, where $|\alpha| + |\beta| = 1$.

3. Lemmas

The following lemma is due to Barrero and Egozcue [2].

LEMMA 3.1 ([2]). *Let $z_1, z_2, \ldots, z_n$ be n, not necessary distinct, complex numbers. Then the polynomial $\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)$ has all its zeros in or on the convex hull of the zeros of $P(z)$.*

Lemmas 3.2 and 3.3 are ascribed to Rather et al. [10].

LEMMA 3.2 ([10]). *Every convex set containing all the zeros of $P(z)$ also contains the zeros of $P^{\gamma}(z)$ for all $\gamma \in \mathbb{R}^n_+$.*

LEMMA 3.3 ([10]). *If all the zeros of the polynomial $P(z)$ lie in a circular region $\mathcal{C}$ and if ξ is a zero of $D^{\gamma}_{\alpha}P(z)$ for some $\gamma \in \mathbb{R}^n_+$, then at most one of the points ξ and α may lie outside of $\mathcal{C}$.*

Lemmas 3.4 and 3.5 are due to Rather et al. [9].

LEMMA 3.4 ([9]). *If $P(z)$ is a polynomial of degree n and $Q(z) = z^n\, \overline{P\left(\frac{1}{\bar{z}}\right)}$, then for $|z| = 1$: $|Q^{\gamma}(z)| = |\Lambda P(z) - zP^{\gamma}(z)|$ and $|P^{\gamma}(z)| = |\Lambda Q(z) - zQ^{\gamma}(z)|$.*

LEMMA 3.5 ([9]). *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq k$ where $k \leq 1$, then $k|P^{\gamma}(z)| \geq |Q^{\gamma}(z)|$ for $|z| = 1$, where $Q(z) = z^n\, \overline{P\left(\frac{1}{\bar{z}}\right)}$.*

Lemma 3.6. *If $P(z)$ is a polynomial of degree n such that* $\max_{|z|=1}|P(z)|=1$ *and α is a complex number with $|\alpha|\geq 1$, then for $|z|\geq 1$*

$$|D_\alpha^\gamma Q(z)|+|D_\alpha^\gamma P(z)|\leq \Lambda\{|\alpha z^{n-1}|+1\}, \tag{14}$$

and

$$|D_\alpha^\gamma Q(z)|+|D_\alpha^\gamma P(z)|\leq \Lambda\{|\alpha z^{n-1}|+1\}, \tag{15}$$

for $|\alpha|\leq 1$, $|z|\leq 1$,where $Q(z)=z^n\ \overline{P\left(\frac{1}{\bar z}\right)}$.

Proof. According to Rouche's theorem, the polynomial $F(z)=P(z)-\beta z^n$ has all its zeros in $|z|<1$ for any complex number β satisfying $|\beta|>1$. Therefore, the polynomial

$$T(z)=z^n\ \overline{F\left(\frac{1}{\bar z}\right)}=z^n\ \overline{P\left(\frac{1}{\bar z}\right)}-\overline{\beta}=Q(z)-\overline{\beta}$$

has no zeros in $|z|\leq 1$ and according to the maximum modulus principle $|T(z)|\leq |F(z)|$, for $|z|\geq 1$.

From Rouche's theorem it follows again that for every μ, $|\mu|>1$, the polynomial $T(z)-\mu F(z)$ has all zeros in $|z|<1$, which according to Lemma 3.3 implies by the assumption of $\mathcal{C}=|z|<1$, that for every complex number α with $|\alpha|\geq 1$ the polynomial $D_\alpha^\gamma[T(z)-\mu F(z)]$ has all its zeros in $|z|<1$, this results in

$$|D_\alpha^\gamma T(z)|\leq |D_\alpha^\gamma F(z)|, \qquad \text{for} \quad |z|\geq 1. \tag{16}$$

Therefore, it follows from (16) that

$$|D_\alpha^\gamma Q(z)-\Lambda\overline{\beta}|\leq |D_\alpha^\gamma P(z)-\beta\Lambda\alpha z^{n-1}|, \qquad \text{for} \quad |z|\geq 1. \tag{17}$$

Now the inequality (8), which also applies to $|z|\geq 1$, shows that

$$\max_{|z|=1}|D_\alpha^\gamma P(z)|\leq \Lambda|\alpha|\max_{|z|=1}|P(z)|. \tag{18}$$

With respect to (18), we can choose an argument of β in (17) such that for $|z|\geq 1$:

$$|D_\alpha^\gamma Q(z)|-\Lambda|\beta|\leq \Lambda|\beta\alpha z^{n-1}|-|D_\alpha^\gamma P(z)|.$$

If we set $|\beta|\to 1$, we get

$$|D_\alpha^\gamma Q(z)|+|D_\alpha^\gamma P(z)|\leq \Lambda\{|\alpha z^{n-1}|+1\}, \quad \text{for} \quad |z|\geq 1.$$

The proof of (14) is thus complete and the proof of the inequality (15) follows in a similar way. □

4. Proofs of the main results

Proof (of Theorem 2.1). By the hypothesis $P(z)\neq 0$ in $|z|<1$, and we have from Theorem 1.2:

$$\alpha\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)\neq z\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)-cP(z) \tag{19}$$

for $|z|<1$ and for any $\alpha\in\mathbb{C}$ with $|\alpha|<1$ and $c\geq 1$. For any fixed z, we can choose $\arg\alpha$ in (19) accordingly to get

$$|\alpha|\ |\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)|\neq |z\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)-cP(z)|.$$

This results for $|z| < 1$ and $|\alpha| < 1$:

$$|\alpha|\ |\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| < |z\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z) - cP(z)| \tag{20}$$

because otherwise the inequality (20) is violated, i.e. if (20) is not true, then there exists $z = z_0$ with $|z_0| < 1$ such that

$$|\alpha|\ |\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z_0)| \geq |z\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z_0) - cP(z_0)|,$$

but for small values of α would contradict our claim. Therefore (20) holds and making $|\alpha| \to 1$ and $|z| \to 1$ in (20), one has:

$$|\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| \leq |z\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z) - cP(z)| \tag{21}$$

for $|z| = 1$. On the other hand, let $P(z)$ be a polynomial of degree n and $M = \max_{|z|=1}|P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. This gives for every λ with $|\lambda| > 1$, $|P(z)| < |\lambda|M$ for $|z| = 1$. Therefore, according to Rouche's theorem, $F(z) = P(z) - \lambda M$ has no zeros in $|z| < 1$. From (21), we get for $|z| = 1$

$$|\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}F(z)| \leq |z\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}F(z) - cF(z)|,$$

or

$$|\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| \leq |z\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z) - c(P(z) - \lambda M)|,$$

it implies for $|z| = 1$:

$$|\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| \leq |cP(z) - z\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z) - c\lambda M)|. \tag{22}$$

An argument of λ is chosen such that

$$|cP(z) - z\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z) - c\lambda M)| = c|\lambda|M - |cP(z) - z\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)|.$$

For $|z| = 1$ and $|\lambda| \to 1$, we thus get from (22)

$$|\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| + |cP(z) - z\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| \leq c\max_{|z|=1}|P(z)|. \tag{23}$$

From (21) and (23) we now obtain

$$2\max_{|z|=1}|\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| \leq c\max_{|z|=1}|P(z)|,$$

which is (4), and this concludes the proof. □

Proof (of Theorem 2.4). Since $M = \max_{|z|=1}|P(z)|$, we have for $|z| = 1$, $|P(z)| \leq M$. This gives for every complex number λ, with $|\lambda| > 1$ and $|z| = 1$, $|P(z)| < |\lambda|\ |z|^n M$. From Rouche's theorem it follows that $\lambda M z^n$ and $P(z) - \lambda M z^n$ have the same number of zeros within $|z| = 1$. Since all zeros of $\lambda M z^n$ lie in $|z| < 1$, therefore it follows that $S(z) = P(z) - \lambda M z^n$ has all its zeros in $|z| < 1$ for all λ with $|\lambda| > 1$. Due to Lemma 3.1, all zeros of

$$\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}S(z) = \mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z) - \lambda M\sum_{k=1}^{n}\alpha_k z^{n-1} \tag{24}$$

are also in $|z| < 1$ for all λ with $|\lambda| > 1$. This results in

$$|\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| \leq M|z|^{n-1}\sum_{k=1}^{n}\alpha_k, \quad \text{for } |z| \geq 1.$$

If this is not true in general, then there exists $z = z_0$ with $|z_0| \geq 1$ such that

$$|\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z_0)| > M|z_0|^{n-1}\sum_{k=1}^{n}\alpha_k.$$

Now we take $\lambda = \frac{\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z_0)}{Mz_0^{n-1}\sum_{k=1}^{n}\alpha_k}$ and we see that λ is a well-defined quantity with $|\lambda| > 1$. From (24) and the chosen λ we get

$$\begin{aligned}\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}S(z_0) &= \mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z_0) - \lambda M\sum_{k=1}^{n}\alpha_k z_0^{n-1}\\ &= \mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z_0) - \frac{\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z_0)}{Mz_0^{n-1}\sum_{k=1}^{n}\alpha_k}M\sum_{k=1}^{n}\alpha_k z_0^{n-1} = 0,\end{aligned}$$

which is a contradiction to the fact that all zeros of $\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}S(z)$ lie in $|z| < 1$. We must therefore have

$$|\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| \leq M|z|^{n-1}\sum_{k=1}^{n}\alpha_k, \quad \text{for } |z| \geq 1.$$

Consequently,

$$\max_{|z|=1}|\mathcal{D}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}P(z)| \leq \sum_{k=1}^{n}\alpha_k \max_{|z|=1}|P(z)|, \quad \text{for } |z| = 1. \qquad \square$$

Proof (of Theorem 2.7). The proof is similar to the proof of Theorem 2.4 but instead of using Lemma 3.1 we use Lemma 3.2. □

Proof (of Theorem 2.8). If $P(z)$ is a polynomial of degree n and $M = \max_{|z|=1}|P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. According to Rouche's theorem, the polynomial $F(z) = P(z) - \lambda M z^n$ vanishes in $|z| < 1$ if $|\lambda| > 1$. If we apply Lemma 3.5 for $k = 1$, we get $|Q^\gamma(z)| \leq |F^\gamma(z)|$, where $Q(z) = z^n\,\overline{P\left(\frac{1}{\bar{z}}\right)}$. This gives $|Q^\gamma(z)| \leq |P^\gamma(z) - \lambda M\Lambda z^{n-1}|$. From this we get for $|z| = 1$:

$$|P^\gamma(z)| + |Q^\gamma(z)| \leq \Lambda M. \tag{25}$$

Now by (25)

$$\begin{aligned}|D_\alpha^\gamma P(z)| &= |\Lambda P(z) + (\alpha - z)P^\gamma(z)| = |\Lambda P(z) - zP^\gamma(z) + \alpha P^\gamma(z)|\\ &\leq |\Lambda P(z) - zP^\gamma(z)| + |\alpha|\;|P^\gamma(z)| = |Q^\gamma(z)| + |\alpha|\;|P^\gamma(z)|\\ &\leq |\alpha|(|Q^\gamma(z)| + |P^\gamma(z)|) \leq |\alpha|\Lambda M.\end{aligned}$$

Consequently, $\max_{|z|=1}|D_\alpha^\gamma P(z)| \leq \Lambda|\alpha|\max_{|z|=1}|P(z)|$. □

Proof (of Theorem 2.10). Since $P(z)$ vanishes in $|z| \geq 1$ and therefore $Q(z) = z^n\,\overline{P\left(\frac{1}{\bar{z}}\right)}$ vanishes in $|z| \leq 1$. According to Rouche's theorem, the polynomial $P(z) - \beta Q(z)$ for all $\beta \in \mathbb{C}$ with $|\beta| > 1$ has all its zeros in $|z| \leq 1$. If s is any real number such that $s > 1$, then we see that the polynomial $P(sz) - \beta Q(sz)$ has all its zeros in $|z| \leq 1/s < 1$. Therefore, it follows from Lemma 3.3 with $\mathcal{C} = |z| < 1$ that for every complex number α with $|\alpha| \geq 1$ the polynomial $D_\alpha^\gamma[P(sz) - \beta Q(sz)]$ has all its zeros in $|z| < 1$, that is, all zeros of $D_\alpha^\gamma P(sz) - \beta D_\alpha^\gamma Q(sz)$ lie in $|z| < 1$. This clearly implies that for $|z| \geq 1$:

$$|D_\alpha^\gamma P(sz)| \leq |D_\alpha^\gamma Q(sz)|. \tag{26}$$

Letting $s \to 1$ in (26) we obtain for $|z| \geq 1$

$$|D_\alpha^\gamma P(z)| \leq |D_\alpha^\gamma Q(z)|. \tag{27}$$

If we now combine (27) with the inequality (14) of Lemma 3.6, we get (9). □

Proof (of Theorem 2.13). Since $P(z)$ is a self-inverse polynomial, we have $P(z) = \zeta Q(z)$, where $|\zeta| = 1$ and

$$Q(z) = z^n \ \overline{P\left(\frac{1}{\bar{z}}\right)}.$$

Therefore, for every real or complex number α with $|\alpha| \geq 1$:

$$|D_\alpha^\gamma P(z)| = |D_\alpha^\gamma Q(z)|. \tag{28}$$

With (28) in (14) of Lemma 3.6 we obtain $2|D_\alpha^\gamma P(z)| \leq \Lambda\{|\alpha z^{n-1}| + 1\}$, and this proves (10) for $|z| \geq 1$ and $|\alpha| \geq 1$.

If we now use the equality (28) in (15) of Lemma 3.6, we obtain for every real or complex number α with $|\alpha| \leq 1$ and $|z| \leq 1$:

$$2|D_\alpha^\gamma P(z)| \leq \Lambda\{|\alpha z^{n-1}| + 1\},$$

and this proves (10) for $|z| \leq 1$ and $|\alpha| \leq 1$. □

Remark 4.1. On combining Lemma 3.5 for $k = 1$, Lemma 3.4 and inequality (25), we have for $|z| = 1$ and $Q(z) = z^n \ \overline{P\left(\frac{1}{\bar{z}}\right)}$:

$$\begin{aligned}|\Lambda Q(z)| &= |\Lambda Q(z) - zQ^\gamma(z) + zQ^\gamma(z)| \leq |\Lambda Q(z) - zQ^\gamma(z)| + |zQ^\gamma(z)| \\ &\leq |P^\gamma(z)| + |P^\gamma(z)| = 2|P^\gamma(z)|.\end{aligned}$$

Therefore,

$$|P^\gamma(z)| \geq \frac{\Lambda}{2}|P(z)|, \tag{29}$$

Note that, inequality (29) holds for all polynomials vanishes in $|z| \leq 1$.

Proof (of Theorem 2.14). The essence of the proof lies in the inequality (29) which is being verified. Hence, we omit the details. □

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