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A CONVENIENT CATEGORY OF NEARNESS STRUCTURES IN TEXTURE SPACES

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Abstract. Nearness spaces were defined for the goal of unifying several types of topological structures by H. Herrlich. The basic motivation of the theory of textures is to find a convenient point set based setting for fuzzy sets. This is the second of three papers which develop various fundamental aspects of the concept of dinearness texture spaces in a categorical setting and present important links with the theory of nearness spaces. Further, it is proved that the category **Near** of nearness spaces is to isomorphic to the full subcategory of dinearness texture spaces.

1. Introduction

As is well known, the concept of nearness space was introduced by Herrlich [12] as an axiomatization of the concept of nearness of an arbitrary collection of sets with the aim of unifying different kinds of topological structures [7] such as uniformity, proximity [15] and metric space; as the author says in [13]:

"The aim of this approach is to find a basic topological concept - if possible intuitively accessible- by means of which any topological concept or idea can be expressed".

Nearness spaces are defined on the basis of covering. Further nearness spaces and uniformly continuous maps form a category labelled **Near**. To achieve the above goal, various relationships have been established between the category **Near** and symmetric topological spaces and continuous maps, uniform spaces and uniformly continuous maps, and proximal spaces and proximal maps.

Texture spaces were introduced by L. M. Brown as a point-based setting of fuzzy sets. In addition, some properties of fuzzy lattices (i.e. the Hutton algebra) can be discussed in terms of textures [4, 5]. The concept of ditopology on textures, which is more general than general topology, and fuzzy topology in the sense of Chang were introduced as a natural extension of the work on the representation of lattice-valued topologies by bitopologies without the set complementation in [2].

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On the other hand, texture spaces provide an abstract model for rough set and soft fuzzy rough set theory [8] and the Hutton closure space [9].

This paper continues the study of the nearness structure on texture spaces in [11]. In the first paper, the concept of nearness structure on texture was introduced under the name of dinearness structure, and some properties were presented. The main aim of the second paper is to consider dinearness texture spaces in a categorical setting. In this context, the notion of uniformly bicontinuous divergences between dinearness texture spaces is introduced, and their basic properties are given in Section 3. Section 4 is devoted to the category of dinearness texture spaces and uniformly bicontinuous difunctions, and some connections to the category **Near** of nearness spaces are given.

An overview of texture spaces and difunctions is given in the next section, and the reader is referred to [2–6, 14] for further background material.

2. Texture spaces

Let U be a set. A *texturing* $\mathcal{U}$ of U is a subset of $\mathcal{P}(U)$ which is a point-separating, complete, completely distributive lattice containing U and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(U, \mathcal{U})$ is then called *texture space*, or *texture* for short.

For $u \in U$ the *p-sets* and, as dually, the *q-sets* are defined by

$$P_u = \bigcap\{A \in \mathcal{U} \mid u \in A\}, \quad Q_u = \bigvee\{A \in \mathcal{U} \mid u \notin A\}.$$

A mapping $\sigma_U : \mathcal{U} \to \mathcal{U}$ is called a *complementation* on $(U, \mathcal{U})$ if it satisfies the conditions $\sigma_U(\sigma_U(A)) = A$ for all $A \in \mathcal{U}$ and $A \subseteq B \Longrightarrow \sigma_U(B) \subseteq \sigma_U(A)$ for all $A, B \in \mathcal{U}$. In this case, $(U, \mathcal{U}, \sigma)$ is referred to as complemented texture.

Example 2.1. (i) For any set X, $(X, \mathcal{P}(X), \pi)$, $\pi(Y) = X \setminus Y$ for $Y \subseteq X$, is the complemented *discrete* texture that represents the usual set structure of X. It is clear that $P_x = \{x\}$, $Q_x = X \setminus \{x\}$ for all $x \in X$.

(ii) Let $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$ and $\lambda((0, r]) = (0, 1 - r]$, $r \in [0, 1]$. Obviously, $(L, \mathcal{L}, \lambda)$ is the Hutton texture of $(\mathbb{I}, \prime)$, where $\mathbb{I} = [0, 1]$ with its usual order and $r' = 1 - r$ for $r \in \mathbb{I}$. Here, $P_r = Q_r = (0, r]$ for all $r \in L$.

(iii) For $\mathbb{I} = [0, 1]$ define $\mathcal{I} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$, $\iota([0, t]) = [0, 1 - t)$ and $\iota([0, t)) = [0, 1 - t]$, $t \in [0, 1]$. $(\mathbb{I}, \mathcal{I}, \iota)$ is a complemented texture, which we will refer to as *unit interval texture*. Here, $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in \mathbb{I}$.

(iv) For textures $(U, \mathcal{U})$ and $(V, \mathcal{V})$, $\mathcal{U} \otimes \mathcal{V}$ is the product texturing of $U \times V$. Note that the product texturing $\mathcal{U} \otimes \mathcal{V}$ of $U \times V$ consists of arbitrary intersections of sets of the form $(A \times V) \cup (U \times B)$, $A \in \mathcal{U}$ and $B \in \mathcal{V}$.

Ditopology: A pair (τ, κ) of subsets of $\mathcal{U}$ is called a *ditopology* on a texture $(U, \mathcal{U})$ where the *open set* family τ and the *closed set* family κ satisfy

$$U, \emptyset \in \tau, \qquad U, \emptyset \in \kappa,$$

$$G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau, \qquad K_1,\, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa,$$
$$G_i \in \tau, i \in I \implies \bigvee_{i\in I} G_i \in \tau, \qquad K_i \in \kappa, i \in I \implies \bigcap_{i\in I} K_i \in \kappa.$$

A ditopology is therefore essentially a "topology" for which there is no *a priori* relationship between the open and closed sets. Usually the family τ is called a *topology*, and the family κ a *cotopology.*

If σ is a complementation on $(U, \mathcal{U})$ and $\kappa = \sigma(\tau)$, then (τ, κ) is called a complemented ditopology on $(U, \mathcal{U}, \sigma)$.

Direlation: Let $(U, \mathcal{U})$, $(V, \mathcal{V})$ be textures. Consider the product texture $\mathcal{P}(U) \otimes \mathcal{V}$ of the textures $(U, \mathcal{P}(U))$ and $(V, \mathcal{V})$ (see Example 2.1 (iv)). We denote the p-sets and the q-sets by $\overline{P}_{(u,v)}$ and $\overline{Q}_{(u,v)}$, respectively. From the product texturing, it is obtained that $\overline{P}_{(u,v)} = \{u\} \times P_v$ and $\overline{Q}_{(u,v)} = (U \setminus \{u\} \times V) \cup (U \times Q_v)$, where $u \in U$ and $v \in V$. Then:

1. $r \in \mathcal{P}(U) \otimes \mathcal{V}$ is called a *relation* from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ if it satisfies:

(R1) $r \not\subseteq \overline{Q}_{(u,v)}, P_{u'} \not\subseteq Q_u$ implies $r \not\subseteq \overline{Q}_{(u',v)}$.

(R2) $r \not\subseteq \overline{Q}_{(u,v)}$ implies $\exists u' \in U$ such that $P_u \not\subseteq Q_{u'}$ and $r \not\subseteq \overline{Q}_{(u',v)}$.

2. $R \in \mathcal{P}(U) \otimes \mathcal{V}$ is called a *corelation* from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ if it satisfies

(CR1) $\overline{P}_{(u,v)} \not\subseteq R, P_u \not\subseteq Q_{u'}$ implies $\overline{P}_{(u',v)} \not\subseteq R$.

(CR2) $\overline{P}_{(u,v)} \not\subseteq R$ implies $\exists u' \in U$ such that $P_{u'} \not\subseteq Q_u$ and $\overline{P}_{(u',v)} \not\subseteq R$.

3. A pair (r, R), where r is a relation and R a corelation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ is called a *direlation* from $(U, \mathcal{U})$ to $(V, \mathcal{V})$.

The identity direlation (i, I) on $(U, \mathcal{U})$ is defined by $i = \bigvee\{\overline{P}_{(u,u)} \mid u \in U\}$ and $I = \bigcap\{\overline{Q}_{(u,u)} \mid U \not\subseteq Q_u\}$.

The composition of direlations: Let $(U, \mathcal{U})$, $(V, \mathcal{V})$, $(W, \mathcal{W})$ be textures.

1. If p is a relation on $(U, \mathcal{U})$ to $(V, \mathcal{V})$ and q a relation on $(V, \mathcal{V})$ to $(W, \mathcal{W})$ then their *composition* is the relation $q \circ p$ on $(U, \mathcal{U})$ to $(W, \mathcal{W})$ defined by

$$q \circ p = \bigvee\{\overline{P}_{(u,w)} \mid \exists\, v \in V \text{ with } p \not\subseteq \overline{Q}_{(u,v)} \text{ and } q \not\subseteq \overline{Q}_{(v,w)}\}.$$

2. If P is a co-relation on $(U, \mathcal{U})$ to $(V, \mathcal{V})$ and Q a co-relation on $(V, \mathcal{V})$ to $(W, \mathcal{W})$ then their *composition* is the co-relation $Q \circ P$ on $(U, \mathcal{U})$ to $(W, \mathcal{W})$ defined by

$$Q \circ P = \bigcap\{\overline{Q}_{(u,w)} \mid \exists\, v \in V \text{ with } \overline{P}_{(u,v)} \not\subseteq P \text{ and } \overline{P}_{(v,w)} \not\subseteq Q\}.$$

3. With p, q; P, Q as above, the *composition* of the direlations (p, P), (q, Q) is the direlation $(q, Q) \circ (p, P) = (q \circ p, Q \circ P)$.

The complement of a direlation: Let (r, R) be a direlation between the complemented textures $(U, \mathcal{U}, \sigma_U)$ and $(V, \mathcal{V}, \sigma_V)$.

1. The *complement r' of the relation* r is the co-relation

$$r' = \bigcap\{\overline{Q}_{(u,v)} \mid \exists\, w, z,\ r \not\subseteq \overline{Q}_{(w,z)},\ \sigma_U(Q_u) \not\subseteq Q_w \text{ and } P_z \not\subseteq \sigma_V(P_v)\}.$$

2. The *complement R' of the co-relation R* is the relation

$$R' = \bigvee\{\overline{P}_{(u,v)} \mid \exists\, w,z,\ \overline{P}_{(w,z)} \not\subseteq R,\ P_w \not\subseteq \sigma_U(P_u) \text{ and } \sigma_V(Q_v) \not\subseteq Q_z\}.$$

3. The *complement $(r,R)'$ of the direlation (r,R)* is the direlation $(r,R)' = (R', r')$.

A direlation (r,R) on $(U,\mathcal{U})$ is said to be *complemented* if $(r,R)' = (r,R)$.

One of the most useful notions of (ditopological) texture spaces is that of difunction. A difunction is a special type of direlation.

Difunctions: Let (f,F) be a direlation from $(U,\mathcal{U})$ to $(V,\mathcal{V})$. Then (f,F) is called a *difunction from $(U,\mathcal{U})$ to $(V,\mathcal{V})$* if it satisfies the following two conditions.

(DF1) For $u,u' \in U$, $P_u \not\subseteq Q_{u'} \implies \exists\, v \in V$ with $f \not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u',v)} \not\subseteq F$.

(DF2) For $v,v' \in T$ and $u \in U$, $f \not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u,v')} \not\subseteq F \implies P_{v'} \not\subseteq Q_v$.

Clearly, identity direlation (i,I) on $(U,\mathcal{U})$ is a difunction and it is called *identity difunction*.

Image and inverse image: Let $(f,F) : (U,\mathcal{U}) \to (V,\mathcal{V})$ be a difunction.

1. For $A \in \mathcal{U}$, the *image* $f^{\to}A$ and the *co-image* $F^{\to}A$ are defined by

$$f^{\to}A = \bigcap\{Q_v \mid \forall\, u,\ f \not\subseteq \overline{Q}_{(u,v)} \implies A \subseteq Q_u\},$$
$$F^{\to}A = \bigvee\{P_v \mid \forall\, u,\ \overline{P}_{(u,v)} \not\subseteq F \implies P_u \subseteq A\}.$$

2. For $B \in \mathcal{V}$, the *inverse image* $f^{\leftarrow}B$ and the *inverse co-image* $F^{\leftarrow}B$ are defined by

$$f^{\leftarrow}B = \bigvee\{P_u \mid \forall\, v,\ f \not\subseteq \overline{Q}_{(u,v)} \implies P_v \subseteq B\},$$
$$F^{\leftarrow}B = \bigcap\{Q_u \mid \forall\, v,\ \overline{P}_{(u,v)} \not\subseteq F \implies B \subseteq Q_v\}.$$

For a difunction, the inverse image and the inverse co-image are equal, but the image and co-image are usually not.

Injective-surjective difunction: Let $(f,F) : (U,\mathcal{U}) \to (V,\mathcal{V})$ be a difunction. Then (f,F) is called *surjective* if it satisfies the condition

(SUR) For $v,v' \in V$, $P_v \not\subseteq Q_{v'} \implies \exists\, u \in U$ with $f \not\subseteq \overline{Q}_{(u,v')}$ and $\overline{P}_{(u,v)} \not\subseteq F$.

(f,F) is called *injective* if it satisfies the condition

(INJ) For $u,u' \in U$ and $v \in V$, $f \not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u',v)} \not\subseteq F \implies P_u \not\subseteq Q_{u'}$.

If (f,F) is both injective and surjective, then it is called *bijective*.

Bicontinuity: Let $(f,F) : (U,\mathcal{U},\tau_U,\kappa_U) \to (V,\mathcal{V},\tau_V,\kappa_V)$ be a difunction. Then it is called *continuous* if $B \in \tau_V \implies F^{\leftarrow}B \in \tau_U$, *cocontinuous* if $B \in \kappa_V \implies f^{\leftarrow}B \in \kappa_U$, and *bicontinuous* if it is both continuous and cocontinuous.

One of the main categories of texture theory considered to date is the category **dfTex** of textures and difunctions. The other main category **dfDitop** of ditopological texture spaces and bicontinuous difunctions is topological over **dfTex**.

Dicover: Let $(U,\mathcal{U})$ be a texture space. A difamily $\mathcal{C} = \{(A_j,B_j) \mid j \in J\}$ of elements of $\mathcal{U} \times \mathcal{U}$ which satisfies $\bigcap_{j\in J_1} B_j \subseteq \bigvee_{j\in J_2} A_j$ for all partitions (J_1,J_2) of J, including the trivial partitions, is called a *dicover* of $(U,\mathcal{U})$ (see [3]).

An important example is the family $\mathcal{P} = \{(P_u,Q_u) \mid U \not\subseteq Q_u\}$ which is a dicover for any texture $(U,\mathcal{U})$.

Now, we recall some useful results by [10, Proposition 11.1].

NOTE 2.2. Let U be a non-empty set. Then
(a) Let $\mathcal{C} = \{A_i \mid i \in I\} \subseteq \mathcal{P}(U)$. Then $\mathcal{C}$ is a cover of U if and only if $\{(A, X \setminus A) \mid A \in \mathcal{C}\}$ is a dicover of the discrete texture space $(U, \mathcal{P}(U))$.

(b) If a family $\mathcal{D} = \{(A_i, B_i) \mid i \in I\}$ is a dicover of $(U, \mathcal{P}(U))$ then the families $\{A_i\}_{i \in I}$ and $\{X \setminus B_i\}_{i \in I}$ are covers of U.

If $\mathcal{C}$ is a dicover, then we sometimes write $L\mathcal{C}M$ in place of $(L, M) \in \mathcal{C}$. We recall the following definitions for dicovers.
1. $\mathcal{C}$ is a *refinement* of $\mathcal{D}$ if given $j \in J$ we have $L\,\mathcal{D}\,M$ so that $A_j \subseteq L$ and $M \subseteq B_j$. In this case we write $\mathcal{C} \prec \mathcal{D}$.

2. If $\mathcal{C}$, $\mathcal{D}$ are dicovers then $\mathcal{C} \wedge \mathcal{D} = \{(A \cap C, B \cup D) \mid A\,\mathcal{C}\,B,\ C\,\mathcal{D}\,D\}$ is the greatest lower bound (meet) of $\mathcal{C}$, $\mathcal{D}$ with respect to refinement.

Dinearness texture spaces: Let $(U, \mathcal{U})$ be a texture space. Let μ be a non-empty set of non-empty dicovers of $(U, \mathcal{U})$. Then μ is called dinearness structure (see [11]) if it is satisfied the following conditions:
(N1) If $\mathcal{C} \prec \mathcal{D}$ and $\mathcal{C} \in \mu$, then $\mathcal{D} \in \mu$.

(N2) If $\mathcal{C} \in \mu$ and $\mathcal{D} \in \mu$, then $\mathcal{C} \wedge \mathcal{D} \in \mu$, where $\mathcal{C} \wedge \mathcal{D} = \{(A \cap C, B \cup D) \mid A\mathcal{C}B, C\mathcal{D}D\}$.

(N3) If $\mathcal{C} \in \mu$, then $\{(\mathrm{int}_\mu(A), \mathrm{cl}_\mu(B)) \mid A\mathcal{C}B\} \in \mu$ where $A \in \mathcal{U}$,

$$\mathrm{int}_\mu A = \bigvee \{P_u \mid \forall P_u \not\subseteq Q_v,\ \{(A, \emptyset), (\emptyset, P_v)\} \in \mu\},$$

$$\mathrm{cl}_\mu A = \bigcap \{Q_u \mid \forall P_v \not\subseteq Q_u,\ \{(\emptyset, A), (Q_v, \emptyset)\} \in \mu\}.$$

A triple $(U, \mathcal{U}, \mu)$, where μ is a dinearness structure on $(U, \mathcal{U})$, is called dinearness texture space.

If σ is a complementation on $(U, \mathcal{U})$ and $\mu = \sigma(\mu)$, then $(U, \mathcal{U}, \sigma, \mu)$ is called complemented dinearness texture space.

3. Uniformly bicontinuous difunctions

In this section, the notion of uniformly bicontinuous difunctions between dinearness texture spaces are defined, and their some properties are given.

LEMMA 3.1. *Let $(U_j, \mathcal{U}_j)$, $j = 1, 2$ be texture spaces and $(f, F) : (U_1, \mathcal{U}_1) \to (U_2, \mathcal{U}_2)$ be a difunction. If the family $\mathcal{C} = \{(A_i, B_i) \mid i \in I\}$ is a dicover of $(U_2, \mathcal{U}_2)$ then the family $(f, F)^{-1}(\mathcal{C}) = \{(F^{\leftarrow}(A_i), f^{\leftarrow}(B_i)) \mid i \in I, (A_i, B_i) \in \mathcal{C}\}$ is also a dicover of $(U_1, \mathcal{U}_1)$.*

Proof. Let the pair (I_1, I_2) be a partition of I. Since the family $\mathcal{C}$ is a dicover of $(U_2, \mathcal{U}_2)$ we can write $\bigcap_{i \in I_1} B_i \subseteq \bigvee_{i \in I_2} A_i$. From [6, Corollary 2.12], we have

$$\bigcap_{i \in I_1} f^{\leftarrow}(B_i) = f^{\leftarrow}(\bigcap_{i \in I_1} B_i) \subseteq F^{\leftarrow}(\bigvee_{i \in I_2} A_i) = \bigvee_{i \in I_2} F^{\leftarrow}(A_i).$$

Hence, $(f,F)^{-1}(\mathcal{C})$ is a dicover of $(U_1,\mathcal{U}_1)$. □

DEFINITION 3.2. Let $(U_j,\mathcal{U}_j,\mu_j)$, $j=1,2$ be dinearness texture spaces and $(f,F):(U_1,\mathcal{U}_1)\to(U_2,\mathcal{U}_2)$ be a difunction. Then (f,F) is called *uniformly bicontinuous difunction* if $\mathcal{C}\in\mu_2 \Longrightarrow (f,F)^{-1}(\mathcal{C})\in\mu_1$.

PROPOSITION 3.3. *Let $(f,F):(U_1,\mathcal{U}_1)\to(U_2,\mathcal{U}_2)$ be a difunction and the families $\mathcal{C}$ and $\mathcal{D}$ be dicovers of $(U_2,\mathcal{U}_2)$. Then the following are satisfied:*
(i) $\mathcal{C}\prec\mathcal{D}\Longrightarrow(f,F)^{-1}(\mathcal{C})\prec(f,F)^{-1}(\mathcal{D})$.

(ii) $(f,F)^{-1}(\mathcal{C}\wedge\mathcal{D})=(f,F)^{-1}(\mathcal{C})\wedge(f,F)^{-1}(\mathcal{D})$.

Proof. Suppose that $(f,F):(U_1,\mathcal{U}_1)\to(U_2,\mathcal{U}_2)$ is a difunction and the families $\mathcal{C}$ and $\mathcal{D}$ are dicovers of $(U_2,\mathcal{U}_2)$.

(i) Let $(F^{\leftarrow}(A),f^{\leftarrow}(B))\in(f,F)^{-1}(\mathcal{C})$ for $(A,B)\in\mathcal{C}$. Since $\mathcal{C}\prec\mathcal{D}$, there exists $(C,D)\in\mathcal{D}$ such that $A\subseteq C$ and $D\subseteq B$. Then $(F^{\leftarrow}(C),f^{\leftarrow}(D))\in(f,F)^{-1}(\mathcal{D})$ and $F^{\leftarrow}(A)\subseteq F^{\leftarrow}(C)$ and $f^{\leftarrow}(D)\subseteq f^{\leftarrow}(B)$. Hence, we obtain $(f,F)^{-1}(\mathcal{C})\prec(f,F)^{-1}(\mathcal{D})$.

(ii) From the definition of $(f,F)^{-1}$, we have:

$$\begin{aligned}(f,F)^{-1}(\mathcal{C}\wedge\mathcal{D}) =&(f,F)^{-1}\{(A\cap C,B\cup D)\mid A\mathcal{C}B,C\mathcal{D}D\}\\ =&(F^{\leftarrow}(A\cap C),f^{\leftarrow}(B\cup D)\mid A\mathcal{C}B,C\mathcal{D}D\}\\ =&(F^{\leftarrow}(A)\cap F^{\leftarrow}(C),f^{\leftarrow}(B)\cap f^{\leftarrow}(D))\mid A\mathcal{C}B,C\mathcal{D}D\}\\ =&\{(F^{\leftarrow}(A),f^{\leftarrow}(B))\mid(A,B)\in\mathcal{C}\}\wedge\{(F^{\leftarrow}(C),f^{\leftarrow}(D))\mid(C,D)\in\mathcal{D}\}\\ =&(f,F)^{-1}(\mathcal{C})\wedge(f,F)^{-1}(\mathcal{D}).\end{aligned}$$ □

THEOREM 3.4. *(i) The identity difunction on a dinearness texture space $(U,\mathcal{U},\mu)$ is uniformly bicontinuous.*

(ii) The composition of uniformly bicontinuous difunction is uniformly bicontinuous.

(iii) Let $(U_j,\mathcal{U}_j,\sigma_j,\mu_j)$, $j=1,2$ be complemented dinearness texture spaces and $(f,F):(U_1,\mathcal{U}_1,\sigma_1)\to(U_2,\mathcal{U}_2,\sigma_2)$ be a complemented difunction. Then (f,F) is uniformly bicontinuous w.r.t $\mu_1-\mu_2$ if and only if $(f,F)'$ is uniformly bicontinuous w.r.t $\sigma(\mu_1)-\sigma(\mu_2)$.

Proof. (i) Let $(A,B)\in\mathcal{C}$. Since $I^{\leftarrow}(A)=A$ and $i^{\leftarrow}(B)=B$, the desired is obtained immediately.

(ii) Suppose that $(U_k,\mathcal{U}_k,\mu_k)$, $k=1,2,3$ are dinearness texture spaces and $(f,F):(U_1,\mathcal{U}_1)\to(U_2,\mathcal{U}_2)$ and $(g,G):(U_2,\mathcal{U}_2)\to(U_3,\mathcal{U}_3)$ are uniformly bicontinuous difunctions. Now, we show that the composition difunction $(g,G)\circ(f,F)=(g\circ f,G\circ F)$ is uniformly bicontinuous. For $\mathcal{C}\in\mu_3$, since

$$(g,G)^{-1}(\mathcal{C})=\{(G^{\leftarrow}(A),g^{\leftarrow}(B))\mid(A,B)\in\mathcal{C}\}\in\mu_2$$

and $$(f,F)^{-1}\big((g,G)^{-1}(\mathcal{C})\big)=\{F^{\leftarrow}(G^{\leftarrow}A),f^{\leftarrow}(g^{\leftarrow}B))\mid(A,B)\in\mathcal{C}\}\in\mu_1,$$

we have $(G\circ F)^{\leftarrow}(\mathcal{B})=F^{\leftarrow}(G^{\leftarrow}(\mathcal{B}))$ and $(g\circ f)^{\leftarrow}(\mathcal{B})=f^{\leftarrow}(g^{\leftarrow}(\mathcal{B}))$.

(iii) $(\Longrightarrow)$ Let $\sigma_2(\mathcal{C})\in\sigma_2(\mu_2)$ such that $\mathcal{C}\in\mu_2$. Since (f,F) is uniformly bicontinuous, $(f,F)^{-1}(\mathcal{C})=\{(F^{\leftarrow}(A),f^{\leftarrow}(B))\mid(A,B)\in\mathcal{C}\}\in\mu_1$. Hence, we have $\sigma_1((f,F)^{-1}(\mathcal{C})) = \{(\sigma_1(f^{\leftarrow}(B)), \sigma_1(F^{\leftarrow}(A)) \mid (A,B) \in \mathcal{C}\} \in \sigma_1(\mu_1)$. Since $(f,F)' = (F', f')$, by [6, Lemma 2.20] we obtain:

$$\begin{aligned}(F',f')^{-1}(\sigma_2(\mathcal{C})) &= \{(f')^{\leftarrow}(\sigma_2(B)), (F')^{\leftarrow}(\sigma_2(A) \mid (A,B) \in \mathcal{C}\} \\ &= \{\sigma_1(F^{\leftarrow}(B)), \sigma_1(f^{\leftarrow}(B)) \mid (A,B) \in \mathcal{C}\} \in \sigma_1(\mu_1).\end{aligned}$$

($\Longleftarrow$) It is obtained similarly. □

Now, we give bases and subbases for dinearness texture spaces.

Definition 3.5. Let $(U, \mathcal{U}, \mu)$ be a dinearness texture space and $\mu' \subseteq \mu$. Then μ' is called a *base* of μ if there exits a dicover $\mathcal{A}' \in \mu'$ such that $\mathcal{A}' \prec \mathcal{A}$ for all $\mathcal{A} \in \mu$.

As usual, a *subbase of* μ is a subset of μ, the set of finite intersections of which is a base of μ.

Proposition 3.6. *Let $(U_j, \mathcal{U}_j, \mu_j)$, $j = 1,2$ be dinearness texture spaces and $(f,F) : (U_1, \mathcal{U}_1) \to (U_2, \mathcal{U}_2)$ be a difunction. Let μ_2' be a base for dinearness structure μ_2. Then (f,F) uniformly bicontinuous $\Longleftrightarrow$ $(f,F)^{-1}(\mathcal{B}) \in \mu_1, \quad \forall \mathcal{B} \in \mu_2'$.*

Proof. ($\Longrightarrow$) Suppose that (f,F) is uniformly bicontinuous difunction and $\mathcal{B} \in \mu_2'$. Since $\mu_2' \subseteq \mu_2$ and $\mathcal{B} \in \mu_2$, we obtain $(f,F)^{-1}(\mathcal{B}) \in \mu_1$ from assumption.

($\Longleftarrow$) Let $\mathcal{B} \in \mu_2$. Then there exists $\mathcal{B}' \in \mu_2'$ such that $\mathcal{B}' \prec \mathcal{B}$, since μ_2' is a base. From Lemma 3.1, the families $(f,F)^{-1}(\mathcal{B}')$ and $(f,F)^{-1}(\mathcal{B})$ are dicovers of $(U_1, \mathcal{U}_1)$. Further, by Proposition 3.3, $(f,F)^{-1}(\mathcal{B}') \prec (f,F)^{-1}(\mathcal{B})$, and $(f,F)^{-1}(\mathcal{B}) \in \mu_1$ from the condition (N1) of dinearness texture space's definition. That is (f,F) uniformly bicontinuous difunction. □

Recall that [11, Theorem 3.5] if $(U, \mathcal{U}, \mu)$ is a dinearness texture space, then (τ_μ, κ_μ) is a ditopology on $(U, \mathcal{U})$, where $\tau_\mu = \{G \in \mathcal{U} \mid int_\mu(G) = G\}$ and $\kappa_\mu = \{K \in \mathcal{U} \mid cl_\mu(K) = K\}$.

Theorem 3.7. *Let $(U_j, \mathcal{U}_j, \mu_j)$, $j = 1,2$ be dinearness texture spaces and (τ_μ, κ_μ) be the corresponding ditopological spaces and $(f,F) : (U_1, \mathcal{U}_1) \to (U_2, \mathcal{U}_2)$ be a difunction. If (f,F) is uniformly bicontinuous difunction w.r.t $\mu_1 - \mu_2$, then (f,F) is bicontinuous difunction in the sense the corresponding ditopologies $(\tau_{\mu_1}, \kappa_{\mu_1}) - (\tau_{\mu_2}, \kappa_{\mu_2})$.*

Proof. Let $G \in \tau_{\mu_2}$. We show that $F^{\leftarrow}(G) = \operatorname{int}_{\mu_1} F^{\leftarrow}(G)$. By [11, Theorem 3.4], it is clear that $\operatorname{int}_{\mu_1} F^{\leftarrow}(G) \subseteq F^{\leftarrow}(G)$. Therefore, it is sufficient to show that $F^{\leftarrow}(G) \subseteq \operatorname{int}_{\mu_1} F^{\leftarrow}(G)$. Suppose that $F^{\leftarrow}(G) \not\subseteq \operatorname{int}_{\mu_1} F^{\leftarrow}(G)$. Then we have $u \in U_1$ such that $F^{\leftarrow}(G) \not\subseteq Q_u$, and $P_u \not\subseteq \operatorname{int}_{\mu_1} F^{\leftarrow}(G)$. Now we choose $v \in U_1$ where $P_u \not\subseteq Q_v$. Since $F^{\leftarrow}(G) \not\subseteq Q_v$, there exists $t \in U_2$ such that $\overline{P}_{(v,t)} \not\subseteq F$ and $G \not\subseteq Q_t$.

If $G = \operatorname{int}_{\mu_2} G \not\subseteq Q_t$, then $\{(G, \emptyset), (\emptyset, P_t)\} \in \mu_2$. Then $\{(F^{\leftarrow}(G), \emptyset), (\emptyset, f^{\leftarrow}(P_t)\} \in \mu_1$, since (f,F) is uniformly bicontinuous difunction. Because $\overline{P}_{(v,t)} \not\subseteq F$, we have $f^{\leftarrow}(P_t) = F^{\leftarrow}(P_t) \not\subseteq Q_v$, and so $P_v \subseteq f^{\leftarrow}(P_t)$ and $\{(F^{\leftarrow}(G), \emptyset), (\emptyset, f^{\leftarrow}(Q_t)\} \prec \{(F^{\leftarrow}(G), \emptyset), (\emptyset, P_v)\}$. From the condition (N1) of dinearness texture space's definition, we have $\{(F^{\leftarrow}(G), \emptyset), (\emptyset, P_v\} \in \mu_1$. This contradicts $P_u \subseteq \operatorname{int}_{\mu_1} F^{\leftarrow}(G)$. Hence, (f,F) is uniformly continuous.

It is similarly shown that (f,F) is uniformly cocontinuous. □

4. The category DiNear

In this section, we give some links between **Near** and the category of dinearness texture spaces and uniformly bicontinuous difunctions.

In general, we follow the terminology of [1] for general concepts related to category theory. If **A** is a category, denotes $\mathrm{Ob}(\mathbf{A})$ the class of objects and $\mathrm{Mor}\,\mathbf{A}$ the class of morphisms of **A**. We will sometimes use the notation $\hom(A_1, A_2)$ for the set of morphisms in **A** from A_1 to A_2.

THEOREM 4.1. *Dinearness texture spaces and uniformly bicontinuous difunctions form a category.*

Proof. Since uniformly bicontinuity between dinearness texture spaces is preserved under composition of difunction by Theorem 3.4 (ii), and identity difunction on $(U, \mathcal{U}, \mu)$ is uniformly bicontinuous by Theorem 3.4 (i) and the identity difunctions are identities for composition and composition is associative by [6, Proposition 2.17], dinearness texture spaces and uniformly bicontinuous difunctions form a category. □

DEFINITION 4.2. The category whose objects are dinearness texture spaces and whose morphisms are uniformly bicontinuous difunctions will be denoted by **DiNear**.

COROLLARY 4.3. *Complemented dinearness texture spaces and uniformly bicontinuous complemented difunctions form a category.*

Proof. Let $(U, \mathcal{U}, \mu, \sigma)$ be a complemented dinearness space. Then the identity difunction on $(U, \mathcal{U})$ is complemented and uniformly bicontinuous by Theorem 3.4 (iii). Further, the composition of complemented difunction is complemented, and the composition of uniformly bicontinuous complemented difunction is uniformly bicontinuous by Theorem 3.4 (ii). Hence, the proof is completed. □

The category whose objects are complemented dinearness spaces and whose morphisms are uniformly bicontinuous complemented difunctions will be denoted by **cDiNear**.

Now suppose that (X, η) is a nearness space, and $\mathcal{DC}$ is a dicover family of $(X, \mathcal{P}(X))$, i.e. $\mathcal{DC} = \{\mathcal{C} = \{(A_i, B_i) \mid i \in I\} \mid \mathcal{C} \text{ is a dicover of } (X, \mathcal{P}(X))\}$. From [11, Theorem 3.7], the family $\mu = \{\mathcal{C} \in \mathcal{DC} \mid \{A_i\}_{i \in I} \in \eta \text{ and } \{X \setminus B_i\}_{i \in I} \in \eta\}$ is a dinearness structure on the discrete texture space $(X, \mathcal{P}(X))$. On the other hand, it is known that [6] if f is a point function from X to Y, then the pair (f, f') is a difunction from $(X, \mathcal{P}(X))$ to $(Y, \mathcal{P}(Y))$ where $f' = (X \times Y) \setminus f$.

PROPOSITION 4.4. *Let (X, η) and (Y, η') be nearness spaces and $f : X \to Y$ be a point function. Then f is uniformly continuous $\Longleftrightarrow$ (f, f') is uniformly bicontinuous.*

Proof. Suppose that (Y, η') is a nearness space and the family

$$\mu' = \{\{(A_i, B_i) \mid i \in I\} \in \mathcal{DC} \mid \mathcal{A} = \{A_i\}_{i \in I} \in \eta' \text{ and } \mathcal{B} = \{X \setminus B_i\}_{i \in I} \in \eta'\}$$

is the corresponding dinearness structure on the discrete texture $(Y, \mathcal{P}(Y))$. For $\mathcal{C} \in \mu'$, $(f, f')^{-1}(\mathcal{C}) = \{((f')^{\leftarrow}(A), f^{\leftarrow}(B)) \mid (A, B) \in \mathcal{C}\}$ and we can write $(f')^{\leftarrow}(A) =$

$f^{-1}(A)$ ve $X \backslash f^{\leftarrow}(B) = X \backslash (f^{-1})(B)$ by [6, Proposition 2.21]. Since $(f, f')^{-1}(\mathcal{C}) \in \mu \Longleftrightarrow f^{-1}(\mathcal{A}) \in \eta$ and $f^{-1}(\mathcal{B}) \in \eta$ for all for $\mathcal{C} \in \mu'$, the proof is completed. □

In the sense of Herrlich, the category of nearness spaces and uniformly continuous function between nearness spaces is denoted by **Near** [12].

THEOREM 4.5. *The category **Near** is isomorphic to the full subcategory of **DiNear**.*

Proof. Firstly, **D-DiNear** denotes the category of dinear texture spaces on discrete textures and uniformly bicontinuous difunctions. Clearly, it is a full subcategory of **DiNear**. Now consider the mapping $\mathfrak{T}$: **Near** $\to$ **D-DiNear** which is defined by $\mathfrak{T}(U, \eta) = (U, \mathcal{P}(U), \mu)$ and $\mathfrak{T}(f) = (f, f')$ for every morphism $f : (U, \eta) \to (V, \eta')$ in **Near**.

Since (f, f') is uniformly bicontinuous by Proposition 4.4, the difunction (f, f') is a morphism in the category **D-DiNear**. Clearly $\mathfrak{T}$ maps the identity function on U to the identity difunction on $(U, \mathcal{P}(U))$, while composition of morphisms in **Near** corresponds to composition of relations in texture spaces, so $\mathfrak{T}(f \circ g) = \mathfrak{T}(f) \circ \mathfrak{T}(g)$ since $f' \circ g' = (f \circ g)'$. This establishes that $\mathfrak{T}$ is a functor. Obviously, $\mathfrak{T}$ is full and faithful and bijective on objects and so it is an isomorphism functor. □

As a consequence of Theorem 4.5, we have the following.

COROLLARY 4.6. *The category **Near** is full embedable into **cDiNear**.*

PROPOSITION 4.7. *In the category **DiNear**:*
(i) Every section [1] is an unifomly bicontinuous injective difunction.

(ii) Every unifomly bicontinuous injective difunction is a monomorphism [1].

(iii) Every retraction [1] is an unifomly bicontinuous surjective difunction.

(iv) Every unifomly bicontinuous surjective difunction is a epimorphism [1].

(v) A morphism is an isomorphism [1] if and only if it is bijective as a difunction and its inverse is uniformly bicontinuous.

Proof. The results (i)–(iv) are obtained automatically in the category **dfTex** by [6, Proposition 3.14]. Let us prove that the result (v).

Let $(U, \mathcal{U}, \mu)$ and $(V, \mathcal{V}, \eta)$ be objects in **DiNear** and $(f, F) : (U, \mathcal{U}) \to (V, \mathcal{V})$ be a difunction. Then (f, F) is an isomorphism in **dfTex** if and only if it is bijective. Further, its inverse $(f, F)^{\leftarrow}$ is a morphism in **dfTex** and $(f, F)^{\leftarrow} \circ (f, F) = (i_U, I_U)$, $(f, F) \circ (f, F)^{\leftarrow} = (i_V, I_V)$. Hence, (f, F) is $\mu - \eta$ uniformly bicontinuous iff $(f, F)^{\leftarrow}$ is $\eta - \mu$ uniformly bicontinuous. □

LEMMA 4.8. *Let $(U_j, \mathcal{U}_j, \mu_j)$, $j = 1, 2$ be dinearness texture space and $(f, F) : (U_1, \mathcal{U}_1) \to (U_2, \mathcal{U}_2)$ be a difunction and μ' a subbase of μ_2. Then (f, F) is a uniformly bicontinuous if $\forall\, \mathcal{C} \in \mu' \Longrightarrow (f, F)^{-1}(\mathcal{C}) \in \mu_1$.*

Proof. Let $\mathcal{C} \in \mu_2$. Since $\mu' = \{\mathcal{A}_i \mid i \in I\}$ is a subbase of μ_2, the family

$$\mathcal{C}_B = \{ \bigwedge_{j \in J} \mathcal{A}_j \mid \mathcal{A}_j \in \mu', J \subseteq I,\, J \text{ finite} \}$$

is a base for μ_2. From Proposition 3.6, we have to show $(f,F)^{-1}(\mathcal{B}) \in \mu_1$ to complete the proof. Now let $\mathcal{B} \in \mathcal{C}_B$ and $\mathcal{C}_B = \bigwedge_{j\in J} \mathcal{A}_j$. Then $\mathcal{A}_j \in \mu'$ for all $j \in J$, and $(f,F)^{-1}(\mathcal{C}) \in \mu_1$ and by Proposition 3.3 (ii):

$$(f,F)^{-1}(\mathcal{B}) = (f,F)^{-1}(\bigwedge_{j\in J} \mathcal{A}_j) = \bigwedge_{j\in J} (f,F)^{-1}(\mathcal{A}_j). \qquad \square$$

Now we consider the forgetful functor $\mathcal{G}$: **DiNear** $\to$ **dfTex** where

$$\mathcal{G}((U_1,\mathcal{U}_1,\mu_1) \overset{(f,F)}{\longrightarrow} (U_2,\mathcal{U}_2,\mu_2)) = (U_1,\mathcal{U}_1) \overset{(f,F)}{\longrightarrow} (U_2,\mathcal{U}_2).$$

Theorem 4.9. *The source $\mathcal{S} = ((U,\mathcal{U},\mu),((U,\mathcal{U},\mu) \overset{(f_j,F_j)}{\longrightarrow} (U_j,\mathcal{U}_j,\mu_j))_{j\in J})$ in **DiNear** is $\mathcal{G}$–initial if and only if $\mathcal{C} = \{(f_j,F_j)^{-1}(\mathcal{A}) \mid \mathcal{A} \in \mu_j, j \in J\}$ is a subbase for $(U,\mathcal{U},\mu)$. That is, μ is coarsest dinearness structure on $(U,\mathcal{U})$ for which the difunctions (f_j,F_j), $j \in J$, are uniformly bicontinuous.*

Proof. ($\Longrightarrow$) Let $\mathcal{S} = ((U,\mathcal{U},\mu),((U,\mathcal{U},\mu) \overset{(f_j,F_j)}{\longrightarrow} (U_j,\mathcal{U}_j,\mu_j))_{j\in J})$ be a $\mathcal{G}$–initial. Since each (f_j,F_j) is a morphism in **DiNear** it is uniformly bicontinuous, hence $\{(f_j,F_j)^{-1}(\mathcal{A}) \mid \mathcal{A} \in \mu_j, j \in J\} \subseteq \mu$.

Now let μ^* be the dinearness structure on $(U,\mathcal{U})$ with subbase $\mathcal{C}$. Then $\mu^* \subseteq \mu$. Since the given source is $\mathcal{G}$–initial the morphism (i,I) in **dfTex** occurring in the commutative diagram on the right lifts to a morphism in **DiNear** making the diagram on the left commute.

$$\begin{array}{ccc} (S,\mathcal{S},\mu^*) & \overset{(f_j,F_j)}{\searrow} & \\ {\scriptstyle (i,I)}\downarrow & & \\ (U,\mathcal{U},\mu) & \underset{(f_j,F_j)}{\longrightarrow} & (U_j,\mathcal{U}_j,\mu_j) \end{array} \qquad \begin{array}{ccc} (U,\mathcal{U}) & \overset{(f_j,F_j)}{\searrow} & \\ {\scriptstyle (i,I)}\downarrow & & \\ (U,\mathcal{U}) & \underset{(f_j,F_j)}{\longrightarrow} & (U_j,\mathcal{U}_j) \end{array}$$

Hence $\mu \subseteq \mu^*$, which proves $\mu = \mu^*$, as required.

($\Longleftarrow$) Let $\mu = \mu^*$ and $((U',\mathcal{U}',\mu'),((U',\mathcal{U}',\mu') \overset{(h_j,H_j)}{\longrightarrow} (U_j,\mathcal{U}_j,\mu_j))_{j\in J})$ be a source in **DiNear** and consider the following diagrams in **DiNear** and **dfTex**, respectively.

$$\begin{array}{ccc} (U',\mathcal{U}',\mu') & \overset{(h_j,H_j)}{\searrow} & \\ {\scriptstyle (k,K)}\downarrow & & \\ (U,\mathcal{U},\mu) & \underset{(f_j,F_j)}{\longrightarrow} & (U_j,\mathcal{U}_j,\mu_j) \end{array} \qquad \begin{array}{ccc} (U',\mathcal{U}') & \overset{(h_j,H_j)}{\searrow} & \\ {\scriptstyle (k,K)}\downarrow & & \\ (S,\mathcal{S}) & \underset{(f_j,F_j)}{\longrightarrow} & (U_j,\mathcal{U}_j) \end{array}$$

Given that the morphism $(k,K) \in \mathbf{dfTex}((U',\mathcal{U}'),(U,\mathcal{U}))$ makes the right hand diagram commutative, it will clearly be sufficient, in view of the fact that $\mathcal{G}$ is faithful, to show that (k,K) is a morphism in **DiNear**. Since $\mathcal{C}$ is a subbase of μ, the family

$$\mathcal{C}_B = \{ \bigwedge_{j\in J'} (f_j,F_j)^{-1}(\mathcal{A}_j) \mid \mathcal{A}_j \in \mu_j,\ J' \subseteq J,\ J' \text{ finite}\}$$

is a base of μ. For a finite index set $J' \subseteq J$ and $j \in J'$, $\mathcal{A}_j \in \mu_j$, it will be sufficient to show that $(k,K)^{-1}(\mathcal{B}) \in \mu'$ where $\mathcal{B} = \bigwedge (f_j,F_j)^{-1}(\mathcal{A}_j) \in \mathcal{C}_B$. Since $(f_j,F_j) \circ (k,K) = (h_j,H_j)$, we have $K^{\leftarrow}(F_j^{\leftarrow}(M)) = (F_j \circ K)^{\leftarrow}(M) = H_j^{\leftarrow}(M)$ and $k^{\leftarrow}(f_j^{\leftarrow}(N)) = (f_j \circ k)^{\leftarrow}(N) = h_j^{\leftarrow}(N)$ for all $(M,N) \in \mathcal{A}_j$. Since (h_j,H_j) is uniformly bicontinuous for all $j \in J$ and $(h_j, H_j)^{-1}(\mathcal{A}) \in \mu'$ for all $\mathcal{A} \in \mu_j$, we have

$$(k,K)^{-1}(\mathcal{B}) = (k,K)^{-1}(\bigwedge_{j\in J'}(f_j,F_j)^{-1}(\mathcal{A}_j)) = \bigwedge_{j\in J'}((k,K)^{-1}((f_j,F_j)^{-1}(\mathcal{A}_j)))$$
$$= \bigwedge_{j\in J'}((f_j,F_j)\circ(k,K))^{-1}(\mathcal{A}_j) = \bigwedge_{j\in J'}(h_j,H_j)^{-1}(\mathcal{A}_j) \in \mu'.$$

Hence (k, K) is uniformly bicontinuous, as required. □

THEOREM 4.10. *The functor* $\mathcal{G}$: **DiNear** $\to$ **dfTex** *is topological. In other words,* ***DiNear*** *is topological category over* ***dfTex*** *with respect to the functor* $\mathcal{G}$.

Proof. Take $(U_j, \mathcal{U}_j, \mu_j) \in \mathrm{Ob}(\mathbf{DiNear})$, $j \in J$, and $(U,\mathcal{U}) \xrightarrow{(f_j,F_j)} (U_j,\mathcal{U}_j)$ in $\mathcal{G}(\mathbf{DiNear}) = \mathbf{dfTex}$. Let μ be the dinearness structure on $(U,\mathcal{U})$ with subbase $\mathcal{C} = \big\{(f_j,F_j)^{-1}(\mathcal{A}_j) \mid \mathcal{A}_j \in \mu_j, j \in J\big\}$. Then, by Theorem 4.9, $((U,\mathcal{U},\mu)),((U,\mathcal{U},\mu) \xrightarrow{(f_j,F_j)} (U_j,\mathcal{U}_j,\mu_j))_{j\in J})$ is the unique $\mathcal{G}$–initial source, which maps to $((U,\mathcal{U}),((U,\mathcal{U}) \xrightarrow{(f_j,F_j)} (U_j,\mathcal{U}_j))_{j\in J})$ under $\mathcal{G}$. □

In order to characterize the products in **DiNear** we apply the notions of limit, initial source and topological functor in the category theory [1]. Then, as a concequence Theorem 4.9, we obtain the following.

THEOREM 4.11. *The source* $\mathcal{S} = ((U,\mathcal{U},\mu),((U,\mathcal{U},\mu) \xrightarrow{(f_j,F_j)} (U_j,\mathcal{U}_j,\mu_j))_{j\in J})$ *is a product of the family* $(U_j,\mathcal{U}_j,\mu_j)_{j\in J}$ *in* ***DiNear*** *iff* μ *has subbase* $\mathcal{C} = \{(f_j,F_j)^{-1}(\mathcal{A}_j) \mid \mathcal{A}_j \in \mu_j, j \in J\}$ *and* $((U,\mathcal{U}),((U,\mathcal{U}) \xrightarrow{(f_j,F_j)} (U_j,\mathcal{U}_j))_{j\in J})$ *is a product of the family* $(U_j,\mathcal{U}_j)_{j\in J}$ *in* ***dfTex***.

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