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UNITS OF THE SEMISIMPLE GROUP ALGEBRAS OF GROUPS OF ORDER 162

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Abstract. In this paper, we consider all the non-metabelian groups of order 162 and characterize the structure of the unit group of the corresponding group algebras. Overall, there are 55 non-isomorphic groups having order 162 and 11 among them are non-metabelian. We study the unit group of the semisimple group algebras over any finite field whose characteristic does not divide the order of these eleven groups.

1. Introduction

The group algebra, denoted by $K\mathcal{G}$ over the field K with p^k elements for a prime number p and $k \in \mathbb{Z}^+$, is the linear combination of elements from $\mathcal{G}$ with coefficients from K, where $\mathcal{G}$ is a finite group. From Maschke's theorem [12] it follows that the group algebra $K\mathcal{G}$ is semisimple if $\mathrm{char}(K) \nmid n$. Consequently, $K\mathcal{G}$ is isomorphic to the direct sum of matrix algebras over division rings according to the Wedderburn decomposition theorem [12], i.e, $K\mathcal{G} \cong M(n_1, D_1) \oplus \cdots \oplus M(n_l, D_l),\ n_i, l \in \mathbb{Z}^+$. The structure of the unit group of $K\mathcal{G}$ can be derived directly from the above isomorphism. Recall that the unit group consists of all invertible elements in $K\mathcal{G}$ and is denoted by $\mathcal{U}(K\mathcal{G})$. For some of the notable recent research on the structure of the unit group, see [13, 15, 20]. Research in this direction is important because of the applications of units in number theory [8], coding theory [9], etc.

With respect to the study of the unit group of all group algebras, the groups can be categorized into two divisions: metabelian and non-metabelian. The first case has been studied in detail by Bakshi et al. in [4]. Therefore, we only need to deal with the non-metabelian groups. Pazderski [17] identified the possible orders of non-metabelian groups. With the help of [17] one can easily find that the smallest non-metabelian group has order 24, and the unit groups of the corresponding group algebras are studied by Khan et al. [10] and Maheshwari et al. [11]. In the same

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direction, Mittal et al. and Arvind et al. classified the unit group of group algebras of non-metabelian groups up to order 120 in [3,13,16,18]. Recently, Abhilash et al. [1,2] and Mittal et al. [14] have completed the work for all groups up to order 144.

Using [17] it can be identified that there are non-metabelian groups of order 162. The number of non-isomorphic groups of order 162 is 55, of which 11 are non-metabelian (see Section 3). The aim of this paper is to consider these 11 groups and compute the unit groups of their group algebras using the Wedderburn decomposition (see [12]).

The flow of this paper is as follows. The important definitions, results and the 11 non-metabelian groups studied in this paper are introduced in Section 2 and Section 3, respectively. In addition, results on the unit groups of semisimple group algebras are proved in Section 3. The fourth section concludes the paper. Finally, in the appendix, we present the semi-direct products involved in the construction of various non-isomorphic groups that play a role in our work.

2. Preliminaries

This section contains the prerequisite definitions and results required to prove the main results. The following notations hold throughout this paper.

K	finite field of order $q = p^k$ with characteristic p and $k \geq 1$
K_d	extension field of K with degree of extension $d, d \in \mathbb{N}$
$\mathcal{G}$	finite group of order n with $p \nmid n$
e	exponent of the group $\mathcal{G}$
ω	primitive e-th root of unity over K
$\mathbb{G}$	Galois group of $K(\omega)$ over K
$\mathcal{T}_{\mathcal{G},K}$	collection of all s such that $\sigma(\omega) = \omega^s$, where $\sigma \in \mathbb{G}$
C_x	conjugacy class of x
$[x, y]$	denote the commutator $x^{-1}y^{-1}xy$ of $x, y \in \mathcal{G}$
1	identity element of $\mathcal{G}$

Definition 2.1 ([6]). (i) For any prime p, an element $x \in \mathcal{G}$ is said to be p'-element if order of x is not divisible by p.

(ii) For any p'-element $x \in \mathcal{G}$, the cyclotomic K-class of $\gamma_x = \sum_{h \in C_x} h$ is the set $S_K(\gamma_x) = \{\gamma_{x^s} \mid s \in \mathcal{T}_{\mathcal{G},K}\}$.

The proposition given below discusses about the total count of cyclotomic K-classes, whereas Lemma 2.3 discusses the number of elements in a particular cyclotomic K-class.

Proposition 2.2 ([6]). *The set of simple components of $K\mathcal{G}/J(K\mathcal{G})$ and the set of cyclotomic K-classes in $\mathcal{G}$, where $J(K\mathcal{G})$ is the Jacobson radical of $K\mathcal{G}$, are in 1-1 correspondence.*

LEMMA 2.3 ([6]). *Let l be the number of cyclotomic K-classes in $\mathcal{G}$. If $K^{(1)}, K^{(2)}, \ldots,$ $K^{(l)}$ are the simple components of $Z\big(K\mathcal{G}/J(K\mathcal{G})\big)$ and $S_1, S_2, \ldots, S_l$ are the cyclotomic K-classes of $\mathcal{G}$, then $|S_i| = [K^{(i)} : K]$ with a suitable ordering of the indices, assuming that $\mathbb{G}$ is cyclic.*

In order to uniquely characterize the Wedderburn decompositions of the semisimple group algebras, we need the following result. See [12, Chapter 3] for its proof.

LEMMA 2.4. *(i) Let $K\mathcal{G}$ be a semisimple group algebra and let $\mathcal{N} \trianglelefteq \mathcal{G}$. Then $K\mathcal{G} \cong K(\mathcal{G}/\mathcal{N}) \oplus \Delta(\mathcal{G}, \mathcal{N})$, where $\Delta(\mathcal{G}, \mathcal{N})$ is an ideal of $K\mathcal{G}$ generated by the set $\{n - 1 : n \in \mathcal{N}\}$.*

(ii) If $\mathcal{N} = \mathcal{G}'$ in part (i), then $K(\mathcal{G}/\mathcal{G}')$ is the sum of all commutative simple components of $K\mathcal{G}$ and $\Delta(\mathcal{G}, \mathcal{G}')$ is the sum of all others.

Furthermore, we discuss a necessary condition for the dimension of the matrix algebra in the Wedderburn decomposition (see [5]).

LEMMA 2.5. *If $\oplus_{i=1}^{t} M_{n_i}(K_i)$ is the summand of the semisimple group algebra $K\mathcal{G}$ and p is the characteristic of K, where K_i is a finite field extension of K for each i. Then p does not divide any of the n_i.*

Next, we discuss two very important results which help us to find the unique Wedderburn decomposition. For the proof of Lemma 2.6 we refer to [19].

LEMMA 2.6. *Let p_1 and p_2 be two primes. Let $\mathbb{F}_{q_1}$ be a field with $q_1 = p_1^{k_1}$ elements and let $\mathbb{F}_{q_2}$ be a field with $q_2 = p_2^{k_2}$ elements, where $k_1, k_2 \geq 1$. Let both the group algebras $\mathbb{F}_{q_1}\mathcal{G}, \mathbb{F}_{q_2}\mathcal{G}$ be semisimple. Suppose that $\mathbb{F}_{q_1}\mathcal{G} \cong \oplus_{i=1}^{t} M(n_i, \mathbb{F}_{q_1})$, $n_i \geq 1$ and $M\left(n, \mathbb{F}_{q_2^r}\right)$ is a Wedderburn component of the group algebra $\mathbb{F}_{q_2}\mathcal{G}$ for some $r \geq 2$ and any positive integer n, i.e. $\mathbb{F}_{q_2}\mathcal{G} \cong \oplus_{i=1}^{s-1} M(m_i, \mathbb{F}_{q_{2,i}}) \oplus M(n, \mathbb{F}_{q_2^r})$, $m_i \geq 1$. Here, $\mathbb{F}_{q_{2,i}}$ is a field extension of $\mathbb{F}_{q_2}$. Then $M(n, \mathbb{F}_{q_1})$ must be a Wedderburn component of the group algebra $\mathbb{F}_{q_1}\mathcal{G}$ and it appears at least r times in the Wedderburn decomposition of $\mathbb{F}_{q_1}\mathcal{G}$.*

LEMMA 2.7 ([3, Corollary 3.8]). *Let $K\mathcal{G}$ be a finite semisimple group algebra. Then if there exists an irreducible representation of degree n over K, then one of the Wedderburn components of $K\mathcal{G}$ is $M_n(K)$. Moreover, if there exist k irreducible representations of degree n over K, then $M_n(K)^k$ is a summand of the group algebra $K\mathcal{G}$.*

If $\mathcal{G}$ has j conjugacy classes, then the representatives of the conjugacy classes are denoted by $g_1(= 1), g_2, g_3, \ldots, g_j$ in this paper. Furthermore, let $K = \mathbb{F}_q$ and $K_i = \mathbb{F}_{q^i}$ for $i \geq 2$.

3. Unit groups

In this section, we discuss the structure of the unit group of the group algebras of non-metabelian groups of order 162. We recall that a group is non-metabelian if its derived subgroup is non-abelian. There are 55 non-isomorphic groups of order 162 (this is known via GAP software [7]) and out of these 55, the following eleven are non-metabelian:

1. $\mathcal{G}_1$:=$((C_3 \times C_3 \times C_3) \rtimes C_3) \rtimes C_2$
2. $\mathcal{G}_2$:=$((C_9 \times C_3) \rtimes C_3) \rtimes C_2$
3. $\mathcal{G}_3$:=$((C_9 \times C_3) \rtimes C_3) \rtimes C_2$
4. $\mathcal{G}_4$:=$((C_9 \times C_3) \rtimes C_3) \rtimes C_2$
5. $\mathcal{G}_5$:=$((C_3 \times C_3 \times C_3) \rtimes C_3) \rtimes C_2$
6. $\mathcal{G}_6$:=$((C_9 \times C_3) \rtimes C_3) \rtimes C_2$
7. $\mathcal{G}_7$:=$((C_9 \times C_3) \rtimes C_3) \rtimes C_2$
8. $\mathcal{G}_8$:=$(C_3 \cdot ((C_3 \times C_3) \rtimes C_3)) \rtimes C_2$
9. $\mathcal{G}_9$:=$C_3 \times (((C_3 \times C_3) \rtimes C_3) \rtimes C_2)$
10. $\mathcal{G}_{10}$:=$((C_9 \times C_3) \rtimes C_3) \rtimes C_2$
11. $\mathcal{G}_{11}$:=$(C_3 \times ((C_3 \times C_3) \rtimes C_3)) \rtimes C_2$

Despite having similar structures, the groups $\mathcal{G}_1$ and $\mathcal{G}_5$ are not isomorphic to each other due to the different group actions involved in their semi-direct product. This can be seen as follows. The structure of both the groups $\mathcal{G}_1$ and $\mathcal{G}_5$ involve an action of C_2 on the group $\mathcal{G} := (C_3 \times C_3 \times C_3) \rtimes C_3$. The presentation of $\mathcal{G}$ is

$$\langle f_1, f_2, f_3, f_4 \mid f_1^3, [f_2, f_1]f_3^{-1}, [f_3, f_1]f_4^{-1}, [f_4, f_1], f_2^3, [f_3, f_2], [f_4, f_2], f_3^3, [f_4, f_3], f_4^3\rangle.$$

The group actions corresponding to $\mathcal{G}_1$ and $\mathcal{G}_5$ are given below: (let Aut($\mathcal{G}$) denote the automorphism group of a group $\mathcal{G}$)

• We consider $\sigma_1 \in \mathrm{Aut}((C_3 \times C_3 \times C_3) \rtimes C_3)$ which maps f_1, f_2, f_3, f_4 to $f_1^2 f_4^2$, $f_2 f_3$, f_3^2, f_4, respectively. Also, we can note that σ_1 is an element of order 2 in $\mathrm{Aut}((C_3 \times C_3 \times C_3) \rtimes C_3)$. Therefore, the group action $C_2 \mapsto \mathrm{Aut}((C_3 \times C_3 \times C_3) \rtimes C_3)$ generates $\mathcal{G}_1$ via σ_1.

• We consider $\sigma_2 \in \mathrm{Aut}((C_3 \times C_3 \times C_3) \rtimes C_3)$ f_1, f_2, f_3, f_4 to $f_1^2 f_3 f_4, f_2^2 f_3^2, f_3, f_4^2$, respectively. Also, σ_2 is an element of order 2 in $\mathrm{Aut}((C_3 \times C_3 \times C_3) \rtimes C_3)$. The group action $C_2 \mapsto \mathrm{Aut}((C_3 \times C_3 \times C_3) \rtimes C_3)$ generates $\mathcal{G}_5$ via σ_2.

Similarly, we can see that the groups $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_6, \mathcal{G}_7, \mathcal{G}_{10}$ are not isomorphic to each other due to the different group actions involved in their semi-direct product (please see appendix for more details). Therefore, all the eleven groups mentioned above are non-isomorphic.

3.1 $\mathcal{G}_1 := ((C_3 \times C_3 \times C_3) \rtimes C_3) \rtimes C_2$

The group $\mathcal{G}_1$ has the following presentation:

$$\mathcal{G}_1 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2, x_1], [x_3, x_1]x_3^{-1}, [x_4, x_1]x_5^{-1}x_4^{-1}, [x_5, x_1], x_2^3, [x_3, x_2]x_4^{-1}, [x_4, x_2], [x_5, x_2], x_3^3, [x_4, x_3]x_5^{-1}, [x_5, x_3], x_4^3, [x_5, x_4], x_5^3\rangle.$$

The sizes (S), orders (O) and the representatives (R) of the 22 conjugacy classes of $\mathcal{G}_1$ are:

R	1	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_4	x_1x_5	x_2^2	x_2x_3	x_2x_4	x_5^2
S	1	9	3	18	6	1	9	9	9	3	18	3	1
O	1	2	3	3	3	3	6	6	6	3	9	3	3

$x_1x_2^2$	$x_1x_2x_3$	$x_1x_2x_4$	$x_2^2x_3$	$x_2^2x_5$	$x_2x_4x_5$	$x_1x_2^2x_3$	$x_1x_2^2x_5$	$x_2^2x_4^2$
9	9	9	18	3	3	9	9	3
6	6	6	9	3	3	6	6	3

It is clear that the exponent of $\mathcal{G}_1$ is 18.

THEOREM 3.1. *The unit group of the group algebra $K\mathcal{G}_1$ is as follows:*
1. For $q \equiv \{1, 7, 13\}$ mod 18, $\mathcal{U}(K\mathcal{G}_1) \cong K^{*6} \oplus GL_2(K)^3 \oplus GL_3(K)^{12} \oplus GL_6(K)$.

2. For $q \equiv \{5, 11, 17\}$ mod 18, $\mathcal{U}(K\mathcal{G}_1) \cong K^{*2} \oplus {K_2}^{*2} \oplus GL_2(K) \oplus GL_2(K_2) \oplus GL_3(K_2)^6 \oplus GL_6(K)$.

Proof. The group algebra $K\mathcal{G}_1$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}_1' \cong (C_3 \times C_3) \rtimes C_3$ and $\frac{\mathcal{G}_1}{\mathcal{G}_1'} \cong C_6$. Since $q = p^k$ and $p > 3$, we split the proof in the following 2 cases.

Case 1. $q \equiv \{1, 7, 13\}$ mod 18. In this case, $T_{\mathcal{G},K} = \{1, 7, 13\}$. So, for any $g \in \mathcal{G}_1$, we have $S_K(\gamma_g) = \{\gamma_{g^t} \mid t \in T_{\mathcal{G},K}\} \Rightarrow S_K(\gamma_g) = \{\gamma_g, \gamma_{g^7}, \gamma_{g^{13}}\}$. Further, we note that $\mathcal{G}_1$ contains elements of orders $1, 2, 3, 6, 9$. For any element $g \in \mathcal{G}_1$ of order 2 or 3 or 6, we note that $g^7 = g^{13} = g$. This means that for such elements, we have $S_K(\gamma_g) = \{\gamma_g\}$. Furthermore, if g is any element of $\mathcal{G}_1$ of order 9, then it can be verified that g^7 and g^{13} belong to the conjugacy class of g. Consequently, $S_K(\gamma_g) = \{\gamma_g\}$ for any $g \in \mathcal{G}_1$ of order 9. Therefore, we conclude that the cardinality of every cyclotomic K-class is 1. So, the decomposition of the group algebra by using Proposition 2.2 and Lemma 2.3 is $K\mathcal{G}_1 \cong K \oplus_{i=1}^{21} M_{n_i}(K), n_i \geq 1$. By applying Lemma 2.4 (ii), we further deduce that $K\mathcal{G}_1 \cong K^6 \oplus_{i=1}^{16} M_{n_i}(K)$, $n_i \geq 2$. Since the dimensions of both the sides are same, we end up with $156 = \sum_{i=1}^{16} n_i^2$. This equation has 13 different solutions. By incorporating Lemma 2.5, we conclude that p can not be 5 and 7. This means that we are remaining with 6 choices of n_i's given as follows:

$$(2^{14}, 6, 8), (2^{12}, 3^3, 9), (2^{10}, 3^4, 4, 8), (2^{10}, 4^5, 6), (2^6, 3^4, 4^6), (2^3, 3^{12}, 6).$$

Here, the notation a^b means the b-tuple $(a, a, \cdots, b\text{-times})$. We observe that the subgroup $N := \langle x_5 \rangle$ is normal in $\mathcal{G}_1$ and $F = \mathcal{G}_1/N \cong (C_3 \times C_3) \rtimes C_6$. Using [4], we note that the Wedderburn decomposition of KF for this case is $KF \cong K^6 \oplus M_2(K)^3 \oplus M_6(K)$. Due to Lemma 2.4 (i), we are remaining with $(2^{14}, 6, 8), (2^{10}, 4^5, 6), (2^3, 3^{12}, 6)$ choices of n_i's. Next, we define the group homomorphism $f : \mathcal{G}_1 \to GL_3(\mathbb{F}_7)$, where $\mathbb{F}_7$ is a finite field having 7 elements, as follows:

$$x_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ x_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix},\ x_3 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ x_4 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},\ x_5 \mapsto 2I_3,$$

where I_3 is 3×3 identity matrix. One can verify that f is an irreducible representation of $\mathcal{G}_1$ of degree 3. Therefore, Lemma 2.7 implies that $M_3(\mathbb{F}_7)$ must be a summand of $\mathbb{F}_7\mathcal{G}_1$. This confirms that $(2^3, 3^{12}, 6)$ is the final choice of the values of n_i's. Therefore, we have $K\mathcal{G}_1 \cong K^6 \oplus M_2(K)^3 \oplus M_3(K)^{12} \oplus M_6(K)$.

Case 2. $q \equiv \{5, 11, 17\}$ mod 18. In this case, the cyclotomic K-classes corresponding to g_3, g_6, g_7 and g_8, respectively, include g_{10}, g_{13}, g_{14} and g_9. Further, the cyclotomic

K-classes corresponding to $g_{11}, g_{12}, g_{15}, g_{16}$ and g_{18}, respectively, include g_{17}, g_{22}, g_{20}, g_{21} and g_{19}, while the rest of the g_i's form individual classes. Therefore, per Proposition 2.2, Lemma 2.3 and Lemma 2.4 (ii), the decomposition for this scenario is given by $K\mathcal{G}_1 \cong K^2 \oplus K_2^2 \oplus_{i=1}^{2} M_{n_i}(K) \oplus_{i=3}^{9} M_{n_i}(K_2)$, $n_i > 1$ which means $156 = n_1^2 + n_2^2 + 2\sum_{i=3}^{9} n_i^2$. Also, [4] implies that $KF \cong K^2 \oplus K_2^2 \oplus M_2(K) \oplus M_2(K_2) \oplus M_6(K)$. Thus, Lemma 2.4 (i) derives that $K\mathcal{G}_1 \cong K^2 \oplus K_2^2 \oplus M_2(K) \oplus M_6(K) \oplus M_2(K_2) \oplus_{i=1}^{6} M_{n_i}(K_2)$ and $54 = \sum_{i=1}^{6} n_i^2$, where $n_i \geq 2$. By using the simple substitution, we observe that the only possible choice of n_i's fulfilling above is (3^6). □

3.2 $\mathcal{G}_2 := ((C_9 \times C_3) \rtimes C_3) \rtimes C_2$

The group $\mathcal{G}_2$ has the following presentation:

$$\mathcal{G}_2 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2, x_1], [x_3, x_1]x_3^{-1}, [x_4, x_1]x_5^{-2}x_4^{-1}, [x_5, x_1], x_2^3 x_5^{-1}, [x_3, x_2]x_4^{-1}, [x_4, x_2], [x_5, x_2], x_3^3, [x_4, x_3]x_5^{-2}, [x_5, x_3], x_4^3, [x_5, x_4], x_5^3\rangle.$$

The sizes, orders and the representatives of the 22 conjugacy classes of $\mathcal{G}_2$ are given below:

R	1	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_4	x_2^2	x_2x_3	x_2x_4	x_2x_5	x_5^2
S	1	9	3	18	6	1	9	9	3	18	3	3	1
O	1	2	9	3	3	3	18	6	9	9	9	9	3

$x_1x_2^2$	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_4^2$	$x_2^2x_3$	$x_2^2x_5$	$x_1x_2^2x_3$	$x_2^2x_5^2$	$x_1x_2^2x_3x_5$
9	9	9	9	18	3	9	3	9
18	18	18	6	9	9	18	9	18

It is clear that the exponent of $\mathcal{G}_2$ is 18.

THEOREM 3.2. *The unit group of the group algebra $K\mathcal{G}_2$ is given below:*

1. For $q \equiv \{1\}$ mod 18, $\mathcal{U}(K\mathcal{G}_2) \cong K^{*6} \oplus GL_2(K)^3 \oplus GL_3(K)^{12} \oplus GL_6(K)$.

2. For $q \equiv \{5, 11\}$ mod 18, $\mathcal{U}(K\mathcal{G}_2) \cong K^{*2} \oplus {K_2}^{*2} \oplus GL_2(K) \oplus GL_2(K_2) \oplus GL_3(K_6)^2 \oplus GL_6(K)$.

3. For $q \equiv \{7, 13\}$ mod 18, $\mathcal{U}(K\mathcal{G}_2) \cong K^{*6} \oplus GL_2(K)^3 \oplus GL_3(K_3)^4 \oplus GL_6(K)$.

4. For $q \equiv \{17\}$ mod 18, $\mathcal{U}(K\mathcal{G}_2) \cong K^{*2} \oplus {K_2}^{*2} \oplus GL_2(K) \oplus GL_2(K_2) \oplus GL_3(K_2)^6 \oplus GL_6(K)$.

Proof. The group algebra $K\mathcal{G}_2$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}_2' \cong (C_3 \times C_3) \rtimes C_3$ and $\frac{\mathcal{G}_2}{\mathcal{G}_2'} \cong C_6$. Since $q = p^k$ and $p > 3$, we split the proof in the following 4 cases.

Case 1. $q \equiv \{1\}$ mod 18. In this case, the cardinality of every cyclotomic K-class is 1. We proceed exactly on the similar lines of **Case 1.** of previous theorem to note that $(2^{14}, 6, 8), (2^{10}, 4^5, 6), (2^3, 3^{12}, 6)$ are the only possible choices of n_i's (this group also has a normal subgroup generated by x_5 with factor group isomorphic to $(C_3 \times C_3) \rtimes C_6$). To this end, we define the group homomorphism $f : \mathcal{G}_2 \to GL_3(\mathbb{F}_{19})$

as follows:

$$x_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ x_2 \mapsto \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix},\ x_3 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, x_4 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 7 \end{pmatrix},\ x_5 \mapsto 7I_3.$$

The group homomorphism f is an irreducible representation of $\mathcal{G}_2$ of degree 3. Therefore, Lemma 2.7 implies that $M_3(\mathbb{F}_{19})$ must be a summand of $\mathbb{F}_{19}\mathcal{G}_2$. This confirms that $(2^3, 3^{12}, 6)$ is the only possible choice of the values of n_i's. Therefore, we have $K\mathcal{G}_2 \cong K^6 \oplus M_2(K)^3 \oplus M_3(K)^{12} \oplus M_6(K)$.

Case 2. $q \equiv \{5, 11\} \bmod 18$. In this case, the cyclotomic K-class corresponding to g_3 includes $g_9, g_{11}, g_{12}, g_{19}, g_{21}$ and g_7 includes $g_{14}, g_{15}, g_{16}, g_{20}, g_{22}$. Similarly, g_6 includes g_{13}, g_8 includes g_{17} and g_{10} includes g_{18}, while rest of the g_i's form individual classes. Therefore, per Proposition 2.2, Lemma 2.3 and Lemma 2.4 (ii), the decomposition is $K\mathcal{G}_2 \cong K^2 \oplus K_2^2 \oplus_{i=1}^{2} M_{n_i}(K) \oplus M_{n_3}(K_2) \oplus_{i=4}^{5} M_{n_i}(K_6)$ and $156 = \sum_{i=1}^{2} n_i^2 + 2 \cdot n_3^2 + 6\sum_{i=4}^{5} n_i^2$, where $n_i > 1$. The possible choices of n_i's fulfilling above equation are $(2,6,2,3,3), (3,3,3,2,4), (3,7,5,2,2), (3,9,3,2,2), (6,8,2,2,2)$. We observe that the subgroup $N := \langle x_5 \rangle$ is normal in $\mathcal{G}_2$ and $F = \mathcal{G}_2/N \cong (C_3 \times C_3) \rtimes C_6$. Using [4], we recall that $KF \cong K^2 \oplus K_2^2 \oplus M_2(K) \oplus M_2(K_2) \oplus M_6(K)$. Therefore, $M_2(K)$ and $M_6(K)$ must be the Wedderburn components of $K\mathcal{G}_2$ per Lemma 2.4 (i). So, $(2,6,2,3,3)$ is the required choice, which means that $K\mathcal{G}_2 \cong K^2 \oplus K_2^2 \oplus M_2(K) \oplus M_2(K_2) \oplus M_3(K_6)^2 \oplus M_6(K)$.

Case 3. $q \equiv \{7, 13\} \bmod 18$. In this case, the cyclotomic K-class corresponding to g_3 includes g_{11}, g_{12} and that of g_9 includes g_{19}, g_{21}. Similarly, the cyclotomic K-class corresponding to g_7 includes g_{15}, g_{16} and that of g_{14} includes g_{20}, g_{22}. The rest of the g_i's form individual classes. Therefore, per Proposition 2.2, Lemma 2.3 and Lemma 2.4 (ii), the decomposition for this scenario is given by $K\mathcal{G}_2 \cong K^6 \oplus_{i=1}^{4} M_{n_i}(K) \oplus_{i=5}^{8} M_{n_i}(K_3)$, $n_i > 1$ with $156 = \sum_{i=1}^{4} n_i^2 + 3\sum_{i=5}^{8} n_i^2$. Again, using [4], we note that $KF \cong K^6 \oplus M_2(K)^3 \oplus M_6(K)$. This and Lemma 2.4 (i) imply that $K\mathcal{G}_2 \cong K^6 \oplus M_2(K)^3 \oplus M_3(K_3)^4 \oplus M_6(K)$.

Case 4. $q \equiv \{17\} \bmod 18$. In this case, the cyclotomic K-class corresponding to g_3 includes g_{21} and g_6 includes g_{13}. Similarly, g_7 incudes g_{22} and g_8 includes g_{17}. Also, the cyclotomic K-class corresponding to g_{11} includes g_9 and g_{10} includes g_{18}. Moreover, g_{15} includes g_{14} and g_{16} includes g_{20}. Similarly, g_{12} includes g_{19}, while the rest of the g_i's form individual classes. Therefore, per Proposition 2.2, Lemma 2.3 and (Lemma 2.4 (ii), the decomposition for this case is given by $K\mathcal{G}_2 \cong K^2 \oplus K_2^2 \oplus_{i=1}^{2} M_{n_i}(K) \oplus_{i=3}^{9} M_{n_i}(K_2)$, $n_i > 1 \Rightarrow 156 = n_1^2 + n_2^2 + \sum_{i=3}^{9} 2 \cdot n_i^2$. The possible choices of n_i's fulfilling above are given below:

$$(2^2, 2^2, 3^2, 4^3), (2, 2^2, 3^5, 5), (2, 6, 2, 3^6), (3^2, 2^3, 3, 4^3), (3^2, 2^2, 3^4, 5), (3, 5, 2^5, 4, 5),$$
$$(3, 7, 2^6, 5), (3, 9, 2^6, 3), (4^2, 2^3, 3^2, 4^2), (4, 6, 2^5, 4^2)(4, 8, 2^5, 3^2), (5^2, 2^2, 3^5), (6, 8, 2^7).$$

We observe that the subgroup $N := \langle x_5 \rangle$ is normal in $\mathcal{G}_2$ and $F = \mathcal{G}_2/N \cong (C_3 \times C_3) \rtimes C_6$. Due to case 2 of Theorem 3.1, we know that $KF \cong K^2 \oplus K_2^2 \oplus M_2(K) \oplus M_2(K_2) \oplus M_6(K)$. Therefore, $M_2(K)$ and $M_6(K)$ must be a Wedderburn component of $K\mathcal{G}_2$. So, $(2,6,2,3^6)$ is the required choice, which means that $K\mathcal{G}_2 \cong K^2 \oplus K_2^2 \oplus M_2(K) \oplus M_2(K_2) \oplus M_3(K_2)^6 \oplus M_6(K)$. □

3.3 $\mathcal{G}_3 := ((C_9 \times C_3) \rtimes C_3) \rtimes C_2$

The group $\mathcal{G}_3$ has the following presentation:

$$\mathcal{G}_3 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2,x_1], [x_3,x_1]x_3^{-1}, [x_4,x_1]x_5^{-1}x_4^{-1}, [x_5,x_1], x_2^3x_5^{-1}, [x_3,x_2]x_4^{-1}, [x_4,x_2], [x_5,x_2], x_3^3, [x_4,x_3]x_5^{-1}, [x_5,x_3], x_4^3, [x_5,x_4], x_5^3\rangle.$$

The sizes, orders and the representatives of the 22 conjugacy classes of $\mathcal{G}_3$ are given below:

R	1	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_4	x_1x_5	x_2^2	x_2x_3	x_2x_4	x_5^2
S	1	9	3	18	6	1	9	9	9	3	18	3	1
O	1	2	9	3	3	3	18	6	6	9	3	9	3

$x_1x_2^2$	$x_1x_2x_3$	$x_1x_2x_4$	$x_2^2x_3$	$x_2^2x_5$	$x_2x_4x_5$	$x_1x_2^2x_3$	$x_1x_2^2x_5$	$x_2^2x_4^2$
9	9	9	18	3	3	9	9	3
18	18	18	3	9	9	18	18	9

It is clear that the exponent of $\mathcal{G}_3$ is 18.

THEOREM 3.3. *The unit group of the group algebra $K\mathcal{G}_3$ is given below:*
1. For $q \equiv \{1\}$ mod 18, $\mathcal{U}(K\mathcal{G}_3) \cong K^{*6} \oplus GL_2(K)^3 \oplus GL_3(K)^{12} \oplus GL_6(K)$.

2. For $q \equiv \{5,11\}$ mod 18, $\mathcal{U}(K\mathcal{G}_3) \cong K^{*2} \oplus {K_2}^{*2} \oplus GL_2(K) \oplus GL_2(K_2) \oplus GL_3(K_6)^2 \oplus GL_6(K)$.

3. For $q \equiv \{7,13\}$ mod 18, $\mathcal{U}(K\mathcal{G}_3) \cong K^{*6} \oplus GL_2(K)^3 \oplus GL_3(K_3)^4 \oplus GL_6(K)$.

4. For $q \equiv \{17\}$ mod 18, $\mathcal{U}(K\mathcal{G}_3) \cong K^{*2} \oplus {K_2}^{*2} \oplus GL_2(K) \oplus GL_2(K_2) \oplus GL_3(K_2)^6 \oplus GL_6(K)$.

Proof. The group algebra $K\mathcal{G}_3$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}_3' \cong (C_3 \times C_3) \rtimes C_3$ and $\frac{\mathcal{G}_3}{\mathcal{G}_3'} \cong C_6$.
Case 1. $q \equiv \{1\}$ mod 18. In this case, the cardinality of every cyclotomic K-class is 1. Therefore, by proceeding on the similar lines of **Case 1.** of Theorem 3.2 and considering the group homomorphism from $\mathcal{G}_3$ to $GL_3(\mathbb{F}_{19})$ as

$$x_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ x_2 \mapsto \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{pmatrix},\ x_3 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ x_4 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 11 \end{pmatrix},\ x_5 \mapsto 7I_3,$$

we can conclude that $K\mathcal{G}_3 \cong K^6 \oplus M_2(K)^3 \oplus M_3(K)^{12} \oplus M_6(K)$. The rest of the proof can be done in similar ways of Theorem 3.2. □

3.4 $\mathcal{G}_4 := ((C_9 \times C_3) \rtimes C_3) \rtimes C_2$

The group $\mathcal{G}_4$ has the following presentation:

$$\mathcal{G}_4 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2,x_1]x_4^{-1}x_2^{-1}, [x_3,x_1]x_3^{-1}, [x_4,x_1]x_4^{-1}, [x_5,x_1], x_2^3x_4^{-2},$$

$[x_3, x_2]x_5^{-1}, [x_4, x_2], [x_5, x_2], x_3^3, [x_4, x_3], [x_5, x_3], x_4^3, [x_5, x_4], x_5^3\rangle.$

The sizes, orders and the representatives of the 21 conjugacy classes of $\mathcal{G}_4$ are given below:

R	1	x_1	x_2	x_3	x_4	x_5	x_1x_5	x_2^2	x_2x_3	x_3x_4	x_4x_5	x_5^2
S	1	27	6	6	2	1	27	6	6	6	2	1
O	1	2	9	3	3	3	6	9	9	3	3	3

$x_1x_5^2$	$x_2^2x_3$	$x_2x_3^2$	$x_2x_3x_4$	$x_2x_4^2$	$x_3^2x_4$	$x_4x_5^2$	$x_2x_3x_4^2$	$x_2^2x_3x_4^2$
27	6	6	6	6	6	2	6	6
6	9	9	9	9	3	3	9	9

It is clear that the exponent of $\mathcal{G}_4$ is 18.

THEOREM 3.4. *The unit group of the group algebra $K\mathcal{G}_4$ is given below:*
1. For $q \equiv \{1\}$ mod 18, $\mathcal{U}(K\mathcal{G}_4) \cong K^{*2} \oplus GL_2(K)^{13} \oplus GL_3(K)^4 \oplus GL_6(K)^2$.

2. For $q \equiv \{5, 11\}$ mod 18, $\mathcal{U}(K\mathcal{G}_4) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_2(K_3^3) \oplus GL_3(K_2)^2 \oplus GL_6(K_2)$.

3. For $q \equiv \{7, 13\}$ mod 18, $\mathcal{U}(K\mathcal{G}_4) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_2(K_3^3) \oplus GL_3(K)^4 \oplus GL_6(K)^2$.

4. For $q \equiv \{17\}$ mod 18, $\mathcal{U}(K\mathcal{G}_4) \cong K^{*2} \oplus GL_2(K)^{13} \oplus GL_3(K_2)^2 \oplus GL_6(K_2)$.

Proof. The group algebra $K\mathcal{G}_4$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}_4' \cong (C_9 \times C_3) \rtimes C_3$ and $\frac{\mathcal{G}_4}{\mathcal{G}_4'} \cong C_2$.

Case 1. $q \equiv \{1\}$ mod 18. In this case, the cardinality of every cyclotomic K-class is 1. Therefore, the decomposition of the group algebra is $K\mathcal{G}_4 \cong K^2 \oplus_{i=1}^{19} M_{n_i}(K_i)$, where $160 = \sum_{i=1}^{19} n_i^2,\ n_i > 1$. We consider the normal subgroup $N_1 = \langle x_5 \rangle$ of $\mathcal{G}_4$ with $\mathcal{G}_4/N_1 \cong (C_9 \times C_3) \rtimes C_2$. Using [4], we see that $K(\mathcal{G}_4/N_1) \cong K^2 \oplus M_2(K)^{13}$. This and Lemma 2.4 (i) imply that $K\mathcal{G}_4 \cong K^2 \oplus M_2(K)^{13} \oplus_{i=1}^{6} M_{n_i}(K)$ with $108 = \sum_{i=1}^{6} n_i^2,\ n_i > 1$. Next, we consider the normal subgroup $N_2 := \langle x_4 \rangle$ of $\mathcal{G}_4$. The corresponding factor group $\mathcal{G}_4/N_2 \cong ((C_3 \times C_3) \rtimes C_3) \rtimes C_2$. Again, by [4], we know that $K(\mathcal{G}_4/N_2) \cong K^2 \oplus M_2(K)^4 \oplus M_3(K)^4$. This and Lemma 2.4 (i) give $K\mathcal{G}_4 \cong K^2 \oplus M_2(K)^{13} \oplus M_3(K)^4 \oplus_{i=1}^{2} M_{n_i}(K)$, with $72 = \sum_{i=1}^{2} n_i^2,\ n_i > 1$. The last equation has a unique solution given by (6^2).

Case 2. $q \equiv \{5, 11\}$ mod 18. In this case, the cyclotomic K-class of g_3 includes g_8, g_{17}, whereas g_9 includes g_{16}, g_{20} and g_{14} includes g_{15}, g_{21}. Similarly, g_7, g_{13} come under same class, g_6, g_{12} come under same class and g_{11}, g_{19} come under same class. The rest of the g_i's form individual classes. The decomposition for this scenario is given by $K\mathcal{G}_4 \cong K^2 \oplus_{i=1}^{4} M_{n_i}(K) \oplus_{i=5}^{7} M_{n_i}(K_2) \oplus_{i=8}^{10} M_{n_i}(K_3)$ with $160 = \sum_{i=1}^{4} n_i^2 + 2\sum_{i=5}^{7} n_i^2 + 3\sum_{i=8}^{10} n_i^2$ where $n_i > 1$. Using [4], we know that $K(\mathcal{G}_4/N_1) \cong K^2 \oplus M_2(K)^4 \oplus M_2(K_3)^3$. This and Lemma 2.4 (i) derive that $K\mathcal{G}_4 \cong K^2 \oplus M_2(K)^4 \oplus_{i=1}^{3} M_{n_i}(K_2) \oplus M_2(K_3)^3$, which means $54 = \sum_{i=1}^{3} n_i^2$. The last equation has two solutions namely $(2, 5^2)$ and $(3^2, 6)$. Using Lemma 2.6, if $M_5(K_2)$ is the Wedderburn component in this case, then $M_5(K)$ must be a Wedderburn component in case 1 which is not so. Hence, $(2^4, 3^2, 6, 2^3)$ is the unique choice.

Case 3. $q \equiv \{7, 13\}$ mod 18. In this case, the cyclotomic K-class of g_3 includes g_8, g_{17}, whereas g_9 includes g_{16}, g_{20} and g_{14} includes g_{15}, g_{21}. Other representatives forms individual classes. The group algebra $K\mathcal{G}_4$ decomposes as follows: $K\mathcal{G}_9 \cong K^2 \oplus_{i=1}^{10} M_{n_i}(K) \oplus_{i=11}^{13} M_{n_i}(K_3)$ with $160 = \sum_{i=1}^{10} n_i^2 + 3\sum_{i=11}^{13} n_i^2$, $n_i > 1$. By following the procedure as in the previous case (i.e., by considering the same normal subgroup N_1 and using the Wedderburn decomposition of $K(\mathcal{G}_4/N_1)$ for this case), we can show that $K\mathcal{G}_4 \cong K^2 \oplus M_2(K)^4 \oplus M_2(K_3)^3 \oplus M_3(K)^4 \oplus M_6(K)^2$.

Case 4. $q \equiv \{17\}$ mod 18. By following the procedure as in previous cases, one can show the result in this case. This completes the proof. □

3.5 $\mathcal{G}_5 := ((C_3 \times C_3 \times C_3) \rtimes C_3) \rtimes C_2$

The group $\mathcal{G}_5$ has the following presentation:

$$\mathcal{G}_5 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2, x_1]x_2^{-1}, [x_3, x_1]x_3^{-1}, [x_4, x_1]x_5^{-2}, [x_5, x_1]x_5^{-1}, x_2^3, [x_3, x_2]x_4^{-1}, [x_4, x_2]x_5^{-1}, [x_5, x_2], x_3^3, [x_4, x_3], [x_5, x_3], x_4^3, [x_5, x_4], x_5^3\rangle.$$

The sizes, orders and the representatives of the 13 conjugacy classes of $\mathcal{G}_5$ are given below:

R	1	x_1	x_2	x_3	x_4	x_5	x_1x_4	x_2x_3	x_3x_5	x_4^2	$x_1x_4^2$	$x_2^2x_3$	$x_3^2x_4$
S	1	27	18	6	3	2	27	18	6	3	27	18	6
O	1	2	3	3	3	3	6	9	3	3	6	9	3

It is clear that the exponent of $\mathcal{G}_5$ is 18.

Theorem 3.5. *The unit group of the group algebra $K\mathcal{G}_5$ is given below:*
1. For $q \equiv \{1, 7, 13\}$ mod 18, $\mathcal{U}(K\mathcal{G}_5) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K)^4 \oplus GL_6(K)^3$.
2. For $q \equiv \{5, 11, 17\}$ mod 18, $\mathcal{U}(K\mathcal{G}_5) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K_2) \oplus GL_6(K)^3$.

Proof. The group algebra $K\mathcal{G}_5$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}_5' \cong (C_3 \times C_3 \times C_3) \rtimes C_3$ and $\frac{\mathcal{G}_5}{\mathcal{G}_5'} \cong C_2$. Since $q = p^k$ and $p > 3$, we split the proof in the following 2 cases.
Case 1. $q \equiv \{1, 7, 13\}$ mod 18. In this case, the cardinality of every cyclotomic K-class is 1. Therefore, the decomposition of the group algebra by using Proposition 2.2 and Lemma 2.3 is $K\mathcal{G}_5 \cong K \oplus_{i=1}^{12} M_{n_i}(K), n_i \geq 1$. By applying Lemma 2.4 (ii), we further deduce that $K\mathcal{G}_5 \cong K^2 \oplus_{i=1}^{11} M_{n_i}(K)$, $n_i \geq 2$. Since the dimensions of both the sides are same, we end up with $160 = \sum_{i=1}^{11} n_i^2$. We observe that the subgroup $N_1 := \langle x_5 \rangle$ is normal in $\mathcal{G}_5$ and $F_1 = \mathcal{G}_5/N_1 \cong ((C_3 \times C_3) \rtimes C_3) \rtimes C_2$. Using [4], we note that $K(\mathcal{G}_4/N_1) \cong K^2 \oplus M_2(K)^4 \oplus M_3(K)^4$. This and Lemma 2.4 (i) deduce that $K\mathcal{G}_5 \cong K^2 \oplus M_2(K)^4 \oplus M_3(K)^4 \oplus_{i=1}^{3} M_{n_i}(K)$ with $\sum_{i=1}^{3} n_i^2 = 108$. The last equation has two solutions given by $(2^2, 10)$ and (6^3). Also, by incorporating Lemma 2.5, we conclude $M_{10}(K)$ can not be a Wedderburn component as p can be 5. This means that $(2^4, 3^4, 6^3)$ is the only possible choice of the values of n_i's.

Case 2. $q \equiv \{5, 11, 17\}$ mod 18. In this case, the cyclotomic K-class corresponding to g_5 includes g_{10} and g_7 includes g_{11}, while the rest of the g_i's form individual classes. Therefore, per Proposition 2.2, Lemma 2.3 and Lemma 2.4 (ii), the decomposition in this case is $K\mathcal{G}_5 \cong K^2 \oplus_{i=1}^{7} M_{n_i}(K) \oplus_{i=8}^{9} M_{n_i}(K_2)$, $n_i > 1$, where $160 = \sum_{i=1}^{7} n_i^2 + 2\sum_{i=8}^{9} n_i^2$. Using [4], we recall that $KF_1 \cong K^2 \oplus M_2(K)^4 \oplus M_3(K)^2$. From this and Lemma 2.4 (i), for $K\mathcal{G}$, $(2^6, 10, 3^2)$ and $(2^4, 6^3, 3^2)$ are the only choices. Also, by Lemma 2.5, $M_5(K)$ can not be a Wedderburn component as p can be 5. So, $(2^4, 6^3, 3^2)$ is the unique value of n_i's. □

3.6 $\mathcal{G}_6 := ((C_9 \times C_3) \rtimes C_3) \rtimes C_2$

The group $\mathcal{G}_6$ has the following presentation:

$$\mathcal{G}_6 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2, x_1]x_2^{-1}, [x_3, x_1]x_5^{-2}x_3^{-1}, [x_4, x_1]x_5^{-2}, [x_5, x_1]x_5^{-1}, x_2^3, [x_3, x_2]x_4^{-1}, [x_4, x_2]x_5^{-1}, [x_5, x_2], x_3^3x_5^{-1}, [x_4, x_3], [x_5, x_3], x_4^3, [x_5, x_4], x_5^3\rangle.$$

By constructing the table of conjugacy classes as in the case of previous groups, we can note that the exponent of $\mathcal{G}_6$ is 18 and it has 13 conjugacy classes.

Theorem 3.6. *The unit group of the group algebra $K\mathcal{G}_6$ is given below:*

1. *For $q \equiv \{1\}$ mod 18, $\mathcal{U}(K\mathcal{G}_6) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K)^4 \oplus GL_6(K)^3$.*
2. *For $q \equiv \{5, 11\}$ mod 18, $\mathcal{U}(K\mathcal{G}_6) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K_2)^2 \oplus GL_6(K_3)$.*
3. *For $q \equiv \{7, 13\}$ mod 18, $\mathcal{U}(K\mathcal{G}_6) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K)^4 \oplus GL_6(K_3)$.*
4. *For $q \equiv \{17\}$ mod 18, $\mathcal{U}(K\mathcal{G}_6) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K_2)^2 \oplus GL_6(K)^3$.*

Proof. We note that the decompositions can be calculated using the same procedure as in Theorems 3.5 and 3.4. Therefore, we skip the proof. □

3.7 $\mathcal{G}_7 := ((C_9 \times C_3) \rtimes C_3) \rtimes C_2$

The group $\mathcal{G}_7$ has the following presentation:

$$\mathcal{G}_7 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2, x_1]x_2^{-1}, [x_3, x_1]x_3^{-1}, [x_4, x_1], [x_5, x_1]x_5^{-1}, x_2^3, [x_3, x_2]x_4^{-1}, [x_4, x_2]x_5^{-1}, [x_5, x_2], x_3^3, [x_4, x_3]x_5^{-2}, [x_5, x_3], x_4^3, [x_5, x_4], x_5^3\rangle.$$

We can note that the exponent of $\mathcal{G}_7$ is 18 and it has 13 conjugacy classes. The proof of the following theorem follows similarly as that of Theorem 3.6.

Theorem 3.7. *The unit group of the group algebra $K\mathcal{G}_7$ is given below:*

1. *For $q \equiv \{1\}$ mod 18, $\mathcal{U}(K\mathcal{G}_7) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K)^4 \oplus GL_6(K)^3$.*
2. *For $q \equiv \{5, 11\}$ mod 18, $\mathcal{U}(K\mathcal{G}_7) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K_2)^2 \oplus GL_6(K_3)$.*
3. *For $q \equiv \{7, 13\}$ mod 18, $\mathcal{U}(K\mathcal{G}_7) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K)^4 \oplus GL_6(K_3)$.*
4. *For $q \equiv \{17\}$ mod 18, $\mathcal{U}(K\mathcal{G}_7) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K_2)^2 \oplus GL_6(K)^3$.*

3.8 $\mathcal{G}_8 := (C_3 \cdot ((C_3 \times C_3) \rtimes C_3)) \rtimes C_2$

The group $\mathcal{G}_8$ has the following presentation:

$$\mathcal{G}_8 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2, x_1]x_5^{-1}x_2^{-1}, [x_3, x_1]x_5^{-1}x_3^{-1}, [x_4, x_1]x_5^{-2}, [x_5, x_1]x_5^{-1}, x_2^3x_5^{-2}, [x_3, x_2]x_4^{-1}, [x_4, x_2]x_5^{-1}, [x_5, x_2], x_3^3x_5^{-2}, [x_4, x_3], [x_5, x_3], x_4^3, [x_5, x_4], x_5^3\rangle.$$

By constructing the table of conjugacy classes as in the case of previous groups, we can note that the exponent of $\mathcal{G}_8$ is 18 and it has 13 conjugacy classes.

THEOREM 3.8. *The unit group of the group algebra $K\mathcal{G}_8$ is given below:*
1. *For $q \equiv \{1\}$ mod 18, $\mathcal{U}(K\mathcal{G}_8) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K)^4 \oplus GL_6(K)^3$.*
2. *For $q \equiv \{5, 11\}$ mod 18, $\mathcal{U}(K\mathcal{G}_8) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K_2)^2 \oplus GL_6(K_3)$.*
3. *For $q \equiv \{7, 13\}$ mod 18, $\mathcal{U}(K\mathcal{G}_8) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K)^4 \oplus GL_6(K_3)$.*
4. *For $q \equiv \{17\}$ mod 18, $\mathcal{U}(K\mathcal{G}_8) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_3(K_2)^2 \oplus GL_6(K)^3$.*

Proof. The proof is similar to that in Theorems 3.5 and 3.4. We therefore omit it. □

3.9 $\mathcal{G}_9 := C_3 \times (((C_3 \times C_3) \rtimes C_3) \rtimes C_2)$

One can note that the exponent of the group $(C_3 \times C_3) \rtimes C_3) \rtimes C_2$ is 6. This means that the exponent of $\mathcal{G}_9$ is 6.

THEOREM 3.9. *The unit group of the group algebra $K\mathcal{G}_9$ is given below:*
1. *For $q \equiv \{1\}$ mod 6, $\mathcal{U}(K\mathcal{G}_9) \cong K^{*6} \oplus GL_2(K)^{12} \oplus GL_3(K)^{12}$.*
2. *For $q \equiv \{5\}$ mod 6, $\mathcal{U}(K\mathcal{G}_9) \cong K^{*2} \oplus K_2^{*2} \oplus GL_2(K)^4 \oplus GL_2(K_2)^4 \oplus GL_3(K_2)^6$.*

Proof. Since $\mathcal{G}_9$ is a direct product of two groups, its decomposition can be deduced using decompsition of the groups $(C_3 \times C_3) \rtimes C_3) \rtimes C_2$ and C_3 through tensor product as in [13]. □

3.10 $\mathcal{G}_{10} := ((C_9 \times C_3) \rtimes C_3) \rtimes C_2$

The group $\mathcal{G}_{10}$ has the following presentation:

$$\mathcal{G}_{10} = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2, x_1], [x_3, x_1]x_3^{-1}, [x_4, x_1]x_4^{-1}, [x_5, x_1], x_2^3x_5^{-1}, [x_3, x_2], [x_4, x_2], [x_5, x_2], x_3^3, [x_4, x_3]x_5^{-1}, [x_5, x_3], x_4^3, [x_5, x_4], x_5^3\rangle.$$

By constructing the table of conjugacy classes as in the case of previous groups, we can note that the exponent of $\mathcal{G}_{10}$ is 18 and it has 30 conjugacy classes.

THEOREM 3.10. *The unit group of the group algebra $K\mathcal{G}_{10}$ is given below:*
1. *For $q \equiv \{1\}$ mod 18, $\mathcal{U}(K\mathcal{G}_{10}) \cong K^{*6} \oplus GL_2(K)^{12} \oplus GL_3(K)^{12}$.*
2. *For $q \equiv \{5, 11\}$ mod 18, $\mathcal{U}(K\mathcal{G}_{10}) \cong K^{*2} \oplus K_2^{*2} \oplus GL_2(K)^4 \oplus GL_2(K_2)^4 \oplus GL_3(K_6)^2$.*
3. *For $q \equiv \{7, 13\}$ mod 18, $\mathcal{U}(K\mathcal{G}_{10}) \cong K^{*6} \oplus GL_2(K)^{12} \oplus GL_3(K_3)^4$.*
4. *For $q \equiv \{17\}$ mod 18, $\mathcal{U}(K\mathcal{G}_{10}) \cong K^{*2} \oplus K_2^{*2} \oplus GL_2(K)^4 \oplus GL_2(K_2)^4 \oplus GL_3(K_2)^6$.*

Proof. The group algebra $K\mathcal{G}_{10}$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}'_{10} \cong (C_3 \times C_3) \rtimes C_3$ and $\frac{\mathcal{G}_{10}}{\mathcal{G}'_{10}} \cong C_6$.

Case 1. $q \equiv \{1\}$ mod 18. In this case, the cardinality of every cyclotomic K-class is 1. Therefore, the decomposition of the group algebra by using Proposition 2.2 and Lemma 2.3 is $K\mathcal{G}_9 \cong K \oplus_{i=1}^{29} M_{n_i}(K), n_i \geq 1$. By applying Lemma 2.4 (ii), we further deduce that $K\mathcal{G}_9 \cong K^6 \oplus_{i=1}^{24} M_{n_i}(K),\ n_i \geq 2$. Since the dimensions of both the sides are same, we end up with $156 = \sum_{i=1}^{24} n_i^2$. This equation has 5 different solutions given as $(2^{23}, 8), (2^{20}, 3^3, 7), (2^{19}, 4^5), (2^{18}, 3^3, 4^2, 5), (2^{12}, 3^{12})$.

By incorporating Lemma 2.5, we conclude that p can not be 5 and 7. This means that we are remaining with 3 choices of n_i's. We define the group homomorphism $f : \mathcal{G}_{10} \to GL_3(\mathbb{F}_{19})$ as follows:

$$x_1 \mapsto \begin{pmatrix} 18 & 0 & 0 \\ 0 & 0 & 18 \\ 0 & 18 & 0 \end{pmatrix}, x_2 \mapsto \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, x_3 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, x_4 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 11 \end{pmatrix}, x_5 \mapsto 7I_3.$$

The group homomorphism f is an irreducible representation of $\mathcal{G}_{10}$ of degree 3. Therefore, Lemma 2.7 implies that $M_3(\mathbb{F}_{19})$ must be a summand of $\mathbb{F}_{19}\mathcal{G}_{10}$. This confirms that $(2^{12}, 3^{12})$ is the only possible choice of the values of n_i's.
The rest of the cases can be done on the similar lines of the cases of Theorem 3.4. □

3.11 $\mathcal{G}_{11} := (C_3 \times ((C_3 \times C_3) \rtimes C_3)) \rtimes C_2$

The group $\mathcal{G}_{11}$ has the following presentation:

$$\mathcal{G}_{11} = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2, x_1]x_2^{-1}, [x_3, x_1]x_3^{-1}, [x_4, x_1]x_4^{-1}, [x_5, x_1], x_2^3, [x_3, x_2], [x_4, x_2], [x_5, x_2], x_3^3, [x_4, x_3]x_5^{-1}, [x_5, x_3], x_4^3, [x_5, x_4], x_5^3\rangle.$$

By constructing the table of conjugacy classes as in the case of previous groups, we can note that the exponent of $\mathcal{G}_{11}$ is 6 and it has 21 conjugacy classes.

Theorem 3.11. *The unit group of the group algebra $K\mathcal{G}_{11}$ is given below:*
1. For $q \equiv \{1\}$ mod 6, $\mathcal{U}(K\mathcal{G}_{11}) \cong K^{*2} \oplus GL_2(K)^{13} \oplus GL_3(K)^4 \oplus GL_6(K)^2$.
2. For $q \equiv \{5\}$ mod 6, $\mathcal{U}(K\mathcal{G}_{11}) \cong K^{*2} \oplus GL_2(K)^{13} \oplus GL_3(K_2)^2 \oplus GL_6(K_2)$.

Proof. The group algebra $K\mathcal{G}_{11}$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}'_{11} \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$ and $\frac{\mathcal{G}_{11}}{\mathcal{G}'_{11}} \cong C_2$.

Case 1. $q \equiv \{1\}$ mod 6. In this case, the cardinality of every cyclotomic K-class is 1. Therefore, the decomposition of the group algebra is $K\mathcal{G}_{11} \cong K \oplus_{i=1}^{20} M_{n_i}(K), n_i \geq 1$. By applying Lemma 2.4 (ii), we further deduce that $K\mathcal{G}_{11} \cong K^2 \oplus_{i=1}^{19} M_{n_i}(K),\ n_i \geq 2$. Since the dimensions of both the sides are same, we end up with $160 = \sum_{i=1}^{19} n_i^2$. We observe that the subgroup $N_1 := \langle x_5 \rangle$ is normal in $\mathcal{G}_{11}$ and $F_1 = \mathcal{G}_{11}/N_1 \cong (C_3 \times C_3 \times C_3) \rtimes C_2$. Using [4] and Lemma 2.4 (i), we reach at $K\mathcal{G}_{11} \cong K^2 \oplus M_2(K)^{13} \oplus_{i=1}^{6} M_{n_i}(K),\ n_i \geq 2$. Again consider the subgroup $N_2 := \langle x_2 \rangle$ which is normal in $\mathcal{G}_{11}$ and $F_2 = \mathcal{G}_{11}/N_2 \cong ((C_3 \times C_3) \rtimes C_3) \rtimes C_2$. Again using [4] and Lemma 2.4 (i), we conclude that $K\mathcal{G}_{11} \cong K^2 \oplus M_2(K)^{13} \oplus M_3(K)^4 \oplus M_6(K)^2$.

Case 2. $q \equiv \{5\}$ mod 6 can be shown by analogy with **Case 2.** Theorem 3.5. □

4. Conclusion

We have studied the unit group of semisimple group algebras of non-metabelian groups of order 162, and this paper completes the study of unit groups of semisimple group algebras of all groups up to order 162 except those of groups of order 150 and 160. It is clear that as the size of the group increases, new techniques are needed to uniquely characterize the unit groups. This paper motivates the researchers to find an algorithm that computes the Wedderburn decomposition of a semisimple group algebra of any finite group.

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Appendix

Group actions involved in the semi-direct product

In this section, we discuss the group actions involved in the semi-direct products of the groups $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_6, \mathcal{G}_7, \mathcal{G}_{10}$. Basically, we explain the homomorphisms from C_2 to $(C_9 \times C_3) \rtimes C_3$, which generate each group. Using GAP, it can be noted that there are 4 non-isomorphic groups having similar structure given by $(C_9 \times C_3) \rtimes C_3$. We call them as H_1, H_2, H_3 and H_4. The presentations of these groups H_1, H_2, H_3 and H_4 can be written as:

$$H_1 = \langle f_1, f_2, f_3, f_4 \mid f_1^3 f_4^{-1}, [f_2, f_1] f_3^{-1}, [f_3, f_1], [f_4, f_1], f_2^3, [f_3, f_2], [f_4, f_2], f_3^3, [f_4, f_3], f_4^3 \rangle,$$
$$H_2 = \langle f_1, f_2, f_3, f_4 \mid f_1^3, [f_2, f_1] f_3^{-1}, [f_3, f_1] f_4^{-1}, [f_4, f_1], f_2^3 f_4^{-1}, [f_3, f_2], [f_4, f_2], f_3^3, [f_4, f_3], f_4^3 \rangle,$$
$$H_3 = \langle f_1, f_2, f_3, f_4 \mid f_1^3, [f_2, f_1] f_3^{-1}, [f_3, f_1] f_4^{-1}, [f_4, f_1], f_2^3 f_4^{-2}, [f_3, f_2], [f_4, f_2], f_3^3, [f_4, f_3], f_4^3 \rangle,$$
$$H_4 = \langle f_1, f_2, f_3, f_4 \mid f_1^3, [f_2, f_1] f_4^{-1}, [f_3, f_1], [f_4, f_1], f_2^3, [f_3, f_2], [f_4, f_2], f_3^3 f_4^{-1}, [f_4, f_3], f_4^3 \rangle.$$

Finally, we discuss the group actions corresponding to the groups $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_6, \mathcal{G}_7$ and $\mathcal{G}_{10}$.

• We consider $\sigma_3 \in \mathrm{Aut}(H_1)$ which maps f_1, f_2, f_3, f_4 to $f_1^2 f_4^2, f_2 f_3, f_3^2, f_4$, respectively. Also σ_3 is an element of order 2 in $\mathrm{Aut}(H_1)$. Therefore, the group homomorphism $C_2 \mapsto H_1$ generates $\mathcal{G}_2$ via σ_3.

• We consider $\sigma_4 \in \mathrm{Aut}(H_2)$ which maps f_1, f_2, f_3, f_4 to $f_1^2 f_2^2 f_3^2, f_2, f_3^2 f_4, f_4$, respectively. Moreover, we consider $\sigma_5 \in \mathrm{Aut}(H_2)$ which maps f_1, f_2, f_3, f_4 to $f_1^2 f_3^2 f_4^2, f_2^2$, f_3, f_4^2, respectively. We note that the order of elements $\sigma_4, \sigma_5 \in \mathrm{Aut}(H_2)$ is 2. Hence, the group homomorphism $C_2 \mapsto H_2$ generates $\mathcal{G}_3$ via σ_4 and the group homomorphism $C_2 \mapsto H_2$ generates $\mathcal{G}_4$ via σ_5.

• We consider $\sigma_6 \in \mathrm{Aut}(H_3)$ which maps f_1, f_2, f_3, f_4 to f_1, f_2, f_3, f_4 to $f_1^2, f_2^2 f_4^2$, $f_3 f_4^2$, f_4^2, respectively. Moreover, we consider $\sigma_7 \in \mathrm{Aut}(H_2)$ which maps f_1, f_2, f_3, f_4 to f_1^2, $f_2^2 f_4^2$, $f_3 f_4^2$, f_4^2, respectively. Also, the order of elements $\sigma_6, \sigma_7 \in \mathrm{Aut}(H_3)$ is 2. Therefore, the group homomorphism $C_2 \mapsto H_3$ generates $\mathcal{G}_6$ via σ_6 and the group homomorphism $C_2 \mapsto H_3$ generates $\mathcal{G}_7$ via σ_7.

• By considering $\sigma_8 \in \mathrm{Aut}(H_4)$ which maps f_1, f_2, f_3, f_4 to $f_1^2, f_2^2 f_4^2, f_3, f_4$, respectively, we note that the group homomorphism $C_2 \mapsto H_4$ generates $\mathcal{G}_{10}$ via σ_8.

Therefore, due to the different group actions mentioned above, we observe that the groups $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_6, \mathcal{G}_7, \mathcal{G}_{10}$ are non-isomorphic to each other.

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