MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 09.09.2024

research paper
оригинални научни рад
DOI: 10.57016/MV-HGLA1043

# ON $gr$-$C$-$2^A$-SECONDARY SUBMODULES

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**Abstract**. Let $\Omega$ be a group with identity $e$, $\Gamma$ be a $\Omega$-graded commutative ring and $\Im$ a graded $\Gamma$-module. In this article, we introduce the concept of $gr$-$C$-$2^A$-secondary submodules and investigate some properties of this new class of graded submodules. A non-zero graded submodule $S$ of $\Im$ is said to be a $gr$-$C$-$2^A$-secondary submodule if whenever $r, s \in h(\Gamma)$, $L$ is a graded submodule of $\Im$, and $rs\,S \subseteq L$, then either $r\,S \subseteq L$ or $s\,S \subseteq L$ or $rs \in Gr(Ann_\Gamma(S))$.

## 1. Introduction

In this article we assume that $\Gamma$ is a commutative $\Omega$-graded ring with identity and $\Im$ is a unitary graded $\Gamma$-module.

Let $\Omega$ be a group with identity $e$ and $\Gamma$ a commutative ring with identity $1_\Gamma$. Then $\Gamma$ is an $\Omega$-*graded ring* if there exist additive subgroups $\Gamma_g$ of $\Gamma$ such that $\Gamma = \bigoplus_{g\in\Omega} \Gamma_g$ and $\Gamma_g\Gamma_h \subseteq \Gamma_{gh}$ for all $g, h \in \Omega$. Furthermore, $h(\Gamma) = \bigcup_{g\in\Omega} \Gamma_g$, (see [13]).

A left $\Gamma$-module $\Im$ is called $\Omega$-*graded* $\Gamma$-*module* if there exists a family of additive subgroups $\{\Im_\alpha\}_{\alpha\in\Omega}$ of $\Im$ such that $\Im = \bigoplus_{\alpha\in\Omega} \Im_\alpha$ and $\Gamma_\alpha\Im_\beta \subseteq \Im_{\alpha\beta}$ for all $\alpha, \beta \in \Omega$. Even if an element of $\Im$ belongs to $\cup_{\alpha\in\Omega}\Im_\alpha = h(\Im)$, it is called homogeneous. We refer to [9, 11–13] for basic properties and more information about graded rings and graded modules. By $L \leq_\Omega \Im$ we mean that $L$ is a $\Omega$-graded submodule of $\Im$.

Let $\Gamma$ be a $\Omega$-graded ring, $\Im$ a graded $\Gamma$-module and $S$ a graded submodule of $\Im$. Then $(S :_\Gamma \Im)$ is defined as $(S :_\Gamma \Im) = \{a \in \Gamma | a\Im \subseteq S\}$. The annihilator of $\Im$ is defined as $(0 :_\Gamma \Im)$ and is denoted by $Ann_\Gamma(\Im)$. Let $\Gamma$ be an $\Omega$-graded ring. The *graded radical* of a graded ideal $L$, denoted by $Gr(L)$, is the set of all $t = \sum_{\alpha\in\Omega} t_\alpha \in \Gamma$, so that for every $\alpha \in \Omega$ there exists $n_\alpha > 0$ with $t_\alpha^{n_\alpha} \in L$, (see [15]). A proper graded submodule $S$ of $\Im$ is called *a completely graded irreducible* if $S = \cap_{\alpha\in\Delta} S_\alpha$, where $\{S_\alpha\}_{\alpha\in\Delta}$ is a family of graded submodules of $\Im$, then $S = S_\beta$ for some $\beta \in \Delta$.

The study of graded rings and modules has long attracted the attention of many researchers, as they have important applications in many fields such as geometry and

*2020 Mathematics Subject Classification*: 13A02, 16W50
*Keywords and phrases*: Graded classical 2-absorbing secondary submodules; graded 2-absorbing submodules; graded 2-absorbing primary submodules.

physics. For example, graded Lie algebra plays an important role in differential geometry, such as the Frolicher-Nijenhuis and Nijenhuis-Richardson brackets (see [10]). In addition, they solve many physical problems related to supermanifolds, supersymmetries and quantizations of systems with symmetry (see [8, 17]).

The notion of graded 2-absorbing ideals was introduced and studied in [1]. Al-Zoubi and Abu-Dawwas in [3] extended graded 2-absorbing ideals to graded 2-absorbing submodules. In [2], the authors introduced the concept of the graded 2-absorbing primary ideal, which is a generalization of the graded primary ideal. The notion of graded 2-absorbing primary submodules as a generalization of graded 2-absorbing primary ideals was introduced and studied in [7]. In [4, 16], the authors introduced the dual notion of graded 2-absorbing submodules (i.e. graded 2-absorbing (resp., graded strongly 2-absorbing) second submodules) of $\Im$ and investigated some properties of these classes of graded modules. In this paper, we introduce the concept of graded classical 2-absorbing secondary submodules as a dual notion of graded 2-absorbing primary submodules. We investigate the basic properties and characteristics of graded classical 2-absorbing secondary submodules.

## 2. Results

Definition 2.1. Let $\Gamma$ be a $\Omega$-graded ring and $\Im$ a graded $\Gamma$-module. A non-zero graded submodule $S$ of $\Im$ is said to be graded classical 2-absorbing secondary (Abbreviated, $gr$-$C$-$2^A$-secondary) submodule of $\Im$ if whenever $r, s \in h(\Gamma)$, $L \leq_\Omega \Im$, and $rs\, S \subseteq L$, then $r\, S \subseteq L$ or $s\, S \subseteq L$ or $rs \in Gr(Ann_\Gamma(S))$.

We say that $\Im$ is a $gr$-$C$-$2^A$-secondary module if $\Im$ is a $gr$-$C$-$2^A$-secondary submodule of itself.

Theorem 2.2. *Let $S$ be a $gr$-$C$-$2^A$-secondary submodule of $\Im$, let $I = \bigoplus_{\alpha\in\Omega} I_\alpha$ and $J = \bigoplus_{\alpha\in\Omega} J_\alpha$ be a graded ideals of $\Gamma$. Then for every $\alpha, \beta \in \Omega$ and $L \leq_\Omega \Im$, with $I_\alpha J_\beta S \subseteq L$ either $I_\alpha S \subseteq L$ or $J_\beta S \subseteq L$ or $I_\alpha J_\beta \subseteq Gr(Ann_\Gamma(S))$.*

*Proof.* Let $\alpha, \beta \in \Omega$ such that $I_\alpha J_\beta S \subseteq L$ for some $L \leq_\Omega \Im$. Assume that $I_\alpha J_\beta \nsubseteq Gr(Ann_\Gamma(S)$. Then there exist $r_\alpha \in I_\alpha$ and $s_\beta \in J_\beta$ such that $r_\alpha s_\beta \notin Gr(Ann_\Gamma(S)$. Now since $r_\alpha s_\beta\, S \subseteq L$, we get $r_\alpha\, S \subseteq L$ or $s_\beta\, S \subseteq L$. We show that either $I_\alpha S \subseteq L$ or $J_\beta S \subseteq L$. On contrary, we suppose that $I_\alpha S \nsubseteq L$ and $J_\beta S \nsubseteq L$. Then there exist $r'_\alpha \in I_\alpha$ and $s'_\beta \in J_\beta$ such that $r'_\alpha\, S \nsubseteq L$ and $s'_\beta\, S \nsubseteq L$. Since $r'_\alpha s'_\beta\, S \subseteq L$ and $S$ be a $gr$-$C$-$2^A$-secondary submodule of $\Im$, $r'_\alpha s'_\beta \in Gr(Ann_\Gamma(S)$. We have three cases:
Case I: Suppose that $r_\alpha\, S \subseteq L$ but $s_\beta\, S \nsubseteq L$. Since $r'_\alpha s_\beta\, S \subseteq L$ and $s_\beta\, S \nsubseteq L$ and $r'_\alpha\, S \nsubseteq L$, this implies $r'_\alpha s_\beta \in Gr(Ann_\Gamma(S))$. Since $r_\alpha\, S \subseteq L$ and $r'_\alpha\, S \nsubseteq L$, we get $(r_\alpha + r'_\alpha)\, S \nsubseteq L$. As $(r_\alpha + r'_\alpha)s_\beta\, S \subseteq L$ and $s_\beta\, S \nsubseteq L$, then $(r_\alpha + r'_\alpha)\, S \nsubseteq L$ implies $(r_\alpha + r'_\alpha)s_\beta \in Gr(Ann_\Gamma(S)$. Since $r'_\alpha s_\beta \in Gr(Ann_\Gamma(S))$, we get $r_\alpha s_\beta \in Gr(Ann_\Gamma(S))$, a contradiction.

Case II: Suppose $s_\beta\, S \subseteq L$ but $r_\alpha\, S \nsubseteq L$. Then similar to the Case I, we get a contradiction.

Case III: Suppose $r_\alpha S \subseteq L$ and $s_\beta S \subseteq L$. Now $s_\beta S \subseteq L$ and $s'_\beta S \nsubseteq L$ imply $(s_\beta + s'_\beta) S \nsubseteq L$. Since $r'_\alpha(s_\beta + s'_\beta) S \subseteq L$ and $(s_\beta + s'_\beta) S \nsubseteq L$ and $r'_\alpha S \nsubseteq L$, we get $r'_\alpha(s_\beta + s'_\beta) \in Gr(Ann_\Gamma(S))$. Now as $r'_\alpha s'_\beta \in Gr(Ann_\Gamma(S))$, we get $r'_\alpha s_\beta \in Gr(Ann_\Gamma(S))$. Again $r_\alpha S \subseteq L$ and $r'_\alpha S \nsubseteq L$ imply $(r_\alpha + r'_\alpha) S \nsubseteq L$. Since $(r_\alpha + r'_\alpha)s'_\beta S \subseteq L$ and $(r_\alpha + r'_\alpha) S \nsubseteq L$ and $s'_\beta S \nsubseteq L$, we have $(r_\alpha + r'_\alpha)s'_\beta \in Gr(Ann_\Gamma(S))$. Since $r'_\alpha s'_\beta \in Gr(Ann_\Gamma(S))$, we get $r_\alpha s'_\beta \in Gr(Ann_\Gamma(S))$. Since $(r_\alpha + r'_\alpha)(s_\beta + s'_\beta) S \subseteq L$ and $(r_\alpha + r'_\alpha) S \nsubseteq L$ and $(s_\beta + s'_\beta) S \nsubseteq L$, we get $(r_\alpha + r'_\alpha)(s_\beta + s'_\beta) \in Gr(Ann_\Gamma(S))$. Since $r_\alpha s'_\beta, r'_\alpha s_\beta, r'_\alpha s'_\beta \in Gr(Ann_\Gamma(S))$, we have $r_\alpha s_\beta \in Gr(Ann_\Gamma(S))$, a contradiction. Thus $I_\alpha S \subseteq L$ or $J_\beta S \subseteq L$. □

Theorem 2.3. *Let $S$ be a gr-C-$2^A$-secondary submodule of $\Im$, then for each $a, b \in h(\Gamma)$ we have $abS = aS$ or $abS = bS$ or $ab \in Gr(Ann_\Gamma(S))$.*

*Proof.* Let $a, b \in h(\Gamma)$, then $abS \subseteq abS$ implies that $aS \subseteq abS$ or $aS \subseteq abS$ or $ab \in Gr(Ann_\Gamma(S))$. Clearly, $abS \subseteq aS$ and $abS \subseteq bS$, so we have $abS = aS$ or $abS = bS$ or $ab \in Gr(Ann_\Gamma(S))$. □

Let $U$ and $P$ be two graded submodules of a graded $\Gamma$-module. To prove that $U \subseteq P$, it suffices to show that if $V$ is a completely graded irreducible submodule of $\Im$ such that $P \subseteq V$, then $U \subseteq V$ (see [4]). A proper graded ideal $L$ of $\Gamma$ is called a graded 2-absorbing primary (abbreviated, *gr*-$2^A$-primary) ideal if whenever $a, b, c \in h(\Gamma)$ with $abc \in L$, then $ab \in L$ or $ac \in Gr(L)$ or $bc \in Gr(L)$.

Theorem 2.4. *Let $S$ be a gr-C-$2^A$-secondary submodule of a graded $\Gamma$-module $\Im$. Then $Ann_\Gamma(S)$ is a gr-$2^A$-primary ideal of $\Gamma$.*

*Proof.* Let $r, s, t \in h(\Gamma)$ wit $rst \in Ann_\Gamma(S)$. Assume that $rs \notin Ann_\Gamma(S)$ and $rt \notin Gr(Ann_\Gamma(S))$. We show that $st \in Gr(Ann_\Gamma(S))$. There exist completely irreducible submodule $J_1$ and $J_2$ of $\Im$ such that $rs\,S \nsubseteq J_1$ and $rt\,S \nsubseteq J_2$. Since $rst\,S = 0 \subseteq J_1 \cap J_2$, $st\,S \subseteq (J_1 \cap J_2 :_\Im r)$. Since $S$ is *gr*-*C*-$2^A$-secondary submodule of $\Im$, we have $rs\,S \subseteq J_1 \cap J_2$ or $rt\,S \subseteq J_1 \cap J_2$ or $st \in Gr(Ann_\Gamma(S))$. If $rs\,S \subseteq J_1 \cap J_2$ or $rt\,S \subseteq J_1 \cap J_2$, then $rs\,S \subseteq J_1$ or $rt\,S \subseteq J_2$ which are contradictions. Therefore $st \in Gr(Ann_\Gamma(S))$. □

A proper graded ideal $L$ of $\Gamma$ is a graded 2-absorbing (abbreviated, *gr*-$2^A$) ideal of $\Gamma$ if whenever $a, b, c \in h(\Gamma)$ with $abc \in L$, then $ab \in L$ or $ac \in L$ or $bc \in L$ (see [1]).

Corollary 2.5. *Let $S$ be a gr-C-$2^A$-secondary submodule of a graded $\Gamma$-module $\Im$. Then $Gr(Ann_\Gamma(S))$ is a gr-$2^A$ ideal of $\Gamma$.*

*Proof.* By Theorem 2.4, $Ann_\Gamma(S)$ is *gr*-$2^A$-primary ideal of $\Gamma$. So by [2, Theorem 2.3], $Gr(Ann_\Gamma(S))$ is *gr*-$2^A$ ideal of $\Gamma$. □

The following example shows that the converse of Theorem 2.4 is not true in general.

Example 2.6. Let $\Gamma = \mathbb{Z}$ and $\Omega = \mathbb{Z}_2$, then $\Gamma$ is a $\Omega$-graded ring with $\Gamma_0 = \mathbb{Z}$ and $\Gamma_1 = \{0\}$. Consider $\Im = \mathbb{Z}_{pq} \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module, where $p, q$ are two prime integers, $\Im$

is a $\Omega$-graded module with $\Im_0 = \mathbb{Z}_{pq} \oplus \{0\}$ and $\Im_1 = \{\bar{0}\} \oplus \mathbb{Q}$. Then $Ann_\Gamma(\Im) = \{0\}$ is a $gr$-$2^A$-primary ideal of $\mathbb{Z}$. But $\Im$ is not $gr$-$C$-$2^A$-secondary $\mathbb{Z}$-module, since $pq\Im \subseteq \{\bar{0}\} \oplus \mathbb{Q}$, but $pM = p\mathbb{Z}_{pq} \oplus \mathbb{Q} \nsubseteq \{\bar{0}\} \oplus \mathbb{Q}$ and $q\Im = q\mathbb{Z}_{pq} \oplus \mathbb{Q} \nsubseteq \{\bar{0}\} \oplus \mathbb{Q}$ and $pq \notin Gr(Ann_\Gamma(\Im))$.

A graded domain $\Gamma$ is called a $gr$-Dedekind ring if every graded ideal of $\Gamma$ factorises into a product of graded prime ideals (see [19]).

A graded $\Gamma$-module $\Im$ is called a $gr$-comultiplication module if for every graded submodule $S$ of $\Im$ there exists a graded ideal $P$ of $\Gamma$ such that $S = (0 :_\Im P)$, or, equivalently, for each graded submodule $S$ of $\Im$, we have $S = (0 :_\Im Ann_\Gamma(S))$ (see [5]).

The $gr$-$C$-$2^A$-secondary submodules of a $gr$-comultiplication module over a $gr$-Dedekind domain are described in the following theorem.

THEOREM 2.7. *Let $\Gamma$ be a gr-Dedekind domain, and $\Im$ be a gr-comultiplication $\Gamma$-module, if $S$ is gr-C-$2^A$-secondary submodule of $\Im$, then $S = (0 :_\Im Ann^n_\Gamma(L))$ or $S = (0 :_\Im Ann^n_\Gamma(L_1)Ann^m_\Gamma(L_2))$, where $L, L_1, L_2$ are graded minimal submodules of $\Im$ and $n, m$ are positive integers.*

*Proof.* By Theorem 2.4, since $S$ is $gr$-$C$-$2^A$-secondary submodule of $\Im$, then $Ann_\Gamma(S)$ is a $gr$-$2^A$-primary ideal of $\Gamma$. Using [18, Theorem 4.1] and [19, Lemma 1.1], we have either $Ann_\Gamma(S) = I^n$ or $Ann_\Gamma(S) = I_1^n I_2^m$, where $I, I_1, I_2$ are graded maximal ideals of $\Gamma$. First assume $Ann_\Gamma(S) = I^n$. If $(0 :_\Im I) = 0$, then $(0 :_\Im I^n) = 0$, and so we conclude that $S = 0$, a contradiction. Now by [5, Theorem 3.9], since $I$ is graded maximal ideal of $\Gamma$, we have $(0 :_\Im I)$ is graded minimal submodule of $\Im$. This implies that $S = (0 :_\Im Ann^n_\Gamma(L))$, where $L = (0 :_\Im I)$. Now assume that $Ann_\Gamma(S) = I_1^n I_2^m$. If $(0 :_\Im I_1) = 0$ and $(0 :_\Im I_2) = 0$, then $S = 0$, a contradiction. Thus either $(0 :_\Im I_1) \neq 0$ or $(0 :_\Im I_2) \neq 0$. Hence one can see that either $S = (0 :_\Im Ann^n_\Gamma(L_1)Ann^m_\Gamma(L_2))$ or $S = (0 :_\Im Ann^n_\Gamma(L_1))$ or $S = (0 :_\Im Ann^m_\Gamma(L_2))$, where $L_1 = (0 :_\Im I_1)$ and $L_2 = (0 :_\Im I_2)$ are graded minimal submodules of $\Im$. □

For a graded $\Gamma$-submodule $S$ of $\Im$, the graded second radical of $S$ is defined as the sum of all $gr$-second $\Gamma$-submodules of $\Im$ contained in $S$, and is denoted by $GSec(S)$. If $S$ does not contain any $gr$-second $\Gamma$-submodule, then $GSec(S) = \{0\}$. The graded second spectrum of $\Im$ is the collection of all $gr$-second $\Gamma$ submodules and is represented by the symbol $GSpec^s(\Im)$. The set of all $gr$-prime $\Gamma$-submodules of $\Im$ is called the graded spectrum of $\Im$ and is denoted by $GSpec(\Im)$. The mapping $\psi : GSpec^s(\Im) \to GSpec(\Gamma/Ann_\Gamma(\Im))$ is defined by $\psi(S) = Ann_\Gamma(S)/Ann_\Gamma(\Im)$ is called the natural mapping of $GSpec^s(\Im)$, see [16]. A graded submodule $S$ of $\Im$ is called a *graded strongly 2-absorbing second* (abbreviated, $gr$-$S$-$2^A$-second) submodule of $\Im$ if whenever $a$, $b \in h(\Gamma)$, $S_1, S_2$ are completely graded irreducible submodules of $\Im$, and $abS \subseteq S_1 \cap S_2$, then $aS \subseteq S_1 \cap S_2$ or $bS \subseteq S_1 \cap S_2$ or $ab \in Ann_\Gamma(S)$, see [4].

It is clear that every $gr$-$S$-$2^A$-second submodule is a $gr$-$C$-$2^A$-secondary submodule of $\Im$, but the converse is generally not true. This is illustrated by the following examples.

EXAMPLE 2.8. Let $\Omega = \mathbb{Z}_2$ and $\Gamma = \mathbb{Z}$ be a $\Omega$-graded ring with $\Gamma_0 = \mathbb{Z}$ and $\Gamma_1 = \{0\}$. Let $\Im = \mathbb{Z}_{p^\infty} = \{\frac{a}{p^n} + \mathbb{Z} : a, n \in \mathbb{Z},\ n \geqslant 0\}$ be a graded $\Gamma$-module with $\Im_0 = \mathbb{Z}_{p^\infty}$

and $\Im_1 = \{0_{\mathbb{Z}_{p^\infty}}\} = \{\mathbb{Z}\}$, where $p$ is a fixed prime number. Consider the graded submodule $N = \langle \frac{1}{p^3} + \mathbb{Z} \rangle$ of $\Im$. Then $N$ is $gr$-$C$-$2^A$-secondary submodule which is not a $gr$-$S$-$2^A$-second submodule.

THEOREM 2.9. *Let $\Im$ be a gr-comultiplication $\Gamma$-module, and the natural map $\psi$ of $GSpec^s(S)$ is surjective, if $S$ is a gr-$C$-$2^A$-secondary submodule of $\Im$, then $GSec(S)$ is a gr-$S$-$2^A$-second submodule of $\Im$.*

*Proof.* Let $S$ be a $gr$-$C$-$2^A$-secondary submodule of $\Im$. By Corollary 2.5, $Gr(Ann_\Gamma(S))$ is $gr$-$2^A$ ideal of $\Gamma$. By [16, Lemma 4.7], $Gr(Ann_\Gamma(S)) = Ann_\Gamma(GSec(S))$. Therefore, $Ann_\Gamma(GSec(S))$ is $gr$-$2^A$ ideal of $\Gamma$. Using [16, Proposition 3.7], $GSec(S)$ is $gr$-$S$-$2^A$-second $\Gamma$-submodule of $\Im$. □

Let $\Gamma$ be a $\Omega$-graded ring, a graded $\Gamma$-module $\Im$ is a $gr$-sum-irreducible if $\Im \neq 0$ and the sum of any two proper graded submodule of $\Im$ is always a proper graded submodule (see [6]).

THEOREM 2.10. *Let $S$ be a gr-$C$-$2^A$-secondary submodule of $\Im$. Then $r\,S = r^2\,S, \forall r \in h(\Gamma)\backslash Gr(Ann_\Gamma(S))$. The converse hold, if $S$ is a gr-sum-irreducible submodule of $\Im$.*

*Proof.* Let $r \in h(\Gamma)\backslash Gr(Ann_\Gamma(S))$. Then $r^2 \in h(\Gamma)\backslash Gr(Ann_\Gamma(S))$. Thus by Theorem 2.3, we have $r\,S = r^2\,S$. Conversely, let $S$ be a $gr$-sum-irreducible submodule of $\Im$ and $rs\,S \subseteq L$, for some $r, s \in h(\Gamma)$ and $L \leq_\Omega \Im$. Suppose that $rs \notin Gr(Ann_\Gamma(S))$. We show that $r\,S \subseteq L$ or $s\,S \subseteq L$. Since $rs \notin Gr(Ann_\Gamma(S))$, we have $r, s \notin Gr(Ann_\Gamma(S))$. Thus $r\,S = r^2\,S$ by assumption. Let $x \in S$, then $rx \in r\,S = r^2\,S$. So $\exists y \in S$ such that $rx = r^2 y$. This implies that $x - ry \in (0 :_S r) \subseteq (L :_S r)$. Thus $x = x - ry + ry \in (L :_S r) + (L :_S s)$. Hence $S \subseteq (L :_S r) + (L :_S s)$. Clearly, $(L :_S r) + (L :_S s) \subseteq S$, as $S$ is $gr$-sum-irreducible submodule of $\Im$, $(L :_S r) = S$ or $(L :_S s) = S$, i.e $r\,S \subseteq L$ or $s\,S \subseteq L$, as needed. □

A graded $\Gamma$-module $\Im$ is called $gr$-multiplication, if for every graded submodule $S$ of $\Im$, there exists a graded ideal $K$ of $\Gamma$ such that $S = K\Im$ (see [14]).

THEOREM 2.11. *Let $S \leq_\Omega \Im$. Then we have the following.*
*(a) If $S$ is a gr-$C$-$2^A$-secondary submodule of $\Im$, then $IC$ is a gr-$C$-$2^A$-secondary submodule of $\Im$, for all graded ideal $I$ of $\Gamma$, with $I \nsubseteq Ann_\Gamma(S)$.*

*(b) If $\Im$ is a gr-multiplication gr-$C$-$2^A$-secondary module, then every non-zero graded submodule of $\Im$ is a gr-$C$-$2^A$-secondary submodule of $\Im$.*

*Proof.* (a) Let $I$ be a graded ideal of $\Gamma$, with $I \nsubseteq Ann_\Gamma(S)$. Then $IC$ is a non-zero graded submodule of $\Im$. Let $r, s \in h(\Gamma)$, $L$ is graded submodule of $\Im$, and $rs\,IC \subseteq L$, then $rs\,S \subseteq (L :_\Im I)$, thus $r\,IC \subseteq L$ or $s\,IC \subseteq L$ or $rs \in Gr(Ann_\Gamma(S)) \subseteq Gr(Ann_\Gamma(IC))$, as desired.

(b) This follows from part (a). □

THEOREM 2.12. *Let $\Gamma$ be $\Omega$-graded ring and $\Im$, $\Im'$ be two graded $\Gamma$-module. Let $\psi : \Im \to \Im'$ be a graded monomorphism.*

*(a) If $S$ is a $gr$-$C$-$2^A$-secondary submodule of $\Im$, then $\psi(S)$ is a $gr$-$C$-$2^A$-secondary submodule of $\Im'$.*

*(b) If $S'$ is a $gr$-$C$-$2^A$-secondary submodule of $\psi(\Im)$, then $\psi^{-1}(S')$ is a $gr$-$C$-$2^A$-secondary submodule of $\Im$.*

*Proof.* (a) As $S \neq 0$, and $\psi$ is a graded monomorphism, we have $\psi(S) \neq 0$, let $r, s \in h(\Gamma)$, $L' \leq_\Omega \Im'$, and $rs\,\psi(S) \subseteq L'$. Then $rs\,S \subseteq \psi^{-1}(L')$. Since $S$ is $gr$-$C$-$2^A$-secondary submodule of $\Im$, $r\,S \subseteq \psi^{-1}(L')$ or $s\,S \subseteq \psi^{-1}(L')$ or $rs \in Gr(Ann_\Gamma(S))$. Therefore, $r\,\psi(S) \subseteq \psi(\psi^{-1}(L')) = \psi(\Im) \cap L' \subseteq L'$ or $s\,\psi(S) \subseteq \psi(\psi^{-1}(L')) = \psi(\Im) \cap L' \subseteq L'$ or $rs \in Gr(Ann_\Gamma(\psi(S)))$, as desired.

(b) If $\psi^{-1}(S') = 0$, then $\psi(\Im) \cap S' = \psi\,\psi^{-1}(S') = \psi(0) = 0$. So $S' = 0$, which is a contradiction. Therefore $\psi^{-1}(S') \neq 0$. Let $r, s \in h(\Gamma)$, $L \leq_\Omega \Im$, and $rs\,\psi^{-1}(S') \subseteq L$. Then $rs\,S' = rs(\psi(\Im) \cap S') = rs\,\psi\,\psi^{-1}(S') \subseteq \psi(L)$. As $S'$ is $gr$-$C$-$2^A$-secondary submodule of $\psi(\Im)$, $r\,S' \subseteq \psi(L)$ or $s\,S' \subseteq \psi(L)$ or $rs \in Gr(Ann_\Gamma(S'))$. Thus $r\,\psi^{-1}(S') \subseteq \psi^{-1}\psi(L) = L$ or $s\,\psi^{-1}(S') \subseteq \psi^{-1}\psi(L) = L$ or $rs \in Gr(Ann_\Gamma(\psi^{-1}(S')))$, as needed. □

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(received 22.03.2023; in revised form 03.05.2024; available online 09.09.2024)

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