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MULTIPLE HOMOCLINIC SOLUTIONS FOR THE DISCRETE $p(X)$-LAPLACIAN PROBLEMS OF KIRCHHOFF TYPE

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Abstract. In this paper we consider the discrete anisotropic difference equation with variable exponent using critical point theory. The study of nonlinear difference equations has now attracted special attention as they have important applications in various research areas such as numerical analysis, computer science, mechanical engineering, cellular neural networks and population growth, cybernetics, etc. In many studies, the authors consider Dirichlet, Neumann or Robin type boundary conditions. However, in this paper, we consider a homoclinic boundary condition, which means that the value of the solution is equal to a constant at infinity. Here we assume that the value of the solution vanishes at infinity. In this paper, we are also interested in the existence of at least one non-trivial homoclinc solution. To achieve this, we apply firstly the direct variational method and secondly the well-known Mountain pass technique, known as the Mountain pass theorem of Ambrosetti and Rabinowitz, to obtain the existence of at least one non-trivial homoclinic solution.

1. Introduction

In this note, we consider the following anisotropic difference equation of Kirchhoff type with homoclinic condition at the boundary

$$\begin{cases} -M(I(u))\left[\Delta(a(k-1,\Delta u(k-1)))-r(k)\phi_{p(k)}(u(k))\right] = \lambda f(k,u(k)), & k\in\mathbb{Z} \\ u(k)\to 0, & |k|\to\infty, \end{cases} \tag{1}$$

where

$$I(u)=\sum_{k\in\mathbb{Z}} A(k-1,\Delta u(k-1))+\sum_{k\in\mathbb{Z}}\frac{r(k)}{p(k)}|u(k)|^{p(k)},$$

$\Delta u(k-1)=u(k)-u(k-1)$ is the forward difference operator; $u(k)\in\mathbb{R}$ for all $k\in\mathbb{Z}$. ϕ is an homomorphism defined by $\phi_{p(k)}(y)=|y|^{p(k)-2}y$, $M:(0,\infty)\to(0,\infty)$ is a non-decreasing continuous function and λ is a positive real number.

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In the last few years, the study of nonlinear difference equations has attracted special attention due to their important applications in various research areas such as numerical analysis, computer science, mechanical engineering, cellular neural networks and population growth, cybernetics, etc. For recent advances in discrete problems, we refer to [2, 3, 6, 20, 23, 24] and the references therein.

To our knowledge, there are only a few papers in the literature that deal with the weak homoclinic solutions of the discrete difference equation with the $p(k)$-Laplacian operator of Kirchhoff type. Recently, Guiro et al. [10] proved the existence of weak homoclinic solutions under competition phenomena between parameters for the following problem using the direct variational method

$$\begin{cases} -\Delta\left(a\left(k-1, \Delta u(k-1)\right)\right)+\alpha(k)|u(k)|^{p(k)-2}u(k)=\delta f(k,u(k)), & k\in\mathbb{Z} \\ \lim_{|k|\to\infty} u(k)=0. \end{cases} \tag{2}$$

In [17] Mihailescu et al. investigated the following problem

$$\begin{cases} \Delta\left(\varphi_{p(.)}\left(\Delta u(k-1)\right)\right)-V(k)|u(k)|^{q(k)-2}u(k)+f(k,u(k))=0, & k\in\mathbb{Z} \\ \lim_{|k|\to\infty} u(k)=0, \end{cases} \tag{3}$$

which is a special case of the problem (1). In their work, they rely on the theory of critical point theory in combination with suitable variational methods, mainly based on the Mountain pass lemma, to prove the existence of at least one non-trivial homoclinic solution.

Note that there are some works dealing with problems like (2) and (3) in the p-Laplacian case (see e.g. [11, 13, 14]).

In the literature, the Mountain pass theorem of Ambrosetti and Rabinowitz [5] was applied to find solutions for various Kirchhoff type equations. In [4], for example, Alves et al. studied the Kirchhoff equations using the variational method and the Mountain pass theorem under the following conditions. There exists a constant m_0 such that $m(t)\geq m_0$, $\forall t>0$, and

$$\int_0^t m(s)\mathrm{d}s \geq tm(t), \quad \forall t\geq 0. \tag{4}$$

Subsequently, using (4) and Polynomial growth condition, Afrouzi et al. in [1] and Chung in [7] proposed the solvability of degenerate Kirchhoff equations. A standard way to deal with problems on unbounded domains is to introduce coercive weight functions due to Omana and Willem [18]. In this paper, we follow the ideas developed in [8, 16, 21, 22] for the study of anisotropic PDEs under the above conditions to prove the existence of homoclinic solutions to the problem (1). More precisely, the classical minimization methods and the Mountain pass lemma are used.

We point out that in Theorem 3.2 and Theorem 4.1 an exact estimate of the parameter λ is given.

This paper is organized as follows. In Section 2, we provide some necessary background material. In Section 3 and Section 4, we give the results of the existence of homoclinic solutions of (1).

2. Mathematical background

In this section we provide some tools that are used throughout the paper. We introduce the spaces

$$l^{p(.)} := \left\{ u : \mathbb{Z} \longrightarrow \mathbb{R} : \quad \rho_{p(.)}(u) = \sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} < \infty \right\},$$

$$l^{r(.),p(.)} := \left\{ u : \mathbb{Z} \longrightarrow \mathbb{R} : \quad \rho_{r(.),p(.)}(u) = \sum_{k\in\mathbb{Z}} r(k)|u(k)|^{p(k)} < \infty \right\}$$

and

$$H^{1,p(.)}_{r(.)} := \left\{ u : \mathbb{Z} \longrightarrow \mathbb{R} : \rho_{r(.),1,p(.)} = \sum_{k\in\mathbb{Z}} r(k)|u(k)|^{p(k)} + \sum_{k\in\mathbb{Z}} |\Delta u(k)|^{p(k)} < \infty \right\},$$

endowed respectevly with the Luxembourg norm

$$\|u\|_{p(.)} := \inf \left\{ \nu > 0; \quad \sum_{k\in\mathbb{Z}} \left| \frac{u(k)}{\nu} \right|^{p(k)} \leq 1 \right\},$$

$$\|u\|_{r(.),p(.)} := \inf \left\{ \nu > 0; \quad \sum_{k\in\mathbb{Z}} r(k) \left| \frac{u(k)}{\nu} \right|^{p(k)} \leq 1 \right\}$$

and $\qquad \|u\|_{r(.),1,p(.)} := \|u\|_{r(.),p(.)} + \|\Delta u\|_{p(.)}$.

The data a respect the following conditions $a(k,.) : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous $\forall k \in \mathbb{Z}$ and there exists a mapping $A : \mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies

$$a(k,\xi) = \frac{\partial}{\partial \xi} A(k,\xi), \qquad A(k,0) = 0, \quad \forall k \in \mathbb{Z}. \tag{5}$$

We also assume that there exists a positive constant $C_1 > 1$ such that

$$|a(k,\xi)| \leq C_1 \left(j(k) + |\xi|^{p(k)-1} \right), \tag{6}$$

for all $\xi \in \mathbb{R}$, where $j(k) \in l^{p'(.)}$ with $\dfrac{1}{p(k)} + \dfrac{1}{p'(k)} = 1$, and

$$|\xi|^{p(k)} \leq a(k,\xi)\xi \leq p(k)A(k,\xi), \quad \forall \xi \in \mathbb{R}. \tag{7}$$

There exists a positive constant $C_2 > 1$ such that for almost every $k \in \mathbb{Z}$ and every $\xi, \eta \in \mathbb{R}$ with $\xi \neq \eta$,

$$(a(k,\xi) - a(k,\eta)).(\xi - \eta) \geq \begin{cases} C_2|\xi - \eta|^{p(k)} & \text{if} \quad |\eta - \xi| \geq 1 \\ C_2|\xi - \eta|^{p^-} & \text{if} \quad |\eta - \xi| < 1. \end{cases} \tag{8}$$

For the function $M : (0,\infty) \longrightarrow (0,\infty)$, we suppose that it's continuous, non-decreasing and there exist positive numbers R_1, R_2 with $R_1 \leq R_2$ and $\alpha > 1$ such that

$$R_1 t^{\alpha-1} \leq M(t) \leq R_2 t^{\alpha-1} \quad \text{for} \quad t > t^* > 0; \tag{9}$$

and there also exists a constant $M_0 > 0$ such that

$$M(t) \geq M_0, \quad \forall t \geq 0. \tag{10}$$

Suppose that there exists a function $r : \mathbb{Z} \longrightarrow \mathbb{R}$ such that

$$r(k) \geq r_0 > 0 \text{ for all } k \in \mathbb{Z} \text{ and } r(k) \longrightarrow \infty \text{ as } |k| \to \infty. \tag{11}$$

Moreover, the continuous functions f, $F : \mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R}$, with F defined by

$$F(k,\xi) = \int_0^{\xi} f(k,s)\mathrm{d}s, \; k \in \mathbb{Z}, \; \xi \in \mathbb{R}, \tag{12}$$

are such that

$$\lim_{s\to 0} \frac{|f(k,s)|}{|s|^{p(k)-1}} = 0, \text{uniformly for all } k \in \mathbb{Z}. \tag{13}$$

Example 2.1. As example of functions which satisfies assumptions (5)-(9), we can give the following.

- $A(k,\xi) = \dfrac{1}{p(k)}\left(\left(1+|\xi|^2\right)^{\frac{p(k)}{2}} - 1\right)$, where $a(k,\xi) = \left(1+|\xi|^2\right)^{\frac{p(k)-2}{2}}\xi$, $k\in\mathbb{Z}$, $\xi\in\mathbb{R}$.
- $A(k,\xi) = \dfrac{1}{p(k)}|\xi|^{p(k)}$, where $a(k,\xi) = |\xi|^{p(k)-2}\xi$; $\xi \in \mathbb{R}$; $k \in \mathbb{Z}$ with $p(k) \geq 2$.
- $M(t) = at^{\alpha-1} + b$ with a and b two positive constants.

In this paper we assume that the function $p(.) : \mathbb{Z} \longrightarrow [2,\infty)$. One denotes by $p^- = \inf_{k\in\mathbb{Z}} p(k)$ and $p^+ = \sup_{k\in\mathbb{Z}} p(k)$.

Remark 2.2 ([9,10]). Let $u \in H^{1,p(.)}_{r(.)}$, then $\lim_{|k|\to\infty} u(k) = 0$.

Proposition 2.3 ([9,17]). *If $u \in l^{p(.)}$ and $p^+ < \infty$ then the following properties hold.*

1. $\|u\|_{p(.)} > 1 \;\Rightarrow \|u\|^{p^-}_{p(.)} \leq \rho_{p(.)}(u) \leq \|u\|^{p^+}_{p(.)}$;
2. $\|u\|_{p(.)} < 1 \;\Rightarrow \|u\|^{p^+}_{p(.)} \leq \rho_{p(.)}(u) \leq \|u\|^{p^-}_{p(.)}$;
3. $\|u_n\|_{p(.)} \to 0 \Leftrightarrow \rho_{p(.)}(u_n) \to 0$ *as* $n \to \infty$.

Proposition 2.4 ([10]). *If $u\in H^{1,p(.)}_{r(.)}$ and $p^+<\infty$ then the following properties hold.*

1. $\|u\|_{r(.),1,p(.)} > 1 \;\Rightarrow \|u\|^{p^-}_{r(.),1,p(.)} \leq \rho_{r(.),1,p(.)}(u) \leq \|u\|^{p^+}_{r(.),1,p(.)}$;
2. $\|u\|_{r(.),1,p(.)} < 1 \;\Rightarrow \|u\|^{p^+}_{r(.),1,p(.)} \leq \rho_{r(.),1,p(.)}(u) \leq \|u\|^{p^-}_{r(.),1,p(.)}$;
3. $\|u_n\|_{r(.),1,p(.)} \to 0 \Leftrightarrow \rho_{r(.),1,p(.)}(u_n) \to 0$ *as* $n \to \infty$.

Theorem 2.5 ([9]). *Let $u \in l^{p(.)}$ and $v \in l^{p'(.)}$ with $\dfrac{1}{p(k)} + \dfrac{1}{p'(k)} = 1$, $\forall k \in \mathbb{Z}$. Then*

$$\sum_{k\in\mathbb{Z}} |u(k)||v(k)| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{p(.)}\|v\|_{p'(.)}.$$

Let us recall some results on critical point theory.

Definition 2.6 ([19]). Let X be a reflexive Banach space. We say that a functional $I : X \longrightarrow \mathbb{R}$ satisfy the Palais-Smale (PS) condition if every sequence $\{u_n\}$ such that $\{I(u_n)\}$ is bounded and $I'(u_n) \longrightarrow 0$, has a convergent subsequence.

Theorem 2.7 ([15]). *Let X be a reflexive Banach space. If a functional $I \in C^1(X, \mathbb{R})$ is weakly lower semi-continous and coercive, then there exists $u_0 \in X$ such that $I(u_0) = \inf_{u \in X} I(u)$ and u_0 is also a critical point of I, i.e. $I'(u_0) = 0$. Moreover, if I is strictly convex, then the critical point is unique.*

Theorem 2.8 ([15]). *Let I satisfy (PS) condition. Suppose that*

1. *$I(0) = 0$;*
2. *there exist $\rho > 0$ and $\alpha > 0$ such that $I(u) \geq \alpha$ for all $u \in X$, with $\|u\| = \rho$;*
3. *there exists $u_1 \in X$ with $\|u_1\| \geq \rho$ such that $I(u_1) < \alpha$.*

Then I has a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u), \text{ where } \Gamma = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = u_1\}.$$

3. Existence of solutions by direct variational method

Definition 3.1. A weak homoclinic solution for problem (1) is a function $u \in H^{1,p(.)}_{r(.)}$ such that

$$M\left(I(u)\right)\left[\sum_{k \in \mathbb{Z}} a(k-1, \Delta u(k-1))\Delta v(k-1) + \sum_{k \in \mathbb{Z}} r(k)|u(k)|^{p(k)-2}u(k)v(k)\right] = \lambda \sum_{k \in \mathbb{Z}} f(k, u(k))v(k); \quad \text{for any } v \in H^{1,p(.)}_{r(.)}.$$

The main result of this paper is given by the following theorem.

Theorem 3.2. *Assume that condition* (9)-(13) *are fulfilled. Then, problem* (1) *has at least one weak homoclinic solution for all $\lambda > 0$ with $\alpha p^- > p^+$.*

For each $\lambda > 0$, the functional corresponding to problem (1) is defined as $J_\lambda : H^{1,p(.)}_{r(.)} \longrightarrow \mathbb{R}$,

$$J_\lambda(u) = \widehat{M}\left(\sum_{k \in \mathbb{Z}} A(k-1, \Delta u(k-1)) + \sum_{k \in \mathbb{Z}} \frac{r(k)}{p(k)}|u(k)|^{p(k)}\right) - \lambda \sum_{k \in \mathbb{Z}} F(k, u(k),$$

where $\widehat{M}(\xi) = \int_0^\xi M(s)\mathrm{d}s$ and $F(\xi) = \int_0^\xi f(s)\mathrm{d}s$ with $\xi \in \mathbb{R}$.

We begin the proof of Theorem 3.2 with some basic properties on functional J_λ.

LEMMA 3.3. *The functional J_λ is well defined on $H^{1,p(.)}_{r(.)}$ and is of class $C^1(H^{1,p(.)}_{r(.)},\mathbb{R})$ with the derivative given by*

$$\begin{aligned}\langle J_\lambda'(u),v\rangle &= M\left(I(u)\right)\left[\sum_{k\in\mathbb{Z}} a(k-1,\Delta u(k-1))\Delta v(k-1)+\sum_{k\in\mathbb{Z}} r(k)|u(k)|^{p(k)-2}u(k)v(k)\right]\\ &\quad -\lambda\sum_{k\in\mathbb{Z}} f(k,u(k))v(k),\end{aligned}$$

for all $u,v\in H^{1,p(.)}_{r(.)}$.

Proof. For any $u\in H^{1,p(.)}_{r(.)}$ and $\lambda>0$, let

$$\Phi(u)=\widehat{M}(I(u))=\widehat{M}\left(\sum_{k\in\mathbb{Z}} A(k-1,\Delta u(k-1))+\sum_{k\in\mathbb{Z}}\frac{r(k)}{p(k)}|u(k)|^{p(k)}\right)$$

and $$\Psi(u)=\sum_{k\in\mathbb{Z}} F(k,u(k)).$$

Then, $J_\lambda(u):=\Phi(u)-\lambda\Psi(u)$. From (5)-(6), we deduce that

$$\begin{aligned}|I(u)| &\le \left|\sum_{k\in\mathbb{Z}} A(k-1,\Delta u(k-1))+\sum_{k\in\mathbb{Z}}\frac{r(k)}{p(k)}|u(k)|^{p(k)}\right|\\
&\le \sum_{k\in\mathbb{Z}}\int_0^{\Delta u(k-1)}|a(k-1,s)|\mathrm{d}s+\frac{1}{p^-}\sum_{k\in\mathbb{Z}} r(k)|u(k)|^{p(k)}\\
&\le \sum_{k\in\mathbb{Z}}\int_0^{\Delta u(k-1)} C_1\left(j(k-1)+|s|^{p(k-1)-1}\right)\mathrm{d}s+\frac{1}{p^-}\sum_{k\in\mathbb{Z}} r(k)|u(k)|^{p(k)}\\
&\le C_1\sum_{k\in\mathbb{Z}}\left(|j(k-1)||\Delta u(k-1)|+\frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)}\right)+\frac{1}{p^-}\sum_{k\in\mathbb{Z}} r(k)|u(k)|^{p(k)}\\
&\le C_1\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|j\|_{p'(.)}\|u\|_{p(.)}+\frac{C_1}{p^-}\sum_{k\in\mathbb{Z}}\left(r(k)|u(k)|^{p(k)}+|\Delta u(k-1)|^{p(k-1)}\right)\\
&\le C_1\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|j\|_{p'(.)}\|u\|_{p(.)}+\frac{C_1}{p^-}\rho_{r(.),1,p(.)}(u)<\infty.\end{aligned}$$

Moreover, we use (9) to get

$$|\Phi(u)|\le|\widehat{M}(I(u))|\le\left|\int_0^{I(u)} M(s)\mathrm{d}s\right|\le\frac{R_2}{\alpha}|I(u)|^\alpha<\infty.$$

Using (13) as in [11], there exists $\delta>0$ such that for $k\in\mathbb{Z}$ and $|t|\le\delta$, $|f(k,t)|\le |t|^{p(x)-1}$. Then,

$$|\Psi(u)|=\left|\sum_{k\in\mathbb{Z}} F(k,u(k)\right|\le\sum_{k\in\mathbb{Z}}|F(k,u(k)|\le\sum_{|k|\le h}|F(k,u(k)|+\sum_{|k|>h}|F(k,u(k)|<\infty.$$

We can conclude that the functional J_λ is well defined on $H^{1,p(.)}_{r(.)}$. From [9] and [12], we have

$$\langle \Phi'(u), v\rangle = M(I(u))\left[\sum_{k\in\mathbb{Z}} a(k-1,\Delta u(k-1))\Delta v(k-1) + \sum_{k\in\mathbb{Z}} r(k)|u(k)|^{p(k)-2}u(k)v(k)\right]$$

and $\langle \Psi'(u), v\rangle = \sum_{k\in\mathbb{Z}} f(k,u(k))v(k)$. □

Assume that $\langle J'_\lambda(u), v\rangle = 0$, which is equivalent to saying

$$-M(I(u))\sum_{k\in\mathbb{Z}}\left[\Delta a(k-1,\Delta u(k-1)) - r(k)\phi_{p(k)}(u) - \lambda f(k,u(k))\right] v(k) = 0, \quad (14)$$

for all $v \in H^{1,p(.)}_{r(.)}$.

For any $k \in \mathbb{Z}$, we define $e_h \in H^{1,p(.)}_{r(.)}$ by putting $e_h(k) = \delta_{hk}$ for $k \in \mathbb{Z}$ such that $\delta_{hk} = 1$ if $k = h$ and $\delta_{hk} = 0$ if $k \neq h$. Taking $v(k) = e_h$ in the equality (14), we obtain $-M(I(u))\left[\Delta a(k-1,\Delta u(k-1)) - r(k)\phi_{p(k)}(u)\right] - \lambda f(k,u(k)) = 0$, $k \in \mathbb{Z}$. Therefore, the critical point u of J_λ satisfies the problem (1).

Lemma 3.4. *The functional $J_\lambda : H^{1,p(.)}_{r(.)} \longrightarrow \mathbb{R}$ is weakly lower semi-continuous.*

Proof. For any $u \in H^{1,p(.)}_{r(.)}$, let

$$I(u) := \sum_{k\in\mathbb{Z}} A(k-1,\Delta u(k-1)) + \sum_{k\in\mathbb{Z}} \frac{r(k)}{p(k)}|u(k)|^{p(k)} = \varphi(u) + \tilde{\varphi}(u)$$

where $\varphi(u) = \sum_{k\in\mathbb{Z}} A(k-1,\Delta u(k-1))$ and $\tilde{\varphi}(u) = \sum_{k\in\mathbb{Z}} \frac{r(k)}{p(k)}|u(k)|^{p(k)}$.

The functional $\tilde{\varphi}$ is completely continuous and weakly lower semi-continuous. We have to prove the semi-continuity of φ.

From (5) and (8), φ is convex. Thus, it is enough to show that φ is lower semi-continuous. Let us fix $u \in H^{1,p(.)}_{r(.)}$ and $\epsilon > 0$. According to the convexity of the functional φ, we have $\varphi(v) \geq \varphi(u) + \langle \varphi'(u), v-u\rangle$, for any $v \in H^{1,p(.)}_{r(.)}$. From [9,12], we obtain

$$\begin{aligned}
\varphi(v) &\geq \varphi(u) + \sum_{k\in\mathbb{Z}} a(k-1,\Delta u(k-1))\left(\Delta v(k-1) - \Delta u(k-1)\right)\\
&\geq \varphi(u) - \sum_{k\in\mathbb{Z}} |a(k-1,\Delta u(k-1)||\Delta v(k-1) - \Delta u(k-1)|\\
&\geq \varphi(u) - C_1\sum_{k\in\mathbb{Z}} |j(k-1) + |\Delta u(k-1)|^{p(k-1)-1}||\Delta(v(k-1) - u(k-1))|.
\end{aligned}$$

Set $h(k-1) = j(k-1) + |\Delta u(k-1)|^{p(k-1)-1}$, we obtain $\varphi(v) \geq \varphi(u) - \epsilon$, for all $v \in H^{1,p(.)}_{r(.)}$ such that $\|u-v\|_{r(.),1,p(.)} < \beta = \dfrac{\epsilon}{S(p^-,(p')^-,C_1)}$.

We conclude that the functional I is weakly lower semi-continuous. As $\widehat{M}$ is continuous and non-decreasing, we deduce that the functional J_λ is also weakly lower semi-continuous. □

Proposition 3.5. *Suppose that assumptions* (9)-(13) *hold and* $\alpha p^- > p^+$. *Then* J_λ *is coercive and bounded from below for all* $\lambda > 0$.

Proof. Let $\|u\|_{r(.),1,p(.)} > 1$; according to (9)-(13), we have

$$\begin{aligned}
J_\lambda(u) &= \widehat{M}\left(\sum_{k\in\mathbb{Z}} A(k-1,\Delta u(k-1)) + \sum_{k\in\mathbb{Z}} \frac{r(k)}{p(k)}|u(k)|^{p(k)}\right) - \lambda \sum_{k\in\mathbb{Z}} F(k,u(k))\\
&\geq \int_0^{\frac{1}{p^+}\rho_{r(.),1,p(.)}(u)} R_1 t^{\alpha-1} - \lambda \sum_{k\in\mathbb{Z}} \frac{1}{p(k)}|u(k)|^{p(k)} - \lambda \sum_{|k|\leq h} F(k,u(k))\\
&\geq \int_0^{\frac{1}{p^+}\rho_{r(.),1,p(.)}(u)} R_1 t^{\alpha-1} - \lambda\left(\sum_{k\in\mathbb{Z}} \frac{1}{p(k)}|u(k)|^{p(k)} + \sum_{k\in\mathbb{Z}} \frac{1}{r_0 p^-}|\Delta u(k-1)|^{p(k-1)}\right) - \tilde{C}\\
&\geq \frac{R_1}{\alpha}\frac{1}{(p^+)^\alpha}(\rho_{r(.),1,p(.)}(u))^\alpha - \frac{\lambda}{r_0 p^-}\rho_{r(.),1,p(.)}(u) - \tilde{C}.
\end{aligned}$$

Finally, we use Proposition 2.4 to get

$$J_\lambda(u) \geq \frac{R_1}{\alpha}\frac{1}{(p^+)^\alpha}\|u\|^{\alpha p^-}_{r(.),1,p(.)} - \frac{\lambda}{r_0 p^-}\|u\|^{p^+}_{r(.),1,p(.)} - \tilde{C}. \tag{15}$$

Since $\alpha p^- > p^+$, $J_\lambda(u) \to \infty$ as $\|u\|_{r(.),1,p(.)} \to \infty$; then, the functional J_λ is coercive.

For all u in $H^{1,p(.)}_{r(.)}$ such that $\|u\|_{r(.),1,p(.)} < 1$, we obtain

$$\begin{aligned}
J_\lambda(u) &\geq \frac{R_1}{\alpha}\frac{1}{(p^+)^\alpha}(\rho_{r(.),1,p(.)}(u))^\alpha - \frac{\lambda}{r_0 p^-}\rho_{r(.),1,p(.)}(u) - \tilde{C}\\
&\geq \frac{R_1}{\alpha}\frac{1}{(p^+)^\alpha}\|u\|^{\alpha p^+}_{r(.),1,p(.)} - \frac{\lambda}{r_0 p^-}\|u\|^{p^-}_{r(.),1,p(.)} - \tilde{C} \geq -C > -\infty.
\end{aligned}$$

Namely, J_λ is bounded from below. □

Proof of Theorem 3.2. By Theorem 2.7, it follows that problem (1) has at least one weak homoclinic solution for all $\lambda > 0$ with $\alpha p^- > p^+$.

Proposition 3.6. *Suppose that* (9)-(13) *are satisfied with* $\alpha p^- = p^+$. *Then, there exists* $\lambda^{(0)} > 0$ *such that for any* $\lambda < \lambda^{(0)}$, *the functional* J_λ *is coercive on* $H^{1,p(.)}_{r(.)}$.

Proof. According to (15) and as $\alpha p^- = p^+$, we get

$$\begin{aligned}
J_\lambda(u) &\geq \frac{R_1}{\alpha}\frac{1}{(p^+)^\alpha}\|u\|^{\alpha p^-}_{r(.),1,p(.)} - \frac{\lambda}{r_0 p^-}\|u\|^{p^+}_{r(.),1,p(.)} - \tilde{C}\\
&\geq \left(\frac{R_1}{\alpha}\frac{1}{(p^+)^\alpha} - \frac{\lambda}{r_0 p^-}\right)\|u\|^{p^+}_{r(.),1,p(.)} - \tilde{C}.
\end{aligned}$$

We put $\lambda^{(0)} = \frac{R_1}{\alpha}\frac{p^- r_0}{(p^+)^\alpha}$. Since $\lambda \in (0,\lambda^{(0)})$, then $J_\lambda(u) \to \infty$ as $\|u\|_{r(.),1,p(.)} \to \infty$. □

Therefore, we immediately deduce the following result.

Corollary 3.7. *Assume that condition* (9)-(13) *are fulfilled with* $\alpha p^- = p^+$. *Then, there exists* $\lambda^{(0)} > 0$ *such that for any* $\lambda \in (0,\lambda^{(0)})$, *problem* (1) *has at least one weak homoclinic solution.*

Proof. Indeed, the functional J_λ is continuous differentiable in the sense of Gâteaux. The assertion follows then from Proposition 3.6 and Theorem 2.7. $\square$

4. Existence of solution by Mountain pass lemma

In this section, we deal with the existence of nontrivial weak homoclinic solutions for the problem (1).

We introduce firstly some assumptions.

(f_0) There exist $\mu > \alpha p^+$ and $t_0 > 0$ such that

$$0 \leq \mu F(k,t) \leq f(k,t)t, \quad |t| \geq t_0, \text{ for all } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

(f_1) There exists $t_0 > 0$ such that $F(k,t) > 0$, for all $k \in \mathbb{Z}$ and all $|t| \geq t_0$.

Our main result is the following theorem.

THEOREM 4.1. *Suppose that* (5)-(13) *and* (f_0)*-*(f_1) *are satisfied.*

Then, there exists $\lambda^{(1)} > 0$ *such that for each* $\lambda \in (0, \lambda^{(1)})$*, problem* (1) *has at least one nontrivial weak homoclinic solutions.*

We need the following auxiliary result for the proof of the main result.

LEMMA 4.2. *(a)* $\left(H^{1,p(.)}_{r(.)}, \|.\|_{1,p(.),r(.)}\right)$ *is a reflexive separable Banach space.*

(b) Suppose that there is a sequence $\{u_n\} \subset H^{1,p(.)}_{r(.)}$ *such that* $u_n \rightharpoonup u$ $H^{1,p(.)}_{r(.)}$*, then the sequence* $\{u_n\}$ *satisfies* $u_n \longrightarrow u$ *in* $l^{p(.)}$*.*

Proof. From [8,17], it is known that $l^{p(.)}$ is a separable Banach space if $p^+ < \infty$ and reflexive if $1 < p^- \leq p^+ < \infty$. Consider the mapping $u \to (u, \Delta u)$, the space $H^{1,p(.)}_{r(.)}$ is a closed subspace of $l^{p(.)}_{r(.)} \times l^{p(.)}$. By [8, Proposition 1.4.4], $H^{1,p(.)}_{r(.)}$ is separable if $p^+ < \infty$ and reflexive if $1 < p^- \leq p^+ < \infty$.

To have (b), let $u_n \rightharpoonup u$ in $H^{1,p(.)}_{r(.)}$. Write $v_n = u_n - u$ then $v_n \rightharpoonup 0$ in $H^{1,p(.)}_{r(.)}$. By Banach-Steinhaus theorem, $\|v_n\|_{r(.),1,p(.)}$ is uniformly bounded.

In the sequel, we follows the results in [13,14]. Let

$$H^{1,p(.)}_{r(.),J} := \left\{ v : J = [-h,h]_{\mathbb{Z}} \to \mathbb{R} : \ \rho_{r(.),1,p(.)} = \sum_{|k| \leq h} r(k)|v(k)|^{p(k)} + \sum_{|k| \leq h} |\Delta v(k)|^{p(k)} < \infty \right\}.$$

The sequence $\{v_n\}$ is bounded in $H^{1,p(.)}_{r(.),J}$ which implies that $\{v_n\}$ is bounded in $l^{p(.)}_J$ where $l^{p(.)}_J$ is the set of all functions $v : J \longrightarrow \mathbb{R}$ such that

$$\|v\|_{J,p(.)} = \inf \left\{ \nu > 0; \quad \rho_{p(.)}\left(\frac{v}{\nu}\right) \leq 1 \right\}.$$

By the uniqueness of the weak limit, we deduce that $v_n \to 0$ in J. So, there is $N \in \mathbb{N}$ such that

$$\sum_{|k|\leq h} |v_n(k)|^{p(k)} < \frac{\epsilon}{3}, \quad \text{for all } n > N. \tag{16}$$

Since $\|v_n\|_{r(.),1,p(.)}$ is uniformly bounded, there exists $K>0$ such that $\|v_n\|_{r(.),1,p(.)} \leq K$. Let

$$\widetilde{K} := \begin{cases} K^{p+} & \text{if } \|v_n\|_{r(.),1,p(.)} > 1 \\ K^{p-} & \text{if } \|v_n\|_{r(.),1,p(.)} < 1. \end{cases}$$

Using (11), we obtain $\dfrac{1}{r(k)} \leq \dfrac{\epsilon}{\widetilde{K}}$ for all $k \in (-\infty, -h)_{\mathbb{Z}} \cup (h, \infty)_{\mathbb{Z}}$. Then,

$$\sum_{|k|>h} |v_n(k)|^{p(k)} < \frac{2\epsilon}{\widetilde{K}} \sum_{|k|>h} r(k)|v_n(k)|^{p(k)} < \frac{2\epsilon}{3}. \tag{17}$$

From (16) and (17), there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\sum_{k\in\mathbb{Z}} |v_n(k)|^{p(k)} = \sum_{|k|>h} |v_n(k)|^{p(k)} + \sum_{|k|\leq h} |v_n(k)|^{p(k)} < \epsilon.$$

Note that ϵ is arbitrary then $\rho_{p(.)}(v_n) \longrightarrow 0$ as $n \to \infty$. From Proposition 2.3 the result follows. □

LEMMA 4.3. *Assume that* (8) *and* (9) *are satisfied and* $\alpha p^- > p^+$. *Then, for all* $\lambda > 0$, *the functional* J_λ *satisfies the Palais-Smale condition.*

Proof. Let $\lambda > 0$ be fixed. Consider $\{u_n\} \subset H^{1,p(.)}_{r(.)}$ be such that $J_\lambda(u_n)$ is bounded and $J'_\lambda(u_n) \longrightarrow 0$. This fact and Proposition 3.5 imply that the sequence $\{u_n\}$ is bounded. By using the preceding result, Lemma 4.2 and passing to a subsequence, we have $u_n \rightharpoonup u$ in $H^{1,p(.)}_{r(.)}$ and $u_n \longrightarrow u$ in $l^{p(.)}$. Then, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|u_n - u\|_{p(.)} < \epsilon \qquad n > N. \tag{18}$$

Taking $g(k, u_n) = (f(k, u_n) - f(k, u))(u_n - u)$, we obtain

$$\begin{aligned} \sum_{k\in\mathbb{Z}} |g(k, u_n)| &\leq \sum_{k\in\mathbb{Z}} |f(k, u_n) - f(k, u)||u_n - u| \\ &\leq \sum_{k\in\mathbb{Z}} |f(k, u_n)||u_n - u| + \sum_{k\in\mathbb{Z}} |f(k, u)||u_n - u|. \end{aligned}$$

We use Discrete Hölder type inequality and (18) to obtain

$$\sum_{k\in\mathbb{Z}} |g(k, u_n) \leq 2C_1\|u_n - u\|_{p(.)} < 2C_1\epsilon \quad \text{if} \quad n > N.$$

Therefore,

$$\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} g(k, u_n) = 0, \quad \text{and} \quad \lim_{n\to\infty} \langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle = 0. \tag{19}$$

$$\begin{aligned}\langle J'_\lambda(u_n), u_n-u\rangle = M(I(u_n))&\left[\sum_{k\in\mathbb{Z}} a(k-1,\Delta u_n(k-1))\Delta(u_n-u)+\sum_{k\in\mathbb{Z}} r(k)\phi(u_n)(u_n-u)\right]\\ &-\lambda\sum_{k\in\mathbb{Z}} f(k,u_n(k))(u_n-u)\end{aligned}$$

and

$$\begin{aligned}\langle J'_\lambda(u_n), u_n-u\rangle = M(I(u))&\left[\sum_{k\in\mathbb{Z}} a(k-1,\Delta u(k-1))\Delta(u_n-u)+\sum_{k\in\mathbb{Z}} r(k)\phi(u)(u_n-u)\right]\\ &-\lambda\sum_{k\in\mathbb{Z}} f(k,u(k))(u_n-u).\end{aligned}$$

From (8) and (9), we deduce that

$$\begin{aligned}\langle J'_\lambda(u_n)-J'_\lambda(u), u_n-u\rangle \geq & M_0\left(\sum_{k\in\mathbb{Z}}\left[a(k-1,\Delta u_n(k-1))-a(k-1,\Delta u(k-1))\right](\Delta u_n-\Delta u)\right)\\ &+\sum_{k\in\mathbb{Z}} M_0 r(k)\left(\phi(u_n)-\phi(u)\right)(u_n-u)-\lambda\sum_{k\in\mathbb{Z}} g(k,u_n)\\ \geq & M_0\left(\sum_{k\in\mathbb{Z}} C_2|\Delta(u_n-u)|^{p(x)}+\sum_{k\in\mathbb{Z}} r(k)|u_n-u|^{p(x)}\right)-\lambda\sum_{k\in\mathbb{Z}} g(k,u_n)\\ \geq & M_0\left(\sum_{k\in\mathbb{Z}} |\Delta(u_n-u)|^{p(x)}+\sum_{k\in\mathbb{Z}} r(k)|u_n-u|^{p(x)}\right)-\lambda\sum_{k\in\mathbb{Z}} g(k,u_n)\\ \geq & M_0\rho_{r(.),1,p(.)}(u_n-u)-\lambda\sum_{k\in\mathbb{Z}} g(k,u_n).\end{aligned}$$

Letting $n\longrightarrow\infty$, from (19) and Proposition 2.4, we obtain $\rho_{r(.),1,p(.)}(u_n-u)\longrightarrow 0$; then $u_n\longrightarrow u$ in $H^{1,p(.)}_{r(.)}$. This show that J_λ satisfies (PS) condition. □

Remark 4.4. For the special case $\alpha p^- = p^+$, for all $\lambda\in(0,\lambda^{(0)})$, the functional J_λ satisfies (PS) condition.

Lemma 4.5. *(A_1) There exists $\lambda^{(1)}>0$ and two positive real numbers θ and η such that for each $\lambda\in(0,\lambda^{(1)})$, $J_\lambda(u)\geq\eta>0$ for all $u\in\left\{u\in H^{1,p(.)}_{r(.)}:\|u\|_{r(.),1,p(.)}=\theta\right\}$.*

(A_2) There exists $u\in H^{1,p(.)}_{r(.)}$ such that for any $\lambda>0$, $\|u\|_{r(.),1,p(.)}>\theta$, $J_\lambda(u)<0$.

Proof. To have (A_1), for $h\in\mathbb{N}$, taking $k_0\in[-h,h]$ and set

$$\mathbf{S}_\theta\left(H^{1,p(.)}_{r(.)}\right):=\left\{u\in H^{1,p(.)}_{r(.)}:\|u\|_{r(.),1,p(.)}=\theta\right\},$$

with $\theta\in(0,1)$. According to (9)-(13), we obtain

$$J_\lambda(u)=\widehat{M}\left(\sum_{k\in\mathbb{Z}} A(k-1,\Delta u(k-1))+\sum_{k\in\mathbb{Z}}\frac{r(k)}{p(k)}|u(k)|^{p(k)}\right)-\lambda\sum_{k\in\mathbb{Z}} F(k,u(k)$$

$$\geq \frac{R_1}{\alpha}\frac{1}{(p^+)^\alpha}(\rho_{r(.),1,p(.)}(u))^\alpha - \frac{\lambda}{r_0 p^-}\rho_{r(.),1,p(.)}(u) - \lambda(2h+1)|F(k_0,u(k_0))|.$$

For $u \in \mathbf{S}_\theta\left(H^{1,p(.)}_{r(.)}\right)$, by Proposition 2.4, we get

$$\begin{aligned} J_\lambda(u) &\geq \frac{R_1}{\alpha}\frac{1}{(p^+)^\alpha}\|u\|^{\alpha p^+}_{r(.),1,p(.)} - \frac{\lambda}{r_0 p^-}\|u\|^{p^-}_{r(.),1,p(.)} - \lambda(2h+1)|F(k_0,u(k_0))| \\ &\geq \frac{R_1}{\alpha}\frac{1}{(p^+)^\alpha}(\theta)^{\alpha p^+} - \frac{\lambda}{r_0 p^-}\theta^{p^-} - \lambda(2h+1)|F(k_0,u(k_0))| \\ &\geq \theta^{p^-}\left[\frac{R_1}{\alpha}\frac{1}{(p^+)^\alpha}(\theta)^{\alpha p^+ - p^-} - \lambda\left(\frac{1}{r_0 p^-} + (2h+1)|F(k_0,u(k_0))|\theta^{-p^-}\right)\right]. \end{aligned}$$

Take

$$\lambda^{(1)} = \frac{\frac{R_1}{\alpha}\frac{1}{(p^+)^\alpha}(\theta)^{\alpha p^+ - p^-}}{\frac{1}{r_0 p^-} + (2h+1)|F(k_0,u(k_0))|\theta^{-p^-}},$$

then, for all $\lambda \in (0,\lambda^{(1)})$ and $u \in \mathbf{S}_\theta\left(H^{1,p(.)}_{r(.)}\right)$, $J_\lambda(u) \geq \eta > 0$.

To obtain (A_2), from (f_0) and (f_1), one has $F(k,t) \geq \dfrac{F(k,t_0)}{t_0^\mu}t^\mu > 0$ for all $t \geq t_0$. For $\delta > 1$ and nonnegative $u \in H^{1,p(.)}_{r(.)}$, set $\Omega_1 := \{k \in \mathbb{Z}, u(k) \geq t_0\}$ and $\Omega_2 := \{k \in \mathbb{Z}, \delta u(k) \geq t_0\}$, we obtain

$$\begin{aligned} \sum_{k\in\mathbb{Z}} F(k,\delta u(k)) &\geq \sum_{k\in\Omega_2} F(k,\delta u(k)) \geq \frac{\delta^\mu}{t_0^\mu}\sum_{k\in\Omega_2} F(k,t_0)u(k)^\mu \\ &\geq \frac{\delta^\mu}{t_0^\mu}\sum_{k\in\Omega_1} F(k,t_0)u(k)^\mu \geq \delta^\mu \sum_{k\in\Omega_1} F(k,t_0) > 0. \end{aligned} \tag{20}$$

Recall that $\Phi(u) = \widehat{M}(I(u)) = \widehat{M}\left(\sum_{k\in\mathbb{Z}} A(k-1,\Delta u(k-1)) + \sum_{k\in\mathbb{Z}}\frac{r(k)}{p(k)}|u(k)|^{p(k)}\right)$.

For all $t \in \mathbb{R}$ such that $|t| \geq t_0$, from (9), we have $\widehat{M}(t) \leq \dfrac{R_2}{\alpha}t^\alpha$. Take $w \in H^{1,p(.)}_{r(.)}\backslash\{0\}$. We have

$$\begin{aligned} J_\lambda(tw) &= \widehat{M}(I(tw)) - \lambda\sum_{k\in\mathbb{Z}} F(k,tw) \\ &\leq \frac{R_2}{\alpha}\left(\sum_{k\in\mathbb{Z}} A(k-1,\Delta tw(k-1)) + \sum_{k\in\mathbb{Z}}\frac{r(k)}{p(k)}|tw(k)|^{p(k)}\right)^\alpha - \lambda\sum_{k\in\mathbb{Z}} F(k,tw). \end{aligned}$$

For each $\lambda > 0$, combining (5)-(6) and (20), we deduce that there exists a constant $\tilde{C} > 0$ such that

$$J_\lambda(tw) \leq \frac{R_2}{\alpha}\left[(\tilde{C} + \frac{1}{r_0 p^-})\left(\sum_{k\in\mathbb{Z}} r(k)|tw(k)|^{p(k)} + |\Delta tw(k-1)|^{p(k-1)}\right)\right]^\alpha$$

$$
\begin{aligned}
&- \lambda \sum_{k\in\Omega_1} F(k,t_0)t^{\mu} \\
&\leq \frac{R_2}{\alpha}\left(\tilde{C}+\frac{1}{r_0 p^-}\right)^{\alpha} t^{\alpha p^+}\left(\sum_{k\in\mathbb{Z}} r(k)|w(k)|^{p(k)}+|\Delta w(k-1)|^{p(k-1)}\right)^{\alpha} \\
&- \lambda \sum_{k\in\Omega_1} F(k,t_0)t^{\mu}.
\end{aligned}
$$

Since $\mu > \alpha p^+$, for sufficiently large $t > 1$, we assert that $J_\lambda(tw) < 0 = J_\lambda(0)$.

Then, for each $\lambda > 0$ and $\theta > 0$, there exists $t > 1$ such that $\|\tilde{u}\|_{r(.),1,p(.)} > \theta$ and $J_\lambda(\tilde{u}) < 0$. □

Finally, we give the proof of Theorem 4.1. From Lemma 4.5 and the fact $J_\lambda(0) = 0$ and as J_λ satisfies the assumptions of Theorem 2.8, problem (1) has at least one nontrivial weak homoclinic solution.

Remark 4.6. In the special case $\alpha p^- = p^+$, arguing again as above, but taking into account Lemma 4.5 and the proof of Theorem 4.1, we can show that for every $\lambda \in (0,\lambda^{(0)}) \cap (0,\lambda^{(1)}) \neq \emptyset$, the same conclusion still holds without any additional assumption. Hence, the problem (1) has at least one nontrivial weak homoclinic solution.

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