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FIXED POINTS OF GENERALIZED φ-QUASI CONTRACTION MAPS

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Abstract. In this paper, we define generalized $\varphi-$quasi contraction map which is more general than strict quadratic quasi contraction map by using an altering distance function φ and prove the existence and uniqueness of fixed points of these maps satisfying asymptotically regular property in the setting of complete metric spaces. We extend these results to T-orbitally complete metric spaces. Examples are provided to illustrate our results. Our results generalize Theorem 4 of [O. Popescu, G. Stan, *Some fixed point theorems for quadratic quasi contractive mappings*, Symmetry, **11** (2019)].

1. Introduction

The study and development of fixed point theory depends mainly on the generalization of the conditions that guarantee the existence and, if possible, the uniqueness of fixed points, as well as on the generalization of the ambient space of the operator under consideration. Since the discovery of the Banach contraction principle, there have been several generalizations.

In 1962 Edelstein [3] proved the following fixed point theorem.

THEOREM 1.1. *Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then, T has a unique fixed point.*

Afterwards in 1973, Hardy and Rogers [5] extended Theorem 1.1 and proved the following theorem.

THEOREM 1.2. *Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping satisfying inequality*

$$d(Tx, Ty) < A \cdot d(x, Tx) + B \cdot d(y, Ty) + C \cdot d(x, y) \tag{1}$$

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for all $x, y \in X$ *and* $x \neq y$*, where* A, B, C *are positive and* $A + B + C = 1$*. Then* T *has a unique fixed point.*

In 1980, Greguš [4] proved the following theorem in Banach spaces.

Theorem 1.3. *Let* X *be a Banach space and* C *a closed convex subset of* X*. Let* $T : X \to X$ *be a mapping satisfying inequality*

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\| \tag{2}$$

for all $x, y \in C$*, where* a, b, c *are positive and* $a+b+c = 1$*. Then* T *has a unique fixed point.*

Recently in 2019, Popescu and Stan [8] introduced the notion of quadratic quasi contraction mapping.

Definition 1.4. A mapping $T : X \to X$ of a metric space X into itself is said to be *quadratic quasi contraction map* if there exists $a \in (0, \frac{1}{2})$ such that

$$d^2(Tx, Ty) \leq a \cdot d^2(x, Tx) + a \cdot d^2(y, Ty) + (1 - 2a) \cdot d^2(x, y) \tag{3}$$

for all $x, y \in X$. T is said to be *strict quadratic quasi contraction map* if

$$d^2(Tx, Ty) < a \cdot d^2(x, Tx) + a \cdot d^2(y, Ty) + (1 - 2a) \cdot d^2(x, y) \tag{4}$$

for all $x, y \in X$ with $x \neq y$.

Remark 1.5. Popescu and Stan [8] observed that if a selfmap T satisfies the inequality (1), then it satisfies (4), but its converse is not true [8, Example 1]. Therefore, strict quadratic quasi contraction maps are more general than the maps that satisfy the inequality (1).

Popescu and Stan [8] proved the following theorems.

Theorem 1.6 ([8, Theorem 4]). *Let* (X, d) *be a compact metric space and let* $T : X \to X$ *be a strict quadratic quasi contraction map. Then* T *has a unique fixed point* $v \in X$*. Moreover if* T *is continuous, then for each* $x \in X$*, the sequence of iterates* $\{T^n x\}$ *converges to* v*.*

Theorem 1.7 ([8, Theorem 5]). *Let* X *be a Banach space and* C *be a closed convex subset of* X*. Let* $T : C \to C$ *be a mapping satisfying the inequality:*

$$\|Tx - Ty\|^2 \leq a\|x - Tx\|^2 + a\|y - Ty\|^2 + b\|x - y\|^2$$

for all $x, y \in C$*, where* $0 < a < \frac{1}{2}, b = 1 - 2a$*. Then* T *has a unique fixed point.*

In 1967, the concept of asymptotic regularity of a selfmap at a point in the space was introduced by Browder and Petryshyn [2].

Definition 1.8. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Let $x \in X$. If $\lim\limits_{n\to+\infty} d(T^n x, T^{n+1}x) = 0$ then T is said to be *asymptotically regular at a point* x in X. If T is asymptotically regular at every point x in X then we say that T is *asymptotically regular* on X.

In 1984, Khan, Swaleh and Sessa [6] considered contraction condition with an altering distance function to prove the existence of fixed points in complete metric spaces.

Definition 1.9. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ ($\mathbb{R}^+ = [0, +\infty)$) be a function. If φ satisfies the conditions
(i) φ is continuous, (ii) φ is monotonically increasing, (iii) $\varphi(t) = 0$ iff $t = 0$,
then φ is said to be an *altering distance function* or *control function*.

We denote the class of all altering distance functions by Φ.

For more details on altering distance functions and results based on altering distance functions, we refer to [7, 10, 11].

Motivated by the work of Popescu and Stan [8], as well as Khan, Swaleh and Sessa [6], in Section 2 we define generalized φ-quasi contraction maps using an altering distance function φ and prove the existence and uniqueness of fixed points of these maps under asymptotically regular property in complete metric spaces. In Section 3 we extend our results to T-orbitally complete metric spaces. In Section 4 we provide examples to illustrate our results. Our results extend and generalize Theorem 1.6 and Theorem 1.7 to complete metric spaces.

2. Fixed points of generalized φ-quasi contraction maps

In the following, we define generalized φ-quasi contraction map.

Definition 2.1. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Assume that there exist $\varphi \in \Phi$ and $a \in (0, \frac{1}{2})$ and a nonnegative real number $r \in (0, a)$ such that

$$\begin{aligned}\varphi(d(Tx, Ty)) \leq\ & a[\varphi(d(x, Tx)) + \varphi(d(y, Ty))] \\ & + (1 - 2a)\varphi(d(x, y)) + r[\varphi(d(x, Ty)) + \varphi(d(y, Tx))] \qquad (5)\end{aligned}$$

for all $x, y \in X$. Then we say that T is a *generalized φ-quasi contraction map.*

We note that if $\varphi(t) = t^2, t \geq 0, r = 0$ then T is a quadratic quasi contraction.

Remark 2.2. The class of generalized φ-quasi contraction maps are more general than strict quadratic quasi contraction maps (see Example 4.2).

Lemma 2.3 ([1, 9]). *Suppose (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to +\infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and*
(i) $\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) = \epsilon$ *(ii)* $\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$,

(iii) $\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$, *(iv)* $\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon$.

Theorem 2.4. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a generalized φ-quasi contraction map. If T is asymptotically regular at some point x_0 in X then T has a unique fixed point in X.*

Proof. Let $T : X \to X$ be asymptotically regular at a point x_0 in X. Let us consider the sequence $\{T^n x_0\}$.

Let $m, n \in \mathbb{N}$ and $m > n$. Now from the inequality (5) and by using the triangle inequality, we have

$$\begin{aligned}
&\varphi(d(T^m x_0, T^n x_0)) = \varphi(d(T(T^{m-1}x_0), T(T^{n-1}x_0)))\\
&\le a[\varphi(d(T^{m-1}x_0, T^m x_0)) + \varphi(d(T^{n-1}x_0, T^n x_0))] + (1-2a)\varphi(d(T^{m-1}x_0, T^{n-1}x_0))\\
&\quad + r[\varphi(d(T^{m-1}x_0, T^n x_0)) + \varphi(d(T^{n-1}x_0, T^m x_0))]\\
&\le a[\varphi(d(T^{m-1}x_0, T^m x_0)) + \varphi(d(T^{n-1}x_0, T^n x_0))] + (1-2a)\varphi(d(T^{m-1}x_0, T^m x_0)\\
&\quad + d(T^m x_0, T^n x_0) + d(T^n x_0, T^{n-1}x_0))\\
&\quad + r[\varphi(d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0)) + \varphi(d(T^{n-1}x_0, T^n x_0) + d(T^n x_0, T^m x_0))].
\end{aligned}$$

On letting $m, n \to +\infty$, we get

$$\begin{aligned}
&\lim_{m,n\to+\infty} \varphi(d(T^m x_0, T^n x_0)) \le\\
&a[\lim_{m\to+\infty} \varphi(d(T^{m-1}x_0, T^m x_0)) + \lim_{n\to+\infty} \varphi(d(T^{n-1}x_0, T^n x_0))]\\
&+(1-2a)[\lim_{m,n\to+\infty} \varphi(d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0) + d(T^n x_0, T^{n-1}x_0))]\\
&+r[\lim_{m,n\to+\infty} \varphi(d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0))\\
&+\lim_{m,n\to+\infty} \varphi(d(T^{n-1}x_0, T^n x_0) + d(T^n x_0, T^m x_0))].
\end{aligned}$$

Since φ is continuous, it follows that

$$\begin{aligned}
&\varphi(\lim_{m,n\to+\infty} d(T^m x_0, T^n x_0)) \le a[\varphi(\lim_{m\to+\infty} d(T^{m-1}x_0, T^m x_0)) + \varphi(\lim_{n\to+\infty} d(T^{n-1}x_0, T^n x_0))]\\
&+(1-2a)\varphi(\lim_{m\to+\infty} d(T^{m-1}x_0, T^m x_0) + \lim_{m,n\to+\infty} d(T^m x_0, T^n x_0)\\
&+\lim_{n\to+\infty} d(T^n x_0, T^{n-1}x_0)) + r[\varphi(\lim_{m\to+\infty} d(T^{m-1}x_0, T^m x_0) + \lim_{m,n\to+\infty} d(T^m x_0, T^n x_0))\\
&+\varphi(\lim_{n\to+\infty} d(T^{n-1}x_0, T^n x_0) + \lim_{m,n\to+\infty} d(T^n x_0, T^m x_0))].
\end{aligned}$$

Now by using the asymptotically regularity of T, we obtain

$$\begin{aligned}
&\varphi(\lim_{m,n\to+\infty} d(T^m x_0, T^n x_0))\\
&\le (1-2a)\varphi(\lim_{m,n\to+\infty} d(T^m x_0, T^n x_0)) + 2r\varphi(\lim_{m,n\to+\infty} d(T^m x_0, T^n x_0))\\
&= ((1-2a) + 2r)\varphi(\lim_{m,n\to+\infty} d(T^m x_0, T^n x_0)),\\
&\quad (1 - ((1-2a) + 2r))\varphi(\lim_{m,n\to+\infty} d(T^m x_0, T^n x_0)) \le 0,
\end{aligned}$$

i.e. $$2(a-r)\varphi(\lim_{m,n\to+\infty} d(T^m x_0, T^n x_0)) \le 0.$$

This implies that $\varphi(\lim_{m,n\to+\infty} d(T^m x_0, T^n x_0)) = 0$, since $a - r > 0$. Therefore,

$\lim_{m,n\to+\infty} d(T^m x_0, T^n x_0) = 0$ and thus $\{T^n x_0\}$ is Cauchy. Since X is complete, there exists a point u in X such that $\lim_{n\to+\infty} T^n x_0 = u$.

We now consider

$$\begin{aligned}\varphi(d(T^n x_0, Tu)) &\le a[\varphi(d(T^{n-1}x_0, T^n x_0)) + \varphi(d(u, Tu))] + (1-2a)\varphi(d(T^{n-1}x_0, u)) \\ &\quad + r[\varphi(d(T^{n-1}x_0, Tu)) + \varphi(d(T^n x_0, u))] \\ &\le a[\varphi(d(T^{n-1}x_0, T^n x_0)) + \varphi(d(u, Tu))] + (1-2a)\varphi(d(T^{n-1}x_0, u)) \\ &\quad + r[\varphi(d(T^{n-1}x_0, u) + d(u, Tu)) + \varphi(d(T^n x_0, u))].\end{aligned}$$

On letting $n \to +\infty$, we have

$$\begin{aligned}\lim_{n\to+\infty} \varphi(d(T^n x_0, Tu)) &\le a[\lim_{n\to+\infty} \varphi(d(T^{n-1}x_0, T^n x_0)) + \varphi(d(u, Tu))] \\ &\quad + (1-2a)\lim_{n\to+\infty} \varphi(d(T^{n-1}x_0, u)) \\ &\quad + r[\lim_{n\to+\infty} \varphi(d(T^{n-1}x_0, u) + d(u, Tu)) + \lim_{n\to+\infty} \varphi(d(T^n x_0, u))].\end{aligned}$$

Since φ is continuous, we have

$$\begin{aligned}\varphi(\lim_{n\to+\infty} d(T^n x_0, Tu)) &\le a[\varphi(\lim_{n\to+\infty} d(T^{n-1}x_0, T^n x_0)) + \varphi(d(u, Tu))] \\ &\quad + (1-2a)\varphi(\lim_{n\to+\infty} d(T^{n-1}x_0, u)) \\ &\quad + r[\varphi(\lim_{n\to+\infty} d(T^{n-1}x_0, u) + d(u, Tu)) + \varphi(\lim_{n\to+\infty} d(T^n x_0, u))]\end{aligned}$$

and hence

$$\varphi(d(u, Tu)) \le a\varphi(d(u, Tu)) + r\varphi(d(u, Tu)),$$

so that $\varphi(d(u, Tu)) \le 0$ (since $(a+r) < 1$). Therefore $\varphi(d(u, Tu)) = 0$, which implies that $d(u, Tu) = 0$. Therefore $Tu = u$ and u is a fixed point of T. Suppose that v is another fixed point of T. Then

$$\begin{aligned}\varphi(d(u, v)) &= \varphi(d(Tu, Tv)) \\ &\le a[\varphi(d(u, Tu)) + \varphi(d(v, Tv))] + (1-2a)\varphi(d(u, v)) + r[\varphi(d(u, Tv)) + \varphi(d(v, Tu))] \\ &= a[\varphi(d(u, u)) + \varphi(d(v, v))] + (1-2a)\varphi(d(u, v)) + r[\varphi(d(u, v)) + \varphi(d(v, u))] \\ &= ((1-2a) + 2r)\varphi(d(u, v)).\end{aligned}$$

This implies that

$$(1 - ((1-2a) + 2r))\varphi(d(u, v)) \le 0, \quad \text{i.e.} \quad 2(a-r)\varphi(d(u, v)) \le 0.$$

Since $r < a$, we have $\varphi(d(u, v)) \le 0$, so that $\varphi(d(u, v)) = 0$ and $d(u, v) = 0$. Therefore $v = u$ and u is the unique fixed point of T. □

THEOREM 2.5. *Let (X, d) be a metric space and T, a mapping of X into itself. Assume that*

(i) T is a generalized φ-quasi contraction map.

(ii) T is asymptotically regular at a point x in X and

(iii) the sequence of iterates $\{T^n x\}$ has a subsequence $\{T^{n_k} x\}$ such that $\{T^{n_k} x\}$ converges to a point z in X,

then z is the unique fixed point of T and $\{T^n x\}$ also converges to z.

Proof. Let T be asymptotically regular at x in X and consider the sequence $\{T^n x\}$.

Suppose that $\{T^{n_k} x\}$ is a subsequence of $\{T^n x\}$ such that $\lim_{k\to+\infty} T^{n_k} x = z$ and $Tz \neq z$. By inequality (5), we have

$$\begin{aligned}
&\varphi(d(T^{n_k}x, Tz)) = \varphi(d(T(T^{n_k-1}x), Tz)) \\
&\leq a[\varphi(d(T^{n_k-1}x, T^{n_k}x)) + \varphi(d(z, Tz))] + (1-2a)\varphi(d(T^{n_k-1}x, z)) \\
&\quad + r[\varphi(d(T^{n_k-1}x, Tz)) + \varphi(d(z, T^{n_k}x))] \\
&\leq a[\varphi(d(T^{n_k-1}x, T^{n_k}x)) + \varphi(d(z, Tz))] + (1-2a)\varphi(d(T^{n_k-1}x, T^{n_k}x) + d(T^{n_k}x, z)) \\
&\quad + r[\varphi(d(T^{n_k-1}x, T^{n_k}x) + d(T^{n_k}x, z) + d(z, Tz)) + \varphi(d(z, T^{n_k}x))].
\end{aligned}$$

On letting $k \to +\infty$, and using the continuity of φ, it follows that

$$\begin{aligned}
\varphi(\lim_{k\to+\infty} d(T^{n_k}x, Tz)) \leq\; & a[\varphi(\lim_{k\to+\infty} d(T^{n_k-1}x, T^{n_k}x)) + \varphi(d(z, Tz))] \\
& + (1-2a)\varphi(\lim_{k\to+\infty} d(T^{n_k-1}x, T^{n_k}x) + \lim_{k\to+\infty} d(T^{n_k}x, z)) \\
& + r[\varphi(\lim_{k\to+\infty} d(T^{n_k-1}x, T^{n_k}x) + \lim_{k\to+\infty} d(T^{n_k}x, z) + d(z, Tz)) \\
& + \varphi(\lim_{k\to+\infty} d(z, T^{n_k}x))].
\end{aligned}$$

Since T is asymptotically regular:

$$\varphi(d(z, Tz)) \leq a\varphi(d(z, Tz)) + r\varphi(d(z, Tz)) = (a+r)\varphi(d(z, Tz))$$

Since $a + r < 1$, $\varphi(d(z, Tz)) \leq 0$ so that $\varphi(d(z, Tz)) = 0$. Therefore $Tz = z$ and z is a fixed point of T.

Uniqueness of fixed point follows trivially by applying the inequality (5).

We now show that $\{T^n x\}$ is Cauchy. Suppose that the sequence $\{T^n x\}$ is not Cauchy. Then from Lemma 2.3, there exist $\epsilon>0$ and subsequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k)>n(k)>k$ such that $d(T^{m(k)}x, T^{n(k)}x)\geq\epsilon$ and $d(T^{m(k)-1}x, T^{n(k)}x)>\epsilon$ and we have $\lim_{k\to+\infty} d(T^{m(k)}, T^{n(k)})=\epsilon$, $\lim_{k\to+\infty} d(T^{m(k)-1}x, T^{n(k)}x)=\epsilon$ and $\lim_{k\to+\infty} d(T^{m(k)-1}x, T^{n(k)-1}x)=\epsilon$. We now consider

$$\begin{aligned}
\varphi(\epsilon) &\leq \varphi(d(T^{m(k)}x, T^{n(k)}x)) = \varphi(d(T(T^{m(k)-1}x), T(T^{n(k)-1}x))) \\
&\leq a[\varphi(d(T^{m(k)-1}x, T^{m(k)}x)) + \varphi(d(T^{n(k)-1}x, T^{n(k)}x))] \\
&\quad + (1-2a)\varphi(d(T^{m(k)-1}x, T^{n(k)-1}x)) + r[\varphi(d(T^{m(k)-1}x, T^{n(k)}x)) \\
&\quad + \varphi(d(T^{n(k)-1}x, T^{m(k)}x))] \\
&\leq a[\varphi(d(T^{m(k)-1}x, T^{m(k)}x)) + \varphi(d(T^{n(k)-1}x, T^{n(k)}x))] \\
&\quad + (1-2a)\varphi(d(T^{m(k)-1}x, T^{n(k)-1}x)) + r[\varphi(d(T^{m(k)-1}x, T^{m(k)}x) \\
&\quad + d(T^{m(k)}x, T^{n(k)}x)) + \varphi(d(T^{n(k)-1}x, T^{n(k)}x) + d(T^{n(k)}x, T^{m(k)}x))].
\end{aligned}$$

On letting $k \to +\infty$, we have

$$\varphi(\epsilon) \leq a[\varphi(\lim_{k\to+\infty} d(T^{m(k)-1}x, T^{m(k)}x)) + \varphi(\lim_{k\to+\infty} d(T^{n(k)-1}x, T^{n(k)}x))]$$

$$+ (1-2a)\varphi(\lim_{k\to+\infty} d(T^{m(k)-1}x, T^{n(k)-1}x)) + r[\varphi(\lim_{k\to+\infty} d(T^{m(k)-1}x, T^{m(k)}x)$$
$$+ \lim_{k\to+\infty} d(T^{m(k)}x, T^{n(k)}x)) + \varphi(\lim_{k\to+\infty} d(T^{n(k)-1}x, T^{n(k)}x)$$
$$+ \lim_{k\to+\infty} d(T^{n(k)}x, T^{m(k)}x))].$$

Since T is asymptotically regular $\varphi(\epsilon) \le (1-2a)\varphi(\epsilon) + 2r\varphi(\epsilon) = ((1-2a)+2r)\varphi(\epsilon)$, i.e. $2(a-r)\varphi(\epsilon) \le 0$, and so $\varphi(\epsilon) \le 0$, which is a contradiction. Therefore $\{T^n x\}$ is Cauchy.

Now, by our assumption (ii), it follows that the sequence $\{T^n x\}$ converges to z in X. Hence the theorem follows. □

REMARK 2.6. Since every strictly quadratic quasi contraction map is a generalized φ-quasi contraction map with $\varphi(t) = t^2,\ t \ge 0$, it follows that Theorem 1.6 follows as a corollary of Theorem 2.4, which in turn follows from Remark 1.5 that Theorem 1.2 also follows as a corollary of Theorem 2.4 in the context of complete metric spaces.

3. Fixed points in orbitally complete metric spaces

Let (X, d) be a metric space and $T : X \to X$. For $x_0 \in X, O(x_0) = \{T^n x_0 / n = 0, 1, 2, \ldots\}$ is called the *orbit of* x_0, where $T^0 = I, I$ the identity map of X.

A metric space X is said to be *T-orbitally complete* if every Cauchy sequence which is contained in $O(x)$ for all x in X converges to a point of X.

Every complete metric space is T-orbitally complete for any T, but every T-orbitally complete metric space need not be a complete metric space. For more details, we refer to [12].

In the following, we prove the existence of fixed points of generalized φ-quasi contraction maps in T-orbitally complete metric spaces.

THEOREM 3.1. *Let (X, d) be a T-orbitally complete metric space. Assume that T is asymptotically regular at some point x_0 in X. If there exist a function $\varphi_{x_0} \in \Phi, a \in (0, \frac{1}{2})$ and $r \in (0, a)$ such that*

$$\varphi_{x_0}(d(Tx, Ty)) \le a\varphi_{x_0}(d(x, Tx)) + a\varphi_{x_0}(d(y, Ty)) + (1-2a)\varphi_{x_0}(d(x, y))$$
$$+ r[\varphi_{x_0}(d(x, Ty)) + \varphi_{x_0}(d(y, Tx))] \tag{6}$$

for all $x, y \in \overline{O(x_0)}$. Then the sequence $\{T^n x_0\}$ is Cauchy in X, $\lim_{n\to+\infty} T^n x_0 = z$, $z \in X$ and z is a fixed point of T. Further, z is unique in the sense that $\overline{O(x_0)}$ contains one and only one fixed point of T.

Proof. If we proceed as in the proof of Theorem 2.4, it follows that $\{T^n x_0\}$ is Cauchy. Since X is T-orbitally complete, we have that there exists z in X such that $z = \lim_{n\to+\infty} T^n x_0$, and $z \in \overline{O(x_0)}$. Now it follows again as in the proof of Theorem 2.4 that z is a fixed point of T, and this z is unique in $\overline{O(x_0)}$. □

Corollary 3.2. *Let (X,d) be a T-orbitally complete metric space. Assume that T is asymptotically regular at some point x_0 in X. If there exist a function $\varphi_{x_0} \in \Phi, a \in (0, \frac{1}{2})$ such that*

$$\varphi_{x_0}(d(Tx,Ty)) \leq a\varphi_{x_0}(d(x,Tx)) + a\varphi_{x_0}(d(y,Ty)) + (1-2a)\varphi_{x_0}(d(x,y)) \qquad (7)$$

for all $x, y \in \overline{O(x_0)}$. Then the sequence $\{T^n x_0\}$ is Cauchy in X, $\lim_{n\to+\infty} T^n x_0 = z, z \in X$ and z is a fixed point of T. Further, z is unique in the sense that $\overline{O(x_0)}$ contains one and only one fixed point of T.

Proof. Since the inequality (7) implies the inequality (6), the conclusion of this corollary follows from Theorem 3.1. □

4. Examples

The following examples are given in support of our results.

Example 4.1. Let $X = [-1, +\infty)$ with the usual metric. We define $T : X \to X$ by $Tx = \begin{cases} \frac{1}{3}, & -1 \leq x \leq \frac{1}{2} \\ 0, & \frac{1}{2} < x < +\infty \end{cases}$ with $\varphi(t) = t^2, t \geq 0$. We choose $a = \frac{5}{11}$ and $r = \frac{4}{11}$. Note that $0 < r < a < \frac{1}{2}$. In the following we show that T satisfies the inequality (5).

Case i) Let $x, y \in [-1, \frac{1}{2}]$. Then $\varphi(d(Tx,Ty)) = 0$. Therefore the inequality (5) holds trivially.

Case ii) Let $x, y \in (\frac{1}{2}, +\infty)$. Then $\varphi(d(Tx,Ty)) = 0$. Therefore the inequality (5) holds trivially.

Case iii) Let $x \in [-1, \frac{1}{2}]$ and $y \in (\frac{1}{2}, +\infty)$, we have $\varphi(d(Tx,Ty)) = \varphi(|\frac{1}{3} - 0|) = \varphi(\frac{1}{3}) = \frac{1}{9}$ and

$$\begin{aligned}
&a[\varphi(d(x,Tx)) + \varphi(d(y,Ty))] + (1-2a)\varphi(d(x,y)) + r[\varphi(d(x,Ty)) + \varphi(d(y,Tx))] \\
&= \frac{5}{11}\varphi(|x - \frac{1}{3}|) + \frac{5}{11}\varphi(|y|) + \frac{1}{11}\varphi(|y-x|) + \frac{4}{11}[\varphi(|x|) + \varphi(y - \frac{1}{3})] \\
&= \frac{5}{11}(x - \frac{1}{3})^2 + \frac{5}{11}y^2 + \frac{1}{11}(y-x)^2 + \frac{4}{11}[x^2 + (y - \frac{1}{3})^2] \\
&\geq \frac{5}{11}(x - \frac{1}{3})^2 + \frac{5}{11}\cdot\frac{1}{4} + \frac{1}{11}(\frac{1}{2} - x)^2 + \frac{4}{11}[x^2 + (\frac{1}{2} - \frac{1}{3})^2] \\
&= \frac{5}{11}(x - \frac{1}{3})^2 + \frac{5}{44} + \frac{1}{11}(\frac{1}{2} - x)^2 + \frac{4}{11}[x^2 + \frac{1}{36}] \geq \frac{5}{44} + \frac{1}{99} > \frac{1}{9} = \varphi(d(Tx,Ty)).
\end{aligned}$$

Case iv) Let $x \in (\frac{1}{2}, +\infty)$ and $y \in [-1, \frac{1}{2}]$. In this case, $\varphi(d(Tx,Ty)) = \varphi(|0 - \frac{1}{3}|) = \varphi(\frac{1}{3}) = \frac{1}{9}$ and

$$\begin{aligned}
&a[\varphi(d(x,Tx)) + \varphi(d(y,Ty))] + (1-2a)\varphi(d(x,y)) + r[\varphi(d(x,Ty)) + \varphi(d(y,Tx))] \\
&= \frac{5}{11}\varphi(|x|) + \frac{5}{11}\varphi(|y - \frac{1}{3}|) + \frac{1}{11}\varphi(|x-y|) + \frac{4}{11}[\varphi(|x - \frac{1}{3}|) + \varphi(y - 0)] \\
&= \frac{5}{11}x^2 + \frac{5}{11}(y - \frac{1}{3})^2 + \frac{1}{11}(x-y)^2 + \frac{4}{11}[(x - \frac{1}{3})^2 + y^2]
\end{aligned}$$

$$\geq \frac{5}{11}\cdot\frac{1}{4}+\frac{5}{11}(y-\frac{1}{3})^2+\frac{1}{11}(\frac{1}{2}-y)^2+\frac{4}{11}[(\frac{1}{2}-\frac{1}{3})^2+y^2]$$
$$=\frac{5}{44}+\frac{5}{11}(y-\frac{1}{3})^2+\frac{1}{11}(\frac{1}{2}-y)^2+\frac{4}{11}[\frac{1}{36}+y^2]\geq\frac{5}{44}+\frac{1}{99}>\frac{1}{9}=\varphi(d(Tx,Ty)).$$

Therefore from Case i) to Case iv), it follows that T satisfies the inequality (5). Also, it is easy to see that T is asymptotically regular on X. Hence T satisfies all the hypotheses of Theorem 2.4 and $\frac{1}{3}$ is the unique fixed point of T.

EXAMPLE 4.2. Let $X = [0,1]$ with the usual metric. We define $T : X \to X$ by $Tx = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x \in (0,1) \\ 0 & \text{if } x = 1 \end{cases}$ with $\varphi(t) = t^2, t \geq 0$. We choose $a = \frac{1}{4}$ and $r = \frac{1}{5}$. Note that $0 < r < a < \frac{1}{2}$. We now show that T satisfies the inequality (5).

Case i) $x = 0, y \in (0,1)$. Then $\varphi(Tx,Ty) = \varphi(|1-\frac{1}{2}|) = (\frac{1}{2})^2 = \frac{1}{4}$ and

$$a[\varphi(d(x,Tx))+\varphi(d(y,Ty))]+(1-2a)\varphi(d(x,y))+r[\varphi(d(x,Ty))+\varphi(d(y,Tx))]$$
$$=\frac{1}{4}\varphi(|0-1|)+\frac{1}{4}\varphi(|y-\frac{1}{2}|)+\frac{1}{2}\varphi(|0-y|)+\frac{1}{5}[\varphi(|0-\frac{1}{2}|)+\varphi(|y-1|)]$$
$$=\frac{1}{4}\cdot 1+\frac{1}{4}(y-\frac{1}{2})^2+\frac{1}{2}y^2+\frac{1}{5}[(\frac{1}{2})^2+(y-1)^2]\geq\frac{1}{4}+\frac{1}{20}>\frac{1}{4}=\varphi(d(Tx,Ty)).$$

Case ii) $x = 1, y \in (0,1)$. Then $\varphi(d(Tx,Ty)) = \varphi(|0-\frac{1}{2}|) = (\frac{1}{2})^2 = \frac{1}{4}$ and

$$a[\varphi(d(x,Tx))+\varphi(d(y,Ty))]+(1-2a)\varphi(d(x,y))+r[\varphi(d(x,Ty))+\varphi(d(y,Tx))]$$
$$=\frac{1}{4}\varphi(|1-0|)+\frac{1}{4}\varphi(|y-\frac{1}{2}|)+\frac{1}{2}\varphi(|1-y|)+\frac{1}{5}[\varphi(|1-\frac{1}{2}|)+\varphi(|y-0|)]$$
$$=\frac{1}{4}+\frac{1}{4}(y-\frac{1}{2})^2+\frac{1}{2}(1-y)^2+\frac{1}{5}[(\frac{1}{2})^2+y^2]\geq\frac{1}{4}+\frac{1}{20}>\frac{1}{4}=\varphi(d(Tx,Ty)).$$

Case iii) $x \in (0,1), y = 0$. Then $\varphi(d(Tx,Ty)) = \varphi(|\frac{1}{2}-1|) = \varphi(\frac{1}{2}) = (\frac{1}{2})^2 = \frac{1}{4}$ and

$$a[\varphi(d(x,Tx))+\varphi(d(y,Ty))]+(1-2a)\varphi(d(x,y))+r[\varphi(d(x,Ty))+\varphi(d(y,Tx))]$$
$$=\frac{1}{4}\varphi(|x-\frac{1}{2}|)+\frac{1}{4}\varphi(|0-1|)+\frac{1}{2}\varphi(|x-0|)+\frac{1}{5}[\varphi(|x-1|)+\varphi(|0-\frac{1}{2}|)]$$
$$=\frac{1}{4}(x-\frac{1}{2})^2+\frac{1}{4}+\frac{1}{2}x^2+\frac{1}{5}[(x-1)^2+(\frac{1}{2})^2]\geq\frac{1}{4}+\frac{1}{20}>\frac{1}{4}=\varphi(d(Tx,Ty)).$$

Case iv) $x \in (0,1), y = 1$. Then $\varphi(d(Tx,Ty)) = \varphi(|\frac{1}{2}-0|) = (\frac{1}{2})^2 = \frac{1}{4}$ and

$$a[\varphi(d(x,Tx))+\varphi(d(y,Ty))]+(1-2a)\varphi(d(x,y))+r[\varphi(d(x,Ty))+\varphi(d(y,Tx))]$$
$$=\frac{1}{4}\varphi(|x-\frac{1}{2}|)+\frac{1}{4}\varphi(|1-0|)+\frac{1}{2}\varphi(|x-1|)+\frac{1}{5}[\varphi(|x-0|)+\varphi(|1-\frac{1}{2}|)]$$
$$=\frac{1}{4}(x-\frac{1}{2})^2+\frac{1}{4}+\frac{1}{2}(x-1)^2+\frac{1}{5}[x^2+(\frac{1}{2})^2]\geq\frac{1}{4}+\frac{1}{20}>\frac{1}{4}=\varphi(d(Tx,Ty)).$$

Case v) $x, y \in (0,1)$. Then $\varphi(d(Tx,Ty)) = 0$. Therefore the inequality (5) holds trivially.

Case vi) $x = 0, y = 1$. Then $\varphi(d(Tx, Ty)) = \varphi(|1 - 0|) = 1$ and

$$a[\varphi(d(x,Tx)) + \varphi(d(y,Ty))] + (1-2a)\varphi(d(x,y)) + r[\varphi(d(x,Ty)) + \varphi(d(y,Tx))]$$
$$= \frac{1}{4}\varphi(|0-1|) + \frac{1}{4}\varphi(|1-0|) + \frac{1}{2}\varphi(|0-1|) + \frac{1}{5}[\varphi(0) + \varphi(0)]$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + \frac{1}{5}(0) = 1 = \varphi(d(Tx,Ty)).$$

Case vii) $x = 1, y = 0$. Then $\varphi(d(Tx, Ty)) = \varphi(|0 - 1|) = 1$ and

$$a[\varphi(d(x,Tx)) + \varphi(d(y,Ty))] + (1-2a)\varphi(d(x,y)) + r[\varphi(d(x,Ty)) + \varphi(d(y,Tx))]$$
$$= \frac{1}{4}\varphi(|1-0|) + \frac{1}{4}\varphi(|0-1|) + \frac{1}{2}\varphi(|1-0|) + \frac{1}{5}[\varphi(0) + \varphi(0)]$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + \frac{1}{5}(0) = 1 = \varphi(d(Tx,Ty)).$$

From all the above cases, we have T satisfies the inequality (5) and hence T is a generalized φ-quasi contraction map. Clearly T is asymptotically regular at any point of $(0, 1)$. Thus T satisfies all the hypotheses of Theorem 2.4 and $\frac{1}{2}$ is the unique fixed point of T.

Here, we observe that T is not a strict quadratic quasi contraction map. For, we choose $x = 0$ and $y = 1$. In this case,

$$\begin{aligned} d(T0, T1) = 1 &\not< a \cdot d^2(0, T0) + a \cdot d^2(1, T1) + (1-2a) \cdot d^2(0, 1) \\ &= a \cdot 1 + a \cdot 1 + (1-2a) \cdot 1 = 1 \end{aligned}$$

for any $a \in (0, \frac{1}{2})$. Hence, the class of generalized φ-quasi contration maps are more general than the class of strict quadratic quasi contraction maps (Remark 2.2).

Now, from Remark 2.6, it follows that Theorem 2.4 is a generalization of Theorem 1.6, which in turn, Theorem 2.4 is also a generalization of Theorem 1.2.

Remark 4.3. If we drop the assumption "T is asymptotically regular at some fixed point x of X" in Theorem 2.4 then the conclusion of Theorem 2.4 may not hold.

Example 4.4. Let $X = \{0, 1\}$ with the usual metric. We define $T : X \to X$ by $Tx = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x = 0 \end{cases}$ with $\varphi(t) = t^2$, $t \geq 0$. For $x = 1$ and $y = 0$, we have $\varphi(d(Tx, Ty)) = \varphi(|0 - 1|) = 1$ and

$$a[\varphi(d(x,Tx)) + \varphi(d(y,Ty))] + (1-2a)\varphi(d(x,y)) + r[\varphi(d(x,Ty)) + \varphi(d(y,Tx))]$$
$$= a \cdot 1 + a \cdot 1 + (1-2a) \cdot 1 + r \cdot 0 = 1 = \varphi(d(Tx,Ty)).$$

Therefore T is a generalized φ-quasi contraction map. But T is not asymptotically regular at any point $x \in X$ and T has no fixed points. This phenomenon indicates the importance of asymptotic regularity of T in our results.

Example 4.5. Let $X = [0, +\infty)$ with the usual metric. We define $T : X \to X$ defined by $T(0) = 1, T(1) = 1 + \frac{1}{2}$, $T(\sum_{i=0}^{n} 2^{-i}) = \sum_{i=0}^{n+1} 2^{-i}$, $T(2) = 2$ with $\varphi_{x_0}(t) = t^2$. Clearly T is asymptotically regular at $x_0 = 0$. We now consider the orbit of $x_0 = 0$, i.e. $O(0) = \{\sum_{i=0}^{n} 2^{-i} : n = 0, 1, 2, \ldots\} \cup \{0\}$, and $\overline{O(0)} = O(0) \cup \{2\}$. Here, it is easy to see that the space X is T-orbitally complete.

In the following, we verify the inequality (7) by choosing $a = \frac{1}{4}$.

Case i) $x = 0, y = 1$. Then

$$\begin{aligned}\varphi_{x_0}(d(Tx, Ty)) &= \varphi_{x_0}(d(1, 1 + \frac{1}{2})) = \varphi_{x_0}(|1 - 1 - \frac{1}{2}|) = \varphi_{x_0}(| - \frac{1}{2}|) = (\frac{1}{2})^2 \\ &= \frac{1}{4} \leq \frac{1}{2} \cdot 1 = (1 - 2a)\varphi_{x_0}(d(x, y)).\end{aligned}$$

Case ii) $x = 0, y = \sum_{i=0}^{n} 2^{-i}$. In this case,

$$\begin{aligned}\varphi_{x_0}(d(Tx, Ty)) &= \varphi_{x_0}(|1 - \sum_{i=0}^{n+1} 2^{-i}|) = (\sum_{i=1}^{n+1} 2^{-i})^2 \\ &= (\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n+1}})^2 = \frac{1}{4}(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n})^2 \\ &\leq \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n})^2 = (1 - 2a)\varphi_{x_0}(d(x, y)).\end{aligned}$$

Case iii) $x = 1, y = \sum_{i=0}^{n} 2^{-i}$. Then

$$\begin{aligned}\varphi_{x_0}(d(Tx, Ty)) &= \varphi_{x_0}(|1 + \frac{1}{2} - \sum_{n=0}^{n+1} 2^{-i}|) = (\sum_{i=2}^{n+1} 2^{-i})^2 \\ &= (\frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n+1}})^2 = \frac{1}{4}(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n})^2 \\ &\leq \frac{1}{2}(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n})^2 = (1 - 2a)\varphi_{x_0}(d(x, y)).\end{aligned}$$

Case iv) $x = 2, y = 0$. Then

$$\varphi_{x_0}(d(Tx, Ty)) = \varphi_{x_0}(|2 - 1|) = 1 \leq 2 = \frac{1}{2} \cdot 4 = (1 - 2a)\varphi_{x_0}(d(x, y)).$$

Case v) $x = 2, y = 1$. Then

$$\begin{aligned}\varphi_{x_0}(d(Tx, Ty)) &= \varphi_{x_0}(|2 - 1 - \frac{1}{2}|) = \varphi_{x_0}(\frac{1}{2}) = (\frac{1}{2})^2 \\ &= \frac{1}{4} \leq \frac{1}{2} = \frac{1}{2} \cdot 1 = (1 - 2a)\varphi_{x_0}(d(x, y)).\end{aligned}$$

Case vi) $x = 2, y = \sum_{i=0}^{n} 2^{-i}$. Then

$$\begin{aligned}\varphi_{x_0}(d(Tx, Ty)) &= \varphi_{x_0}(|2 - \sum_{i=0}^{n+1} 2^{-1}|) = (2 - \sum_{i=0}^{n+1} 2^{-i})^2 = (1 - \sum_{i=1}^{n+1} 2^{-i})^2 \\ &= \frac{1}{4}(\sum_{i=n+1}^{+\infty} 2^{-i})^2 \leq \frac{1}{2}(\sum_{i=n+1}^{+\infty} 2^{-i})^2 \leq (1 - 2a)\varphi_{x_0}(d(x, y)).\end{aligned}$$

Case vii) $x = \sum_{i=0}^{n} 2^{-i}, y = \sum_{i=0}^{m} 2^{-i}$. Let $m > n$

$$\varphi_{x_0}(d(Tx, Ty)) = \varphi_{x_0}(|\sum_{i=0}^{n+1} 2^{-i} - \sum_{i=0}^{m+1} 2^{-i}|) = \varphi_{x_0}(\sum_{i=n+2}^{m+1} 2^{-i}) = \frac{1}{4}(\sum_{i=n+1}^{m} 2^{-i})^2$$

$$\leq \frac{1}{2}(\sum_{i=n+1}^{m} 2^{-i})^2 = (1-2a)\varphi_{x_0}(d(x,y)).$$

Therefore from all the above possible cases, it follows that T satisfies the inequality

$$\begin{aligned}\varphi_{x_0}(d(Tx,Ty)) &\leq (1-2a)\varphi_{x_0}(d(x,y)) \\ &\leq a\varphi_{x_0}(d(x,Tx)) + a\varphi_{x_0}(d(y,Ty)) + (1-2a)\varphi_{x_0}(d(x,y))\end{aligned}$$

for all $x, y \in \overline{O(x_0)}$. Hence the inequality (7) holds with $a = \frac{1}{4}$. Therefore T satisfies all the hypotheses of Corollary 3.2 and 2 is the unique fixed point of T in the closure of the orbit of 0, i.e. $\overline{O(0)}$ and $\lim_{n\to+\infty} T^n 0 = 2$.

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