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DIAGONAL COMPLETION OF QUANTALE-VALUED CAUCHY TOWER SPACES

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Abstract. We study certain diagonal axioms defined for quantale-valued Cauchy tower spaces and their relations to similar diagonal axioms for quantale-valued convergence tower spaces and quantale-valued uniform limit tower spaces. We construct a completion for a quantale-valued Cauchy tower space that preserves a diagonal axiom and show that our construction is the coarsest possible such completion.

1. Introduction

It is well-known that Cauchy spaces [13] are a natural setting for studying completeness and completions [25]. Richardson and Kent introduced probabilistic versions of Cauchy spaces [26] using a whole family of spaces indexed by the unit interval. This line of research was followed and generalized by Nusser [21, 22]. However, despite the name, the connection to probabilistic metric spaces remained unclear. In a previous paper [12] we extended these approaches even further by allowing the index set to be a quantale. This allowed to established natural Cauchy towers in particular for probabilistic metric spaces, but also classical metric spaces are covered. The completion of quantale-valued metric spaces (or L-categories) has been dealt with in many papers, see e.g. [4, 6, 16]. Here we address the completion of a subcategory of the category of quantale-valued Cauchy tower spaces which is close to quantale-valued metric spaces, by studying diagonal conditions. We construct a so-called diagonal completion of a diagonal quantale-valued Cauchy tower space which is in a sense the coarsest of such completions.

The paper is organized as follows. In the next section we collect the necessary background about quantales and filters and we fix the notation. Section 3 reviews quantale-valued Cauchy tower spaces and quantale-valued convergence spaces and

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gives several examples. Suitable diagonal axioms are introduced and studied in Section 4 and in Section 5 we give a completion construction which preserves a diagonal axiom. Finally we draw some conclusions.

2. Preliminaries

A *commutative and integral quantale* is a triple $\mathsf{L} = (L, \leq, *)$ with a complete lattice $(L, \leq)$, with distinct top element $\top$ and bottom element $\bot$, and a commutative semigroup $(L, *)$, with $\top$ as the unit and $*$ is distributive over arbitrary joins, i.e. if we have $(\bigvee_{i\in J} \alpha_i) * \beta = \bigvee_{i\in J}(\alpha_i * \beta)$ for all $\alpha_i, \beta \in L, i \in J$. We will, in the sequel, simply speak of a quantale.

Important examples of quantales are the unit interval $[0,1]$ with a left-continuous *t-norm* [27] or *Lawvere's quantale*, the interval $[0,\infty]$ with the opposite order and addition $\alpha * \beta = \alpha + \beta$ (extended by $\alpha + \infty = \infty + \alpha = \infty$), see e.g. [5]. A further noteworthy example is the quantale of distance distribution functions with a sup-continuous triangle function as semigroup operation, see e.g. [5]. This quantale is important in the theory of probabilistic metric spaces [27].

In some places we need the *well-below relation* in a complete lattice: $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \lhd \beta$ and, for a subset $B \subseteq L$, we have $\alpha \lhd \bigvee_{\beta\in B} \beta$ iff $\alpha \lhd \beta$ for some $\beta \in B$. A complete lattice is completely distributive, if and only if we have $\alpha = \bigvee\{\beta : \beta \lhd \alpha\}$ for any $\alpha \in L$

For a set X, we denote its power set by $\mathsf{P}(X)$ and the set of all (proper) filters $\mathbb{F}, \mathbb{G}, \ldots$ on X by $\mathsf{F}(X)$. The set $\mathsf{F}(X)$ is ordered by set inclusion and maximal elements of $\mathsf{F}(X)$ in this order are called *ultrafilters*. In particular, for each $x \in X$, the *point filter* $[x] = \{A \subseteq X : x \in A\}$ is an ultrafilter. More general, for $A \subseteq X$ we denote $[A] = \{F \subseteq X : A \subseteq X\} \in \mathsf{F}(X)$. If $\mathbb{F} \in \mathsf{F}(X)$ and $f : X \longrightarrow Y$ is a mapping, then we define $f(\mathbb{F}) \in \mathsf{F}(Y)$ by $f(\mathbb{F}) = \{G \subseteq Y : f(F) \subseteq G \text{ for some } F \in \mathbb{F}\}$. For filters $\Phi, \Psi \in \mathsf{F}(X \times X)$ we define Φ^{-1} to be the filter generated by the filter base $\{F^{-1} : F \in \Phi\}$ where $F^{-1} = \{(x,y) \in X \times X : (y,x) \in F\}$ and $\Phi \circ \Psi$ to be the filter generated by the filter base $\{F \circ G : F \in \Phi, G \in \Psi\}$, whenever $F \circ G \neq \emptyset$ for all $F \in \Phi, G \in \Psi$, where $F \circ G = \{(x,y) \in X \times X : (x,s) \in F, (s,y) \in G \text{ for some } s \in X\}$.

For a set J and a *selection function* $\sigma : J \longrightarrow \mathsf{F}(X)$ and $A \subseteq X$ we define $A^\sigma = \{j \in J : A \in \sigma(j)\}$. We then have for $A, B \subseteq X$ that $X^\sigma = J$ and that $A \subseteq B$ implies $A^\sigma \subseteq B^\sigma$; and $(A \cap B)^\sigma = A^\sigma \cap B^\sigma$. We define now for a selection function σ and for a filter $\mathbb{G} \in \mathsf{F}(J)$, the *diagonal filter* $\kappa\sigma(\mathbb{G}) \in \mathsf{F}(X)$ of $(\mathbb{G}, \sigma)$ by $A \in \kappa\sigma(\mathbb{G})$ if $A^\sigma \in \mathbb{G}$.

For details and notation from category theory we refer to [1, 24].

3. L-convergence tower spaces and L-Cauchy tower spaces

DEFINITION 3.1 ([11]). Let $\mathsf{L} = (L, \leq, *)$ be a quantale. A pair $(X, \overline{q} = (q_\alpha)_{\alpha \in L})$ is called an L-*convergence tower space*, if $(q_\alpha \colon \mathsf{F}(X) \longrightarrow \mathsf{P}(X))_{\alpha \in L}$ is a family of mappings satisfying:
(LCTS1) $x \in q_\alpha([x])$, $\forall x \in X, \alpha \in L$;

(LCTS2) $\forall \mathbb{F}, \mathbb{G} \in \mathsf{F}(X)$, with $\mathbb{F} \leq \mathbb{G}$, and $\alpha \in L$ implies $q_\alpha(\mathbb{F}) \subseteq q_\alpha(\mathbb{G})$;

(LCTS3) $\forall \alpha, \beta \in L$ with $\alpha \leq \beta$ implies $q_\beta(\mathbb{F}) \subseteq q_\alpha(\mathbb{F})$, $\forall \mathbb{F} \in \mathsf{F}(X)$;

(LCTS4) $x \in q_\perp(\mathbb{F})$, $\forall x \in X$, $\mathbb{F} \in \mathsf{F}(X)$.
If, moreover, $(X, \overline{q})$ satisfies
(LCTS5) $q_\alpha(\mathbb{F}) \cap q_\alpha(\mathbb{G}) \leq q_\alpha(\mathbb{F} \wedge \mathbb{G})$, $\forall \alpha \in L$, for $\mathbb{F}, \mathbb{G} \in \mathsf{F}(X)$,
then we call $(X, \overline{q})$ an L-*limit tower space.* If $(X, \overline{q})$ satisfies $x \in q_{\vee A}(\mathbb{F})$ whenever $x \in q_\alpha(\mathbb{F})$ $\forall \alpha \in A$, it is called *left-continuous.* A mapping $f \colon (X, \overline{q}) \longrightarrow \big(X', \overline{q'}\big)$ between L-convergence tower spaces is called *continuous* if, for all $x \in X$, and for all $\mathbb{F} \in \mathsf{F}(X)$, $f(x) \in q'_\alpha(f(\mathbb{F}))$ whenever $x \in q_\alpha(\mathbb{F})$. The category of all L-convergence tower spaces and continuous mappings is denoted by $\mathsf{L\text{-}CTS}$.

If $L = \{0, 1\}$, then L-convergence tower spaces can be identified with classical convergence spaces, [3, 24]. For Lawvere's quantale $\mathsf{L} = ([0, \infty], \geq, +)$, an L-limit tower space is a limit tower space [2] and a left-continuous L-limit tower space is an approach limit spaces in the sense of Lowen [17]. For $\mathsf{L} = ([0, 1], \leq, *)$, we obtain probabilistic convergence spaces in the sense of Richardson and Kent, [26] and for the quantale of distance distribution functions an L-convergence tower space is a probabilistic convergence space in the definition of [7].

DEFINITION 3.2 ([12]). Let $\mathsf{L} = (L, \leq, *)$ be a quantale. A pair $(X, \overline{C}) = (X, (C_\alpha)_{\alpha \in L})$ is called an L-*Cauchy tower space*, where $C_\alpha \subseteq \mathsf{F}(X)$ for all $\alpha \in L$, if
(LChyTS1) $[x] \in C_\alpha$ for all $x \in X, \alpha \in L$;

(LChyTS2) $\mathbb{G} \geq \mathbb{F} \in C_\alpha$ implies $\mathbb{G} \in C_\alpha$;

(LChyTS3) $\alpha \leq \beta, \mathbb{F} \in C_\beta$ implies $\mathbb{F} \in C_\alpha$;

(LChyTS4) $C_\perp = \mathbb{F}(X)$.

(LChyTS5) $\mathbb{F} \in C_\alpha, \mathbb{G} \in C_\beta, \mathbb{F} \vee \mathbb{G}$ exists, implies $\mathbb{F} \wedge \mathbb{G} \in C_{\alpha * \beta}$.
An L-Cauchy tower space is called *left-continuous* if $\mathbb{F} \in C_\alpha$ for all $\alpha \in A \subseteq L$ implies $\mathbb{F} \in C_{\bigvee A}$.

We call a mapping $f : (X, \overline{C}) \longrightarrow (X', \overline{C'})$ *Cauchy-continuous* if $\mathbb{F} \in C_\alpha$ implies $f(\mathbb{F}) \in C'_\alpha$. The category with the L-Cauchy tower spaces as objects and Cauchy-continuous mappings as morphisms is denoted by $\mathsf{L\text{-}ChyTS}$.

For $L = \{0, 1\}$ we obtain the classical Cauchy spaces [13]. For $\mathsf{L} = ([0, 1], \leq, *)$ with a continuous t-norm $*$, an L-Cauchy tower space is a probabilistic Cauchy space in the definition of Nusser [21, 22] and if $* = \min$, then we obtain the probabilistic

Cauchy spaces of Kent and Richardson [15]. In case of Lawever's quantale L, an L-Cauchy tower space is a Cauchy tower space in the definition of [20], which can be identified in the left-continuous case with an approach Cauchy space [18, 19].

We note that for $(X, \overline{C}) \in |\mathsf{L\text{-}ChyTS}|$, the "top level" $(X, C_\top)$ is a classical Cauchy space [13].

Proposition 3.3 ([12]). *The category* $\mathsf{L\text{-}ChyTS}$ *is topological. If the quantale operation is the minimum, i.e. if* $* = \wedge$, *then it is also a Cartesian closed category.*

As the category $\mathsf{L\text{-}ChyTS}$ is topological we can do initial constructions as follows, see [12]. For $(f_j : X \longrightarrow (X_j, \overline{C^j}))_{j \in J}$ we define for $\mathbb{F} \in \mathsf{F}(X)$, $\mathbb{F} \in C_\alpha \iff f_j(\mathbb{F}) \in C^j_\alpha$ for all $j \in J$.

We define, for an L-Cauchy tower space the underlying L-limit tower space by [12]

$$x \in q^{\overline{C}}_\alpha(\mathbb{F}) \iff \mathbb{F} \wedge [x] \in C_\alpha.$$

Proposition 3.4. *Let* $(X, \overline{C})$ *be an* L*-Cauchy tower space. Then*

$$x \in q^{\overline{C}}_\alpha(\mathbb{F}) \iff \exists \mathbb{G} \in C_\alpha, \mathbb{G} \leq \mathbb{F}, x \in \bigcap_{G \in \mathbb{G}} G.$$

Proof. Let first $x \in q^{\overline{C}}_\alpha(\mathbb{F})$. Then $\mathbb{G} = \mathbb{F} \wedge [x] \in C_\alpha$ and $\mathbb{G} \leq \mathbb{F}$ and $x \in \bigcap_{G \in \mathbb{G}} G$.

For the converse, we conclude from $\mathbb{G} \in C_\alpha$, $x \in \bigcap_{G \in \mathbb{G}} G$ that $\mathbb{G} \leq [x]$ and hence $\mathbb{G} = \mathbb{G} \wedge [x] \in C_\alpha$, i.e. $x \in q^{\overline{C}}_\alpha(\mathbb{G})$. From (LCTS2) then also $x \in q^{\overline{C}}_\alpha(\mathbb{F})$. □

An L-Cauchy tower space $(X, \overline{C})$ is called a *T1-space* if $(X, \overline{q^{\overline{C}}})$ is a T1-space [9], i.e. if $[x] \wedge [y] \in C_\top$ implies $x = y$. It is called a *T2-space* if $(X, \overline{q^{\overline{C}}})$ is a T2-space [9], i.e. if $\mathbb{F} \wedge [x], \mathbb{F} \wedge [y] \in C_\top$ implies $x = y$. It has been shown in [12] that for L-Cauchy tower spaces, the T1-axiom and the T2-axiom are equivalent.

Example 3.5 (L-metric spaces [4, 16]). Natural examples are given by L-metric spaces (X, d) with $d : X \times X \longrightarrow L$ with the properties $d(x, x) = \top$ and $d(x, y) * d(y, z) \leq d(x, z)$. Whenever we discuss L-metric spaces in this paper, we assume that L is completely distributive. For an L-metric space (X, d) we define $x \in q^d_\alpha(\mathbb{F}) \iff \bigvee_{F \in \mathbb{F}} \bigwedge_{xy \in F} d(x, y) \geq \alpha$. Then $(X, \overline{q^d})$ is an L-convergence tower space, see [11]. Similarly, we define $\mathbb{F} \in C^d_\alpha \iff \bigvee_{F \in \mathbb{F}} \bigwedge_{x, y \in F} d(x, y) \geq \alpha$. Then $(X, \overline{C^d})$ is an L-Cauchy tower space, see [12].

Example 3.6 ([8, L-uniform limit tower spaces]). For a set X we call a family $\overline{\Lambda} = (\Lambda_\alpha)_{\alpha \in L}$, with $\Lambda_\alpha \subseteq \mathsf{F}(X \times X)$, which satisfies the axioms, for all $\alpha, \beta \in L$,

(LUC1) $[(x, x)] \in \Lambda_\alpha$ for all $x \in X$;

(LUC2) $\Psi \in \Lambda_\alpha$ whenever $\Phi \leq \Psi$ and $\Phi \in \Lambda_\alpha$;

(LUC3) $\Phi \wedge \Psi \in \Lambda_\alpha$ whenever $\Phi, \Psi \in \Lambda_\alpha$;

(LUC4) $\Lambda_\beta \subseteq \Lambda_\alpha$ whenever $\alpha \leq \beta$;

(LUC5) $\Phi^{-1} \in \Lambda_\alpha$ whenever $\Phi \in \Lambda_\alpha$;

(LUC6) $\Phi \circ \Psi \in \Lambda_{\alpha * \beta}$ whenever $\Phi \in \Lambda_\alpha$, $\Psi \in \Lambda_\beta$ and $\Phi \circ \Psi$ exists;

(LUC7) $\Lambda_\perp = \mathsf{F}(X \times X)$

an L-*uniform convergence tower on* X and we call the pair $(X, \overline{\Lambda})$ an L-*uniform convergence tower space* [8]. A mapping $f : (X, \overline{\Lambda}) \longrightarrow (X', \overline{\Lambda'})$ between L-uniform convergence tower spaces is called *uniformly continuous* if $(f \times f)(\Phi) \in \Lambda'_\alpha$ whenever $\Phi \in \Lambda_\alpha$. The category of L-uniform convergence tower spaces with uniformly continuous mappings as morphisms is denoted by L-UCTS.

We define, for an L-uniform limit tower space (X, Λ),

$$x \in q_\alpha^{\overline{\Lambda}}(\mathbb{F}) \iff [x] \times \mathbb{F} \in \Lambda_\alpha,$$
$$\mathbb{F} \in C_\alpha^{\overline{\Lambda}} \iff \mathbb{F} \times \mathbb{F} \in \Lambda_\alpha.$$

Then $(X, \overline{q^{\overline{\Lambda}}})$ is an L-convergence tower space and $(X, \overline{C^{\overline{\Lambda}}})$ is an L-Cauchy tower space. Moreover, we have $q_\alpha^{\overline{C^{\overline{\Lambda}}}}(\mathbb{F}) \subseteq q_\alpha^{\overline{\Lambda}}(\mathbb{F}) \subseteq q_{\alpha*\alpha}^{\overline{C^{\overline{\Lambda}}}}(\mathbb{F})$ for all $\mathbb{F} \in \mathsf{F}(X)$ and all $\alpha \in L$, see [10].

4. Diagonal axioms for quantale-valued Cauchy tower spaces

In [9] we introduced the following diagonal axiom for an L-convergence tower space $(X, \overline{q})$ and a mapping $\gamma : L \times L \longrightarrow L$. We say that $(X, \overline{q})$ satisfies the axiom (LF-γ) if

$$\forall J, \psi : J \longrightarrow X, \sigma : J \longrightarrow \mathsf{F}(X), \mathbb{G} \in \mathsf{F}(J), x \in X, \alpha, \beta \in L :$$
$$x \in q_\alpha(\psi(\mathbb{G})), \psi(j) \in q_\beta(\sigma(j)) \forall j \in J \Longrightarrow x \in q_{\gamma(\alpha,\beta)}(\kappa\sigma(\mathbb{G})).$$

Proposition 4.1 ([9]). *Let (X, d) be an L-metric space. Then $(X, \overline{q^d})$ satisfies (LF-γ) with $\gamma(\alpha, \beta) = \alpha * \beta$.*

Let now $(X, \overline{C})$ be an L-Cauchy tower space and let $\gamma : L \times L \longrightarrow L$ be a fixed mapping. We say that $(X, \overline{C})$ satisfies the axiom (LCF-γ) if

$$\forall J, \psi : J \longrightarrow X, \mathbb{G} \in \mathsf{F}(J), \sigma : J \longrightarrow \mathsf{F}(X), \alpha, \beta \in L :$$
$$\psi(\mathbb{G}) \in C_\alpha, \sigma(j) \wedge [\psi(j)] \in C_\beta \forall j \in J \Longrightarrow \kappa\sigma(\mathbb{G}) \in C_{\gamma(\alpha,\beta)}.$$

The next two results show that the axiom (LCF-γ) behaves somewhat complicated, however they also underline why it makes sense to introduce the diagonal axioms depending on a mapping $\gamma : L \times L \longrightarrow L$.

Proposition 4.2. *Let (X, d) be an L-metric space and let $\gamma(\alpha, \beta) = \alpha * \beta * \beta$. Then $(X, \overline{C^d})$ satisfies (LCF-γ).*

Proof. Let $\psi(\mathbb{G}) \in C_\alpha^d$ let $\sigma : J \longrightarrow \mathsf{F}(X)$ such that for all $j \in J$, we have $\sigma(j) \wedge [\psi(j)] \in C_\beta^d$. Let $\alpha' \lhd \alpha$ and $\beta' \lhd \beta$. Then there is $G \in \mathbb{G}$ such that for all $i, j \in G$ we have $d(\psi(i), \psi(j)) \geq \alpha'$ and for every $j \in J$ there is $F_j \in \sigma(j)$ such that for all $u, v \in F_j \cup \{\psi(j)\}$ we have $d(u, v) \geq \beta'$. In particular we have $d(u, \psi(j)) \geq \beta'$ and $d(\psi(j), v) \geq \beta'$ for all $u \in F_j$. We define $H = \bigcup_{j \in G} F_j$. Then $H \in \sigma(j)$ for all $j \in G$, i.e. we have $G \subseteq H^{\sigma}$ and therefore $H^{\sigma} \in \mathbb{G}$ which means that $H \in \kappa\sigma(\mathbb{G})$. For $a, b \in H$ there are $j_a, j_b \in G$ with $a \in F_{j_a}$ and $b \in F_{j_b}$. Hence $d(a, \psi(j_a)) \geq \beta'$, $d(\psi(j_b), b) \geq \beta'$ and $d(\psi(j_a), \psi(j_b)) \geq \alpha'$ and we obtain $d(a, b) \geq d(a, \psi(j_a)) * d(\psi(j_a), \psi(j_b)) * d(\psi(j_b), b) \geq \beta' * \alpha' * \beta'$. We conclude $\bigvee_{H \in \kappa\sigma(\mathbb{G})} \bigwedge_{a,b \in H} d(a, b) \geq \beta' * \alpha' * \beta'$. The complete distributivity then yields $\kappa\sigma(\mathbb{G}) \in C^d_{\alpha*\beta*\beta}$. □

PROPOSITION 4.3. *Let the* L*-Cauchy tower space* $(X, \overline{C})$ *satisfy (LCF-*γ*). Then* $(X, \overline{q^{\overline{C}}})$ *satisfies (LF-*γ'*) with* $\gamma'(\alpha, \beta) = \alpha * \gamma(\alpha, \beta)$*.*

Proof. Let $x \in q^{\overline{C}}_{\alpha}(\psi(\mathbb{G}))$ and $\psi(j) \in q^{\overline{C}}_{\beta}(\sigma(j))$ for all $j \in J$. Then $\psi(\mathbb{G}) \wedge [x] \in C_{\alpha}$ and $\sigma(j) \wedge [\psi(j)] \in C_{\beta}$ for all $j \in J$. By (LChyTS2) then also $\psi(\mathbb{G}) \in C_{\alpha}$ and (LChyTS5) and (LChyTS1) imply $(\sigma(j) \wedge [\psi(j)]) \wedge [\psi(j)] \in C_{\beta}$ for all $j \in J$. From the axiom (LCF-γ) we conclude $\kappa(\sigma(\cdot) \wedge [\psi(\cdot)])(\mathbb{G}) \in C_{\gamma(\alpha,\beta)}$.

We next show that $\kappa(\sigma(\cdot) \wedge [\psi(\cdot)])(\mathbb{G}) \leq \psi(\mathbb{G})$. Let $H \in \kappa(\sigma(\cdot) \wedge [\psi(\cdot)])(\mathbb{G})$. Then there is $G \in \mathbb{G}$ such that for all $j \in G$ we have $H \in \sigma(j) \wedge [\psi(j)] \leq [\psi(j)]$, i.e. we have $\psi(j) \in H$. Hence $\psi(G) \subseteq H$ and we have $H \in \psi(\mathbb{G})$.

Therefore $(\psi(\mathbb{G}) \wedge [x]) \vee \kappa(\sigma(\cdot) \wedge [\psi(\cdot)])(\mathbb{G})$ exists and hence, by (LChyTS5), $\kappa(\sigma(\cdot) \wedge [\psi(\cdot)])(\mathbb{G}) \wedge [x] \in C_{\alpha*\gamma(\alpha,\beta)}$. Now we note that $\kappa\sigma(\mathbb{G}) \geq \kappa(\sigma(\cdot) \wedge [\psi(\cdot)])(\mathbb{G})$ and obtain finally $\kappa\sigma(\mathbb{G}) \wedge [x] \in C_{\alpha*\gamma(\alpha,\beta)}$, i.e. $x \in q^{\overline{C}}_{\alpha*\gamma(\alpha,\beta)}(\kappa\sigma(\mathbb{G}))$. □

For an L-uniform limit tower space (X, Λ) the axiom (LUF-γ) is defined similarly [10]:

$$\forall J, \psi : J \longrightarrow X \times X, \sigma : J \longrightarrow \mathsf{F}(X \times X), \mathbb{G} \in \mathsf{F}(J), \alpha, \beta \in L :$$
$$\psi(\mathbb{G}) \in \Lambda_{\alpha}, \psi(j) \in q^{\overline{\Lambda}}_{\beta} \times q^{\overline{\Lambda}}_{\beta}(\sigma(j)) \forall j \in J \Longrightarrow \kappa\sigma(\mathbb{G}) \in \Lambda_{\gamma(\alpha,\beta)}.$$

PROPOSITION 4.4. *Let the* L*-uniform limit tower space* $(X, \overline{\Lambda})$ *satisfy (LUF-*γ*). Then* $(X, \overline{C^{\overline{\Lambda}}})$ *satisfies (LCF-*γ*).*

Proof. Let J be a set, $\psi : J \longrightarrow X$, $\mathbb{G} \in \mathsf{J}$ and $\sigma : J \longrightarrow \mathsf{F}(X)$ such that $\psi(\mathbb{G}) \in C^{\Lambda}_{\alpha}$ and $\sigma(j) \wedge [\psi(j)] \in C^{\Lambda}_{\beta}$ for all $j \in J$. We define $\widetilde{J} = J \times J$ and $\widetilde{\psi} : \widetilde{J} \longrightarrow X \times X$ by $\widetilde{\psi}(i, j) = (\psi(i), \psi(j))$. Furthermore, we define the selection function $\widetilde{\sigma} : \widetilde{J} \longrightarrow \mathsf{F}(X \times X)$ by $\widetilde{\sigma}(i, j) = \sigma(i) \times \sigma(j)$ and $\widetilde{\mathbb{G}} = \mathbb{G} \times \mathbb{G} \in \mathsf{F}(\widetilde{J})$. Then $\widetilde{\psi}(\widetilde{\mathbb{G}}) = \psi(\mathbb{G}) \times \psi(\mathbb{G}) \in \Lambda_{\alpha}$. Also, as $(\sigma(j) \wedge [\psi(j)]) \times (\sigma(j) \wedge [\psi(j)]) \in \Lambda_{\beta}$, we conclude $[\psi(j)] \times \sigma(j) \in \Lambda_{\beta}$ and hence $\psi(j) \in q^{\overline{\Lambda}}_{\beta}(\sigma(j))$ for all $j \in J$. This implies $\widetilde{\psi}(i, j) \in q^{\overline{\Lambda}}_{\beta} \times q^{\overline{\Lambda}}_{\beta}(\widetilde{\sigma}(i, j))$ for all $(i, j) \in \widetilde{J}$. The axiom (LUF-γ) then yields $\kappa\widetilde{\sigma}(\widetilde{\mathbb{G}}) \in \Lambda_{\gamma(\alpha,\beta)}$ and we finally show that $\kappa\widetilde{\sigma}(\widetilde{\mathbb{G}}) \leq \kappa\sigma(\mathbb{G}) \times \kappa\sigma(\mathbb{G})$. To this end, let $A \in \kappa\widetilde{\sigma}(\widetilde{\mathbb{G}})$. Then $A^{\widetilde{\sigma}} \in \widetilde{\mathbb{G}} = \mathbb{G} \times \mathbb{G}$. For $(i, j) \in A^{\widetilde{\sigma}}$ we have $A \in \widetilde{\sigma}(i, j) = \sigma(i) \times \sigma(j)$. Hence, there are $A_i \in \sigma(i)$ and $A_j \in \sigma(j)$ auch that $A_i \times A_j \subseteq A$. It follows that $(i, j) \in A^{\sigma}_i \times A^{\sigma}_j$ and so we have $A^{\widetilde{\sigma}} \subseteq A^{\sigma}_i \times A^{\sigma}_j$. As $A^{\widetilde{\sigma}} \in \mathbb{G} \times \mathbb{G}$, there is $G \in \mathbb{G}$ such that $G \times G \subseteq A^{\sigma}_i \times A^{\sigma}_j$, i.e. we have $A^{\sigma}_i, A^{\sigma}_j \in \mathbb{G}$, which means that $A_i, A_j \in \kappa\sigma(\mathbb{G})$. Therefore $A \in \kappa\sigma(\mathbb{G}) \times \kappa\sigma(\mathbb{G})$. The axiom (LUC2) yields $\kappa\sigma(\mathbb{G}) \times \kappa\sigma(\mathbb{G}) \in \Lambda_{\gamma(\alpha,\beta)}$ and we conclude $\kappa\sigma(\mathbb{G}) \in C^{\overline{\Lambda}}_{\gamma(\alpha,\beta)}$, which completes the proof. □

As in general the preservation properties of the axiom (LCF-γ) are complicated, weaker diagonal conditions make sense. We say that the L-Cauchy tower space $(X, \overline{C})$ satisfies the axiom (WLCF), or is *diagonal*, if

$$\forall J, \psi : J \longrightarrow X, \mathbb{G} \in \mathsf{F}(J), \sigma : J \longrightarrow \mathsf{F}(X), \alpha \in L :$$
$$\psi(\mathbb{G}) \in C_\alpha, \sigma(j) \wedge [\psi(j)] \in C_\top \ \forall j \in J \Longrightarrow \kappa\sigma(\mathbb{G}) \in C_\alpha.$$

If the mapping $\gamma : L \times L \longrightarrow L$ satisfies $\gamma(\alpha, \top) = \alpha$, then (LCF-$\gamma$) implies (WLCF).

Proposition 4.5. *Let (X, d) be an L-metric space. Then $(X, \overline{C^d})$ satisfies (WLCF).*

Proof. According to Proposition 4.2, $(X, \overline{C^d})$ satisfies (LCF-γ) with $\gamma(\alpha, \beta) = \alpha * \beta * \beta$. As $\gamma(\alpha, \top) = \alpha$, the remark above shows that $(X, \overline{C^d})$ satisfies (WLCF). □

For an L-convergence tower space $(X, \overline{q})$ we similarly say that it satisfies the axiom (WLF) if

$$\forall J, \psi : J \longrightarrow X, \mathbb{G} \in \mathsf{F}(J), \sigma : J \longrightarrow \mathsf{F}(X), \alpha \in L :$$
$$x \in q_\alpha(\psi(\mathbb{G})), \psi(j) \in q_\top(\sigma(j)) \forall j \in J \Longrightarrow x \in q_{\alpha*\alpha}(\kappa\sigma(\mathbb{G})).$$

If the axiom (WLF) is satisfied for $(X, \overline{q})$, a weak pretopological axiom is valid. See in this regard also Proposition 4.6 in [9], the proof of which can easily be adapted.

Proposition 4.6. *If $(X, \overline{q})$ satisfies (WLF) and $x \in q_\top(\mathbb{F}_j)$ for all $j \in J$, then $x \in q_\top(\bigwedge_{j \in J} \mathbb{F}_j)$.*

The next proposition can be seen similar to the case of the axiom (LF), see [9].

Proposition 4.7. *For an L-metric space (X, d), the space $(X, \overline{q^d})$ satisfies (WLF).*

Proposition 4.8. *Let the L-Cauchy tower space $(X, \overline{C})$ satisfy (WLCF). Then $(X, \overline{q^{\overline{C}}})$ satisfies (WLF).*

Proof. This is similar to the proof of Proposition 3.4 and not shown. □

An L-uniform limit tower space $(X, \overline{\Lambda})$ satisfies the axiom (WLUF) if

$$\forall J, \psi : J \longrightarrow X \times X, \sigma : J \longrightarrow \mathsf{F}(X \times X), \mathbb{G} \in \mathsf{F}(J), \alpha \in L :$$
$$\psi(\mathbb{G}) \in \Lambda_\alpha, \psi(j) \in q_\top^{\overline{\Lambda}} \times q_\top^{\overline{\Lambda}}(\sigma(j)) \forall j \in J \Longrightarrow \kappa\sigma(\mathbb{G}) \in \Lambda_\alpha.$$

Again, the proof of the following proposition is not shown because it is similar to the proof of Proposition 4.4.

Proposition 4.9. *Let the L-uniform limit tower space $(X, \overline{\Lambda})$ satisfy (WLUF). Then $(X, \overline{C^{\overline{\Lambda}}})$ satisfies (WLCF).*

The axioms (LCF-γ) and (WLCF) are preserved by initial constructions.

Proposition 4.10. *Let $(X_k, \overline{C^k})$ satisfy the axiom (LCF-γ) (resp. the axiom (WLCF)) for all $k \in K$ and let $(f_k : X \longrightarrow (X_\lambda, \overline{C^k}))_{k \in K}$ be a source and $\overline{C}$ the initial structure on X w.r.t. this source. Then $(X, \overline{C})$ satisfies the axiom (LCF-γ) (resp. the axiom (WLCF)).*

Proof. We only show (LCF-γ), the other axiom is similar. Let $(f_k : X \longrightarrow (X_k, \overline{C^k}))_{k\in K}$ be a source and let all $(X_k, \overline{C^k})$ satisfy the axiom (LCF-γ). Let J be a set and let $\psi : J \longrightarrow X$, $\sigma : J \longrightarrow \mathsf{F}(X)$ and let $\mathbb{G} \in \mathsf{F}(J)$ such that $\psi(\mathbb{G}) \in C_\alpha$ and $\sigma(j)\wedge[\psi(j)] \in C_\beta$ for all $j \in J$. We define, for $k \in K$, $\psi_k = f_k \circ \psi$ and $\sigma_k = f_k \circ \psi$. Then, for all $k \in K$, we have $\psi_k(\mathbb{G}) = f_k(\psi(\mathbb{G})) \in C^k_\alpha$ and $\sigma_k(j) \wedge [\psi_k(j)] = f_k(\sigma(j) \wedge [\psi(j)]) \in C^k_\beta$, and therefore, by (LCF-γ) we conclude $\kappa\sigma_k(\mathbb{G}) \in C^k_{\gamma(\alpha,\beta)}$. We have $A \in \kappa\sigma_k(\mathbb{G})$ if, and only if, $(f_k^{\leftarrow}(A))^\sigma = A^{\sigma_k} \in \mathbb{G}$ if, and only if $A \in f_k(\kappa\sigma(\mathbb{G}))$. Hence, for all $k \in K$, we have $f_k(\kappa\sigma(\mathbb{G})) \in C^k_{\gamma(\alpha,\beta)}$ which means, $\kappa\sigma(\mathbb{G}) \in C_{\gamma(\alpha,\beta)}$. □

In particular the axioms (LCF-γ) and (WLCF) are preserved by the formation of subspaces.

The following result gives yet another axiom which characterizes diagonal spaces with a very special selection function. It will be applied in the next section.

PROPOSITION 4.11. *For an* L*-Cauchy tower space* $(X, \overline{C})$ *the axiom (WLCF) is equivalent to*

$$\text{(WLCG)} \qquad \forall \mathbb{F} \in \mathsf{F}(X), \alpha \in L : \mathbb{F} \in C_\alpha \Longrightarrow \mathbb{U}^\top(\mathbb{F}) \in C_\alpha,$$

where $\mathbb{U}^\top(\mathbb{F}) = \kappa\sigma_{\mathbb{U}}(\mathbb{F})$ *for the selection function* $\sigma_{\mathbb{U}} : X \longrightarrow \mathsf{F}(X)$, $\sigma_{\mathbb{U}}(x) = \mathbb{U}^\top_x$, *with the neighborhood filters* $\mathbb{U}^\top_x = \bigwedge_{\mathbb{G}\wedge[x]\in C_\top} \mathbb{G}$.

Proof. Let first (WLCF) be satisfied and let $\mathbb{F} \in C_\alpha$. We define $J = \{(x,\mathbb{G}) : \mathbb{G} \wedge [x] \in C_\top\}$ and define the mapping $\psi : J \longrightarrow X$ by $\psi((x,\mathbb{G})) = x$. Further we define $\sigma(x,\mathbb{G}) = \mathbb{G}$ for $(x,\mathbb{G}) \in J$. As $[x] \wedge [x] \in C_\top$, the mapping ψ is a surjection and hence $\mathbb{K} = \psi^{\leftarrow}(\mathbb{F}) \in \mathbb{F}(J)$ and $\psi(\mathbb{K}) = \mathbb{F} \in C_\alpha$. The axiom (WLCF) then implies $\kappa\sigma(\mathbb{K}) \in C_\alpha$. We show $\kappa\sigma(\mathbb{K}) \le \mathbb{U}^\top(\mathbb{F})$. Let $A \in \kappa\sigma(\mathbb{K})$. Then $A^\sigma \in \mathbb{K} = \psi^{\leftarrow}(\mathbb{F})$ and there is $F \in \mathbb{F}$ such that $\psi^{\leftarrow}(F) \subseteq A^\sigma$. This means that for all $(x,\mathbb{G}) \in J$, $(x,\mathbb{G}) \in \psi^{\leftarrow}(F)$ implies $(x,\mathbb{G}) \in A^\sigma$. Equivalently, $\mathbb{G} \wedge [x] \in C_\top$ and $x \in F$ implies $A \in \sigma(x,\mathbb{G}) = \mathbb{G}$. Hence, for $x \in F$ we have $A \in \bigwedge_{\mathbb{G}\wedge[x]\in C_\top} \mathbb{G} = \mathbb{U}^\top_x = \sigma_{\mathbb{U}}(x)$, i.e. we have $F \subseteq A^{\sigma_{\mathbb{U}}}$, which implies $A^{\sigma_{\mathbb{U}}} \in \mathbb{F}$ and finally $A \in \kappa\sigma_{\mathbb{U}}(\mathbb{F})$. Therefore, we have $\mathbb{U}^\top(\mathbb{F}) \in C_\alpha$.

For the converse, let $\psi : J \longrightarrow X$, $\mathbb{G} \in \mathsf{F}(J)$ and $\sigma : J \longrightarrow \mathsf{F}(X)$. If $\psi(\mathbb{G}) \in C_\alpha$ and for all $j \in J$, $\sigma(j) \wedge [\psi(j)] \in C_\top$, then $\sigma(j) \ge \mathbb{U}^\top_{\psi(j)}$ for all $j \in J$ and by (WLCG) $\mathbb{U}^\top(\psi(\mathbb{G})) \in C_\alpha$. We show that $\mathbb{U}^\top(\psi(\mathbb{G})) \le \kappa\sigma(\mathbb{G})$. Let $A \in \mathbb{U}^\top(\psi(\mathbb{G}))$. Then $A^{\sigma_{\mathbb{U}}} \in \psi(\mathbb{G})$ and hence there is a set $G \in \mathbb{G}$ such that $\psi(G) \subseteq A^{\sigma_{\mathbb{U}}}$. For $j \in G$ with $\psi(j) = x$ we then have $A \in \sigma_{\mathbb{U}}(x) \le \sigma(j)$, i.e. $j \in A^\sigma$. This means that $\psi(G) \subseteq \psi(A^\sigma)$ and $\psi(A^\sigma) \in \psi(\mathbb{G})$. Therefore $A^\sigma \in \psi^{\leftarrow}(\psi(\mathbb{G})) \le \mathbb{G}$ and we have $A \in \kappa\sigma(\mathbb{G})$. We conclude with (LChyTS2) that $\kappa\sigma(\mathbb{G}) \in C_\alpha$ and the axiom (WLCF) is valid. □

EXAMPLE 4.12. We show a space $(X, \overline{C})$ which satisfies (WLCF) but not (LCF-γ). Let $L = \{\bot, \alpha, \top\}$ with $\bot < \alpha < \top$ and consider the minimum as quantale operation, i.e. we use the quantale $\mathsf{L} = (L, \le, \wedge)$. Let further X be an infinite set and let $\mathbb{F}_0 = \{F \subseteq X : X \setminus F \text{ is finite}\}$ be the complement-finite filter and fix an ultrafilter $\mathbb{U} \ge \mathbb{F}_0$. Then $\mathbb{U} \ne [x]$ for all $x \in X$. We define $\overline{C}$ by $\mathbb{F} \in C_\top$ if $\mathbb{F} = [x]$ for some $x \in X$ or if $\mathbb{F} = \mathbb{U}$ and $\mathbb{F} \in C_\alpha$ if $\mathbb{F} = \mathbb{U}$ or if $\mathbb{F} \ge [A]$ with a non-empty finite set $A \subseteq X$

and $C_\perp = \mathsf{F}(X)$. It is then not difficult to see that $(X, \overline{C})$ is an L-Cauchy tower space and that the T2-axiom is valid. For $x \in X$ we have $\mathbb{U}_x^\top = \bigwedge_{\mathbb{G}\wedge[x]\in C_\top} \mathbb{G} = [x]$ and therefore $\mathbb{U}^\top(\mathbb{F}) = \mathbb{F}$ for all $\mathbb{F} \in \mathsf{F}(X)$ and the axiom (WLCF) is valid. We use the mapping $\gamma(\alpha,\beta) = \alpha \wedge \beta$. If we assume that $(X, \overline{C})$ satisfies (LCF-γ), then $(X, \overline{q^{\overline{C}}})$ satisfies (LF-γ) and this implies, as $\gamma(\top,\alpha) = \alpha$, that $(X, \overline{q^{\overline{C}}})$ is pretoppological [9]. We have in particular $x \in q^{\overline{C}}(\mathbb{U}_x^\alpha)$ with $\mathbb{U}_x^\alpha = \bigwedge_{\mathbb{G}\wedge[x]\in C_\alpha} \mathbb{G}$. As $\bigwedge_{\mathbb{G}\wedge[x]\in C_\alpha} \mathbb{G} \le \bigwedge_{A\subseteq X \text{ finite}}[A] = [X]$ we conclude $\mathbb{U}_x^\alpha = [X]$ and because $[X] \wedge [x] = [X] \notin C_\alpha$, this contradicts the pretopologicalness of $(X, \overline{q^{\overline{C}}})$. Hence $(X, \overline{C})$ does not satisfy (LF-γ).

5. A diagonal completion

We call an L-Cauchy tower space $(X, \overline{C})$ *complete* [12] if for all $\alpha \in L$, $\mathbb{F} \in C_\alpha$ implies the existence of $x \in X$ such that $\mathbb{F} \wedge [x] \in C_\alpha$. With this definition, the "point of convergence" $x = x(\mathbb{F},\alpha)$ not only depends on the filter $\mathbb{F}$ but also on the "level" $\alpha \in L$. In the left-continuous case, this cannot occur and $x = x(\mathbb{F})$ depends only on the filter $\mathbb{F}$ [12].

Let $(X, \overline{C})$ be a non-complete L-Cauchy tower space. A *completion of* $(X, \overline{C})$ is a pair $((X', \overline{C'}), \kappa)$ with a complete L-Cauchy tower space $(X', \overline{C'})$ and an initial and injective mapping $\kappa : (X, \overline{C}) \longrightarrow (X', \overline{C'})$ such that $\kappa(X)$ is dense in $(X', \overline{C'})$. Here, a set $A \subseteq X$ is called *dense in* $(X, \overline{C})$ if A is dense in $(X, \overline{q^{\overline{C}}})$, i.e. if for all $x \in X$ there is $\mathbb{F} \in \mathsf{F}(X)$ with $A \in \mathbb{F}$ and $\mathbb{F} \wedge [x] \in C_\top$. A mapping $\kappa : (X, \overline{C}) \longrightarrow (X', \overline{C'})$ is called *initial* if $\mathbb{F} \in C_\alpha$ if and only if $\kappa(\mathbb{F}) \in C'_\alpha$.

We consider now a non-complete L-Cauchy tower space $(X, \overline{C})$ and define $\mathcal{N}_{\overline{C}} = \{\mathbb{V} \in \mathsf{F}(X) : \mathbb{V} \in C_\top, \mathbb{V} \wedge [x] \notin C_\top \forall x \in X\}$. Furthermore, we consider the following equivalence relation on $C_\top$: $\mathbb{F} \sim \mathbb{G} \iff \mathbb{F} \wedge \mathbb{G} \in C_\top$ and we denote the equivalence class of $\mathbb{F} \in C_\top$ by $\langle \mathbb{F} \rangle = \{\mathbb{G} \in C_\top : \mathbb{F} \sim \mathbb{G}\}$. We define $X^* = \{\langle [x] \rangle : x \in X\} \cup \{\langle \mathbb{V} \rangle : \mathbb{F} \in \mathcal{N}_{\overline{C}}\}$. We note that $\{\langle \mathbb{F} \rangle : \mathbb{F} \in \mathcal{N}_{\overline{C}}\} = \{\langle \mathbb{V} \rangle : \mathbb{V} \in \mathcal{N}_{\overline{C}}, \mathbb{V} \text{ ultra}\}$. It suffices to note that for $\mathbb{F} \in \mathcal{N}_{\overline{C}}$ we can choose an ultrafilter $\mathbb{V} \ge \mathbb{F}$ and then clearly, by (LChyTS5), $\mathbb{V} \in \mathcal{N}_{\overline{C}}$ and $\langle \mathbb{V} \rangle = \langle \mathbb{F} \rangle$.

Furthermore, we say that a space $(X, \overline{C})$ satisfies the completion axiom (LCA) if whenever $\mathbb{F} \in C_\alpha$ with $\mathbb{F} \wedge [x] \notin C_\alpha$ for all $x \in X$, then there exists $\mathbb{V} \in \mathcal{N}_{\overline{C}}$ such that $\mathbb{F} \wedge \mathbb{V} \in C_\alpha$. It is shown in [15, 22] that a space $(X, \overline{C}) \in |\mathsf{L\text{-}ChyTS}|$ has a completion if and only if the axiom (LCA) is satisfied.

Example 5.1. Let X be an infinite set and consider again the complement-finite filter $\mathbb{F}_0 = \{F \subseteq X : X \setminus F \text{ is finite}\}$ and an ultrafilter $\mathbb{U} \ge \mathbb{F}$. Remember that $\mathbb{U} \ne [y]$ for all $y \in X$. We define the following L-Cauchy tower on X. For $\perp < \alpha < \top$ we define $\mathbb{F} \in C_\alpha \iff \mathbb{F} = \mathbb{U}$ or if $\mathbb{F} = [x]$ for some $x \in X$ and for $\alpha = \top$ we define $\mathbb{F} \in C_\top$ if $\mathbb{F} = [x]$ for some $x \in X$. It is not difficult to show that the L-Cauchy tower space $(X, \overline{C})$ satisfies the T2-axiom and is not complete. As $\mathcal{N}_{\overline{C}}$ is empty, $(X, \overline{C})$ cannot have a completion.

In the sequel, we are going to construct a completion which is diagonal for a non-complete space $(X, \overline{C})$. Note that by Proposition 4.10 $(X, \overline{C})$ must be diagonal for this. For a related similar construction in the category of Cauchy spaces, see [14].

We denote the inclusion mapping $\iota : X \longrightarrow X^*$, $x \longmapsto \iota(x) = \langle [x] \rangle$ and note that if $(X, \overline{C})$ is a T2-space, then ι is an injection: If $\iota(x) = \iota(y)$, then $[x] \wedge [y] \in C_\top$ and by (T2) then $x = y$. In the sequel, we will therefore always assume that $(X, \overline{C})$ is a T2-space.

For $A \subseteq X$ we define the set $A^* \subseteq X^*$ by

$$x^* \in A^* \iff \begin{cases} x \in A & \text{if } x^* = \langle [x] \rangle, \\ A \in \mathbb{V}_{\min} & \text{if } x^* = \langle \mathbb{V} \rangle, \end{cases}$$

with $\mathbb{V}_{\min} = \bigwedge_{\mathbb{G} \in \langle \mathbb{V} \rangle} \mathbb{G}$. The following lemma collects the properties of this construction. The proofs are straightforward and not shown.

Lemma 5.2. *Let $A, B \subseteq X$. Then*
(i) $(X)^* = X^*$; *(ii)* $\emptyset^* = \emptyset$; *(iii)* $A \subseteq B \Rightarrow A^* \subseteq B^*$; *(iv)* $(A \cap B)^* = A^* \cap B^*$.

Let now $\Phi \in \mathsf{F}(X^*)$. We define $\widetilde{\Phi} \subseteq P(X)$ by $A \in \widetilde{\Phi} \iff A^* \in \Phi$. From Lemma 5.2 it follows that $\widetilde{\Phi} \in \mathsf{F}(X)$.

Lemma 5.3. *Let $\Phi, \Psi \in \mathsf{F}(X^*)$ ans let $\mathbb{F} \in \mathsf{F}(X)$ and let $x^* \in X^*$.*
(i) If $\Phi \leq \Psi$, then $\widetilde{\Phi} \leq \widetilde{\Psi}$;

(ii) $(\widetilde{\Phi \wedge \Psi}) = \widetilde{\Phi} \wedge \widetilde{\Psi}$;

(iii) if $\Phi \vee \Psi$ exists, then also $\widetilde{\Phi} \vee \widetilde{\Psi}$ exists;

(iv) $\widetilde{\iota(\mathbb{F})} = \mathbb{F}$;

(v) $\widetilde{[x^*]} = \begin{cases} [x] & \text{if } x^* = \langle [x] \rangle, \\ \mathbb{V}_{\min} & \text{if } x^* = \langle \mathbb{V} \rangle. \end{cases}$

Proof. We only show (iv) and (v). For (iv), we have $A \in \widetilde{\iota(\mathbb{F})}$ if, and only if, $A^* \in \iota(\mathbb{F})$ if, and only if, there is $F \in \mathbb{F}$ such that $\iota(F) \subseteq A$. If $x \in F$, then $\langle [x] \rangle \in A^*$ and hence $x \in A$. Therefore we have $F \subseteq A$ and $A \in \mathbb{F}$. Conversely, if $A \in \mathbb{F}$, then $\iota(A) \in \iota(\mathbb{F})$. We show that $\iota(A) \subseteq A^*$. If $\iota(x) \in \iota(A)$, then $x \in A$ and hence $x^* \in A^*$. Therefore, finally, $A \in \iota(\mathbb{F})$.

For (v), let first $x^* = \langle [x] \rangle$. Then $A \in \widetilde{[x^*]}$ iff $A^* \in [x^*]$ iff $x^* \in A^*$ iff $x \in A$ iff $A \in [x]$. Secondly, let $x^* = \langle \mathbb{V} \rangle$. Then similarly, we have $A \in \widetilde{[x^*]}$ iff $A^* \in [x^*]$ iff $x^* \in A^*$ iff $A \in \mathbb{V}_{\min}$. □

Following [15], we call an L-Cauchy tower space $(X, \overline{C})$ *cushioned* if for all $\mathbb{V} \in \mathcal{N}_{\overline{C}}$ we have that $\mathbb{V}_{\min} = \bigwedge_{\mathbb{G} \in \langle \mathbb{V} \rangle} \mathbb{G} \in \langle \mathbb{V} \rangle$.

We show that an L-Cauchy tower space $(X, \overline{C})$ which allows a diagonal completion must be cushioned. In this sense, we will not demand too much later.

PROPOSITION 5.4. *Let the* L*-Cauchy tower space* $(X,\overline{C})$ *have a completion* $((X^+,\overline{C^+}),j)$ *such that* $(X^+,\overline{C^+})$ *is diagonal and T2. Then* $(X,\overline{C})$ *is cushioned, diagonal, T2 and satisfies the completion axiom (LCA).*

Proof. Let $\mathbb{V}\in\mathcal{N}_{\overline{C}}$ and let $\mathbb{G}\sim\mathbb{V}$, i.e. $\mathbb{G}\wedge\mathbb{V}\in C_\top$. Then $j(\mathbb{G})\wedge j(\mathbb{V})\in C^+_\top$ and by completeness there is $x^+\in X^+$ such that $j(\mathbb{G})\wedge j(\mathbb{V})\wedge[x^+]\in C^+_\top$, and hence also $\jmath(\mathbb{V})\wedge[x^+]\in C^+_\top$. The T2-property shows that x^+ does not depend on $\mathbb{G}$ but only on $\mathbb{V}$. The Proposition 4.6 then shows that $\bigwedge_{\mathbb{G}\wedge\mathbb{V}\in C_\top} j(\mathbb{G})\wedge j(\mathbb{V})\wedge[x^+]\in C^+_\top$ and hence also $\bigwedge_{\mathbb{G}\wedge\mathbb{V}\in C_\top} j(\mathbb{G})=j(\bigwedge_{\mathbb{G}\wedge\mathbb{V}\in C_\top}\mathbb{G})=j(\mathbb{V}_{\min})\in C^+_\top$ amd therefore also $\mathbb{V}_{\min}\in C_\top$ and $(X,\overline{C})$ is cushioned. The diagonal and T2 properties are inherited by subspaces and the proof is complete. $\square$

Let now $(X,\overline{C})$ be non-complete, T2 and cushioned and satisfy the completion axiom. We define an L-Cauchy tower $\overline{C^*}$ on X^* by $\Phi\in C^*_\alpha \iff \widetilde{\Phi}\in C_\alpha$.

PROPOSITION 5.5. *Let* $(X,\overline{C})$ *be non-complete, T2 and cushioned and satisfy the completion axiom (LCA). Then*
(i) $(X^*,\overline{C^*})\in|\mathsf{L\text{-}ChyTS}|$.

(ii) If $(X,\overline{C})$ *is left-continuous, then so is* $(X^*,\overline{C^*})$.

(iii) $(X^*,\overline{C^*})$ *is complete.*

(iv) $\iota(\mathbb{F})\in C^*_\alpha \iff \mathbb{F}\in C_\alpha$.

(v) $\iota(X)$ *is dense in* $(X^*,\overline{C^*})$.

(vi) $(X^*,\overline{C^*})$ *is a T2-space.*

(vii) If $(X,\overline{C})$ *satisfies (WLCF), then so does* $(X^*,\overline{C^*})$.

Proof. (i) (LChyTS1) Let $x^*=\langle[x]\rangle$ with $x\in X$. Then $\widetilde{[x^*]}=[x]\in C_\alpha$ and hence $[x^*]\in C^*\alpha$. If $x^*=\langle\mathbb{V}\rangle$ with $\mathbb{V}\in C_\top$, then from the cushionedness we get $\mathbb{V}_{\min}\in\langle\mathbb{V}\rangle$ and hence $\widetilde{[x^*]}=\mathbb{V}_{\min}=\mathbb{V}\wedge\mathbb{V}_{\min}\in C_\top\subseteq C_\alpha$, which implies $[x^*]\in C^*_\alpha$.

(LChyTS2) Let $\Phi\in C^*_\alpha$ and let $\Psi\geq\Phi$. Then $\widetilde{\Psi}\geq\widetilde{\Phi}\in C_\alpha$ and hence $\widetilde{\Psi}\in C_\alpha$ which implies $\Psi\in C^*_\alpha$.

(LChyTS3) If $\alpha\leq\beta$ and $\Phi\in C^*_\beta$, then $\widetilde{\Phi}\in C_\beta\subseteq C_\alpha$ and hence $\Phi\in C^*_\alpha$.

(LChyTS4) Let $\Phi\in\mathbb{F}(X^*)$. Then $\widetilde{\Phi}\in C_\bot$ and hence $\Phi\in C^*_\bot$.

(LChyTS5) Let $\Phi\in C^*_\alpha,\Psi\in C^*_\beta$ and let $\Phi\vee\Psi$ exist. Then $\widetilde{\Phi}\in C_\alpha,\widetilde{\Psi}\in C_\beta$ and $\widetilde{\Phi}\vee\widetilde{\Psi}$ exists. Hence $\widetilde{\Phi\wedge\Psi}=\widetilde{\Phi}\wedge\widetilde{\Psi}\in C_{\alpha*\beta}$ and this implies $\Phi\wedge\Psi\in C^*_{\alpha*\beta}$.

(ii) Let $A\subseteq L$. If $\Phi\in C^*_\alpha$ for all $\alpha\in A$, then $\widetilde{\Phi}\in C_\alpha$ for all $\alpha\in A$ and hence, by left-continuity, $\widetilde{\Phi}\in C_{\bigvee A}$. This implies $\Phi\in C^*_{\bigvee A}$.

(iii) Let $\Phi\in C^*_\alpha$. Then $\widetilde{\Phi}\in C_\alpha$. If $\widetilde{\Phi}\wedge[x]\in C_\alpha$ for some $x\in X$, then $\widetilde{\Phi\wedge[\langle[x]\rangle]}=\widetilde{\Phi}\wedge[x]\in C_\alpha$ and hence $\Phi\wedge[x^*]\in C^*_\alpha$ with $x^*=\langle[x]\rangle$. If $\widetilde{\Phi}\wedge[x]\notin C_\alpha$ for all $x\in X$, then by the completion axiom (LCA) there is $\mathbb{V}\in\mathcal{N}_{\overline{C}}$ such that $\widetilde{\Phi}\wedge\mathbb{V}\in C_\alpha$. As $(\widetilde{\Phi}\wedge\mathbb{V})\vee(\mathbb{V}\wedge\mathbb{V}_{\min})$ exists and $\mathbb{V}_{\min}=\mathbb{V}\wedge\mathbb{V}_{\min}\in C_\top$ we conclude $(\widetilde{\Phi}\wedge\mathbb{V})\wedge(\mathbb{V}\wedge\mathbb{V}_{\min})=$

$\widetilde{\Phi} \wedge \mathbb{V}_{\min} \in C_{\alpha*\top} = C_\alpha$. Hence $\widetilde{\Phi \wedge [\langle \mathbb{V} \rangle]} = \widetilde{\Phi} \wedge \mathbb{V}_{\min} \in C_\alpha$ and we have $\Phi \wedge [x^*] \in C^*_\alpha$ with $x^* = \langle \mathbb{V} \rangle$.

(iv) We have $\iota(\mathbb{F}) \in C^*_\alpha$ if and only if $\mathbb{F} = \widetilde{\iota(\mathbb{F})} \in C_\alpha$.

(v) We need to show that $X^* \subseteq \overline{\iota(X)}$. If $x^* = \langle [x] \rangle$ with $x \in x$, then $\iota(X) = x^* \in \iota(X)$, i.e. $\iota(X) \in [x^*]$ and $[x^*] \in C^*_\top$. If $x^* = \langle \mathbb{V} \rangle$ with $\mathbb{V} \in \mathcal{N}_{\overline{C}}$, then $\iota(X) \in \iota(\mathbb{V})$ and $\iota(\mathbb{V}) \wedge [\langle \mathbb{V} \rangle] \in C^*_\top$, as $\widetilde{\iota(\mathbb{V}) \wedge [\langle \mathbb{V} \rangle]} = \widetilde{\iota(\mathbb{V})} \wedge \widetilde{[\langle \mathbb{V} \rangle]} = \mathbb{V} \wedge \mathbb{V}_{\min} = \mathbb{V}_{\min} \in C_\top$.

(vi) Let $[x^*] \wedge [y^*] \in C^*_\top$. If $x^* = \langle [x] \rangle$ and $y^* = \langle [y] \rangle$ with $x, y \in X$, then $[x] \wedge [y] = \widetilde{[x^*]} \wedge \widetilde{[y^*]} \in C_\top$ and hence $x = y$, i.e. $x^* = y^*$. If $x^* = \langle \mathbb{U} \rangle$ and $y^* = \langle \mathbb{V} \rangle$ with $\mathbb{U}, \mathbb{V} \in \mathcal{N}_{\overline{C}}$, then $\mathbb{U}_{\min} \wedge \mathbb{V}_{\min} \in C_\top$ and hence $\mathbb{U}_{\min} \sim \mathbb{V}_{\min}$ from which $x^* = \langle \mathbb{U} \rangle = \langle \mathbb{V} \rangle = y^*$ follows. The other cases do not occur, as e.g. $x^* = \langle [x] \rangle$ and $y^* = \langle \mathbb{V} \rangle$ with $x \in X$ and $\mathbb{V} \in \mathcal{N}_{\overline{C}}$ implies $[x] \wedge \mathbb{V} = \widetilde{[x^*]} \wedge \widetilde{[y^*]} \in C_\top$, in contradiction to $\mathbb{V} \in \mathcal{N}_{\overline{C}}$.

(vii) We use Proposition 4.11. Let $\Phi \in C^*_\alpha$. Then $\widetilde{\Phi} \in C_\alpha$ and by (WLCF) $\mathbb{U}_\top(\widetilde{\Phi}) \in C_\alpha$. We show that $\mathbb{U}_\top(\widetilde{\Phi}) \leq \widetilde{\mathbb{U}^*_\top(\Phi)}$ with the neighbourhood filters $\mathbb{U}^{*x^*}_\top = \bigwedge_{\Psi \wedge [x^*] \in C^*_\top} \Psi$. Let $A \in \mathbb{U}_\top(\widetilde{\Phi})$. Then $A^{\sigma_{\mathbb{U}}} \in \widetilde{\Phi}$, i.e. we have $(A^{\sigma_{\mathbb{U}}})^* \in \Phi$. We will show that $(A^{\sigma_{\mathbb{U}}})^* \subseteq (A^*)^{\sigma_{\mathbb{U}^*}}$. Consider first $x \in X$ amd let $\langle [x] \rangle \in (A^{\sigma_{\mathbb{U}}})^*$. Then $x \in A^{\sigma_{\mathbb{U}}}$, and this is equivalent to $A \in \sigma_{\mathbb{U}}(x) = \mathbb{U}^\top_x$. This means that $A \in \mathbb{G}$ for all $\mathbb{G} \wedge [x] \in C_\top$ and in particular, $A \in \widetilde{\Psi}$ whenever $\widetilde{\Psi} \wedge [x] \in C_\top$. Therefore we have $A^* \in \bigwedge_{\widetilde{\Psi} \wedge [x] \in C_\top} \Psi = \bigwedge_{\Psi \wedge [\langle [x] \rangle] \in C^*_\top} \Psi = \mathbb{U}^{\top *}_{\langle [x] \rangle} = \sigma_{\mathbb{U}^*}(\langle [x] \rangle)$ and this implies $\langle [x] \rangle \in (A^*)^{\sigma_{\mathbb{U}^*}}$. Let now $\mathbb{V} \in \mathcal{N}_{\overline{C}}$ and let $\langle \mathbb{V} \rangle \in (A^{\sigma_{\mathbb{U}}})^*$. This is equivalent to $A^{\sigma_{\mathbb{U}}} \in \mathbb{V}_{\min}$. Now we note that if $\widetilde{\Psi} \wedge \mathbb{V}_{\min} \in C\top$, then $\widetilde{\Psi} \geq \mathbb{V}_{\min}$ and hence we have $A \in \widetilde{\Psi}$ whenever $\widetilde{\Psi} \wedge \mathbb{V}_{\min} \in C_\top$, i.e. we have $A^* \in \bigwedge_{\widetilde{\Psi} \wedge \mathbb{V}_{\min} \in C_\top} \Psi = \bigwedge_{\Psi \wedge [\langle \mathbb{V} \rangle] \in C^*_\top} \Psi = \mathbb{U}^{\top *}_{\langle \mathbb{V} \rangle} = \sigma_{\mathbb{U}^*}(\langle \mathbb{V} \rangle)$. Therefore $\langle \mathbb{V} \rangle \in (A^*)^{\sigma_{\mathbb{U}^*}}$. We conclude $(A^*)^{\sigma_{\mathbb{U}^*}} \in \Phi$ and hence $A \in \widetilde{\mathbb{U}^*_\top(\Phi)}$. This shows that $\widetilde{\mathbb{U}^*_\top(\Phi)} \in C_\alpha$ which finally implies $\mathbb{U}^*_\top(\Phi) \in C^*_\alpha$. □

We collect the contents of the previous proposition in the following theorem.

Theorem 5.6. *Let the* L*-Cauchy tower space* $(X, \overline{C})$ *be a non-complete, cushioned, diagonal T2-space that satisfies the completion axiom (LCA). Then* $((X^*, \overline{C^*}), \iota)$ *is a completion of* $(X, \overline{C})$ *that is diagonal and a T2-space.*

Note that for $\mathbb{F} \in C_\top$ we have $\iota(\mathbb{F}) \wedge [\langle \mathbb{F} \rangle] \in C^*$. In general, we call a completion $((Y, \overline{D}), \kappa)$ of $(X, \overline{C})$ *in standard form* if $Y = X^*, \kappa = \iota$ and if $\iota(\mathbb{F}) \wedge [\langle \mathbb{F} \rangle] \in D_\top$ for all $\mathbb{F} \in C_\top$, cf. [25].

Proposition 5.7. *For an* L*-Cauchy tower space* $(X, \overline{C})$ *which is diagonal, T2 and cushioned and satisfies the completion axiom (LCA), the completion* $((X^*, \overline{C^*}), \iota)$ *is the coarsest diagonal T2-completion in the sense that for any other diagonal T2-completion in standard form,* $((X^*, \overline{D}), \iota)$*, of* $(X, \overline{C})$ *we have* $D_\alpha \subseteq C^*_\alpha$ *for all* $\alpha \in L$.

Proof. We first show that for $A \subseteq X^*$ we have $A^{\sigma}_{\mathbb{U}^{\overline{D}}} \subseteq (\iota^{\leftarrow}(A))^*$ with $\sigma_{\mathbb{U}^{\overline{D}}}(x^*) = \mathbb{U}^{D\top}_{x^*} = \bigwedge_{\Psi \wedge [x^*] \in D_\top} \Psi$. To this end, let $x^* \in A^{\sigma}_{\mathbb{U}^{\overline{D}}}$. Then $A \in \bigwedge_{\Psi \wedge [x^*] \in D_\top} \Psi$, i.e. if $\Psi \wedge [x^*] \in D_\top$, then $A \in \Psi$. Let first $x^* = \langle [x] \rangle$ with $x \in X$. Then with

$\Psi = [\langle[x]\rangle]$ we have $\Psi \wedge [x^*] \in D_\top$ and hence $A \in [\langle[x]\rangle]$. This means $\langle[x]\rangle \in A$, i.e. $x \in \iota^{\leftarrow}(A)$ and we have $\langle[x]\rangle \in (\iota^{\leftarrow}(A))^*$. For the second case, let $x^* = \langle\mathbb{V}\rangle$ with $\mathbb{V} \in \mathcal{N}_{\overline{C}}$. Then $\langle\mathbb{V}\rangle = \langle\mathbb{V}_{\min}\rangle$ and because $((X^*, \overline{D}), \iota)$ is in standard form we have $\iota(\mathbb{V}_{\min}) \wedge [\langle\mathbb{V}_{\min}\rangle] \in D_\top$, which implies $A \in \iota(\mathbb{V}_{\min})$. Hence $\iota^{\leftarrow}(A) \in \mathbb{V}_{\min}$ and we conclude $\langle\mathbb{V}\rangle = \langle\mathbb{V}_{\min}\rangle \in (\iota^{\leftarrow}(A))^*$.

Let now $\Phi \in D_\alpha$. As $(Y, \overline{D})$ is diagonal, it follows that $\kappa\sigma_{\mathbb{U}^D}(\Phi) \in D_\alpha$. From what we have just shown, we see that $\kappa\sigma_{\mathbb{U}^D}(\Phi) \leq \iota(\widetilde{\Phi})$. In fact, for $A \in \kappa\sigma_{\mathbb{U}^D}(\Phi)$ we have $A^\sigma_{\mathbb{U}^{\overline{D}}} \in \Phi$ and hence $(\iota^{\leftarrow}(A))^* \in \Phi$ which means $\iota^{\leftarrow}(A) \in \widetilde{\Phi}$, i.e. $A \in \iota(\widetilde{\Phi})$. Therefore, $\iota(\widetilde{\Phi}) \in D_\alpha$, which implies $\widetilde{\Phi} \in C_\alpha$ and hence, by definition, $\Phi \in C^*_\alpha$. □

6. Conclusions

We have introduced certain diagonal axioms for quantale-valued Cauchy tower spaces and have shown their relations with diagonal axioms for quantale-valued convergence tower spaces and quantale-valued uniform convergence tower spaces. We gave a completion construction that preserves the diagonal property. At present not known is the construction of a completion that preserves the stronger diagonal axiom (LCF-γ).

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