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SOLVING GENERALIZED EQUILIBRIUM PROBLEMS FOR NONEXPANSIVE MAPPINGS ON HADAMARD MANIFOLDS

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Abstract. In this article, we propose a parallel viscosity iterative method for determining a common solution of a finite family of generalized equilibrium problems and a fixed point of a nonexpansive mapping in the setting of Hadamard manifolds. Under some mild conditions, we prove that the sequence generated by the proposed algorithm converges to a common solution of a finite family of generalized equilibrium problems and a fixed point problem for a nonexpansive mapping. We apply our result to solve a convex minimization problem and present a numerical example to demonstrate the performance of our method. Our results extend and improve many related results on generalized equilibrium problems from linear spaces to Hadamard manifolds.

1. Introduction

Let C be a nonempty closed convex subset of a topological space E, $F : C \times C \to \mathbb{R}$ and $G : C \times C \to \mathbb{R}$ are two bifunctions. The Generalized Equilibrium Problem (in short, GEP) is to find $x \in C$ such that

$$G(x, y) + \langle Fx, y - x\rangle \geq 0, \ \forall \ y \in C. \tag{1}$$

The GEP is known as a generalization of several other problems, for instance, variational inequality problems, minimization problems, fixed point problems, Nash equilibrium problems in noncooperative games and many others see [2, 5, 8]. We denote by Ψ the solution set of (1). If $F = 0$ in (1), then the GEP reduces to the Equilibrium Problem (in short, EP), which is to find $x \in C$ such that $G(x, y) \geq 0, \ \forall \ y \in C$.

It is known that equilibrium problems had a great impact and influence in the development of several topics in science and engineering. It has been shown that the theory of equilibrium problems provides a natural, novel and unified framework for several problems arising in nonlinear analysis, optimization, economics, finance, game

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theory and engineering. In 2008, Yang et al. [28] proposed the following proximal point method for approximating a solution of an EP as follows:

$$\begin{cases} x_1 \in C, \\ G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\, y \in C, \\ x_{n+1} = \alpha_n f(u_n) + (1-\alpha_n)Tu_n, \end{cases}$$

where $G : C \times C \to \mathbb{R}$ is a bifunction, $T : C \to C$ is a nonexpansive mapping, f is a contraction mapping with constant $\alpha \in (0,1)$ with $Fix(T) \bigcap EP(G)$ nonempty. They established a strong convergence theorem.

Several iterative methods have been used to approximate solutions of GEP in real Hilbert spaces. For instance, Takahashi and Takahashi [25] proposed an iterative scheme for approximating the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a generalized equilibrium problem in a real Hilbert space. They proved a strong convergence result. Also, Shehu [24] introduced a modified Halpern method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a generalized equilibrium problem in a real Hilbert space. He also established a strong convergence result. In the settings of a reflexive Banach space, Kazmi et al. [13] proposed a hybrid iterative method for finding a common solution of a generalized equilibrium problem and a fixed point problem for a Bregman relatively nonexpansive mapping. They proved the following theorem:

Theorem 1.1. *Let C be a nonempty, closed and convex subset of a reflexive Banach space X with dual space X^* such that $C \subset int(domf)$. Let $f : X \to (-\infty, +\infty]$ be a coercive Legendre function which is bounded subsets of X. Let $G : C \times C \to \mathbb{R}$ and $F : C \times C \to \mathbb{R}$ be bifunctions. Let $T : C \to C$ be a Bregman relatively nonexpansive mapping. Assume that $\Omega := GEP(G,F) \cap Fix(T)$ is nonempty. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated iteratively by*

$$\begin{cases} x_0, z_0 \in C, \\ u_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(Tx_n)), \\ z_{n+1} = R^f_{G,F} u_n, \\ \theta_n := \{z \in C : D - f(z, z_{n+1}) \leq \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, x_n)\}, \\ Q_n = \{z \in C : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = Proj^f_{\theta_n \cap Q_n} x_0, \ \forall\, n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $Proj^f_\Omega x_0$, where $Proj^f_\Omega x_0$ is the Bregman projection of C onto Ω.

In 2012, Colao et al. [7] introduced the concept of equilibrium problem where the associated bifunction is monotone and proved the existence of its solution on Hadamard manifolds. Wang [27] studied the notion of monotone and accretive vector fields on Riemannian manifolds. Nemeth [18] generalized some basic concepts in existence and uniqueness theorems in the classical theory of variational inequalities from Euclidean spaces to Hadamard manifolds. Zhou and Huang [29] studied the notion of KKM mapping and proved a generalized KKM theorem on a Hadamard manifold. Noor et al. [21] introduced an implicit method for solving equilibrium problem on Hadamard manifolds and Noor et al. [20] proposed an explicit method for solving the equilibrium problem on Hadamard manifolds.

Let C be a nonempty, closed and geodesically convex subset of a Hadamard manifold $\mathbb{M}$. Let $G : C \times C \to \mathbb{R}$ be a bifunction and let $F : C \to T\mathbb{M}$ be a single-valued vector field. The GEP is to find $x \in C$ such that

$$G(x,y) + \langle Fx, \exp_x^{-1} y\rangle \geq 0,\ \forall\ y \in C, \tag{2}$$

where $\exp^{-1}$ is the inverse of the exponential function $\exp : T\mathbb{M} \to \mathbb{M}$ with $T\mathbb{M}$ the tangent bundle of $\mathbb{M}$.

Very recently, the existence of solution of the generalized equilibrium problem on a Hadamard manifold using the KKM lemma was studied [22]. A convergence result for approximating a solution to the GEP, which is also a fixed point of a nonexpansive mapping, was established using the following viscosity iterative method:

$$\begin{cases} y_n = \exp_{x_n}(1-\beta_n)\exp_{x_n}^{-1} Sx_n, \\ x_{n+1} = \exp_{f(x_n)}(1-\alpha_n)\exp_{f(x_n)}^{-1} R_{r_n}^{G,F}, \end{cases}$$

where S is an nonexpansive mapping, $f : \mathbb{M} \to \mathbb{M}$ is an α-contraction and $R_{r_n}^{G,F}$ (defined in section 2) is the resolvent of $GEP(F,\psi)$.

Inspired by the results [7, 22, 24] and other related results in the literature, we propose a viscosity iterative method for a finite family of generalized equilibrium problems which is also a fixed point of a nonexpansive mapping on a Hadamard manifold. Using our proposed iterative method, we prove that the sequence generated by our iterative algorithm converges to a solution of the finite family of generalized equilibrium problems and is a fixed point of a nonexpansive mapping. We present some consequences of our result. In summary, the problems discussed in this article is to find $x \in C$ such that $Fix(\Phi) \cap \bigcap_{j=1}^{N} GEP(F_j,\psi_j)$, where $Fix(\Phi) = \{x \in C : x = \Phi x\}$ is the fixed point set of a nonlinear mapping Φ.

We highlight our contributions in this article, which are the following:

(i) Our result generalizes many related results on EP and GEP from linear spaces to Hadamard manifolds (see [1, 12, 19]).

(ii) The proposed algorithm does not require at each step of the iteration process the computation of subsets of C_n, and Q_n (or C_{n+1}) as in the case in [6, 24] and the computation of the projection of the initial point onto their intersection, which leads to a high computational cost of the iteration processes. The removal of all these restrictions makes our work applicable to a larger class of real world problems.

2. Preliminaries

Let $\mathbb{M}$ be an m-dimensional manifold, let $x \in \mathbb{M}$ and let $T_x\mathbb{M}$ be the tangent space of $\mathbb{M}$ at $x \in \mathbb{M}$. We denote by $T\mathbb{M} = \bigcup_{x\in\mathbb{M}} T_x\mathbb{M}$ the tangent bundle of $\mathbb{M}$. An inner product $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian metric on $\mathbb{M}$ if $\langle\cdot,\cdot\rangle_x : T_x\mathbb{M}\times T_x\mathbb{M}\to\mathbb{R}$ is an inner product for all $x\in\mathbb{M}$. The corresponding norm induced by the inner product $\mathcal{R}_x\langle\cdot,\cdot\rangle$ on $T_x\mathbb{M}$ is denoted by $\|\cdot\|_x$. We will drop the subscript x and adopt $\|\cdot\|$ for the corresponding norm induced by the inner product. A differentiable manifold $\mathbb{M}$ endowed with a Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian manifold. In what follows, we denote the Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle$ when no confusion arises. Given a piecewise smooth curve $\gamma : [a,b]\to\mathbb{M}$ joining x to y (that is, $\gamma(a)=x$ and $\gamma(b)=y$), we define the length $l(\gamma)$ of γ by $l(\gamma):=\int_a^b\|\gamma'(t)\|dt$. The Riemannian distance $d(x,y)$ is the minimal length over the set of all such curves joining x to y. The metric topology induced by d coincides with the original topology on $\mathbb{M}$. We denote by ∇ the Levi-Civita connection associated with the Riemannian metric [23].

Let γ be a smooth curve in $\mathbb{M}$. A vector field X along γ is said to be parallel if $\nabla_{\gamma'}X=\mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector. If γ' itself is parallel along γ, then we say that γ is a geodesic and $\|\gamma'\|$ is a constant. If $\|\gamma'\|=1$, then the geodesic γ is said to be normalized. A geodesic joining x to y in $\mathbb{M}$ is called a minimizing geodesic if its length equals $d(x,y)$. A Riemannian manifold $\mathbb{M}$ equipped with a Riemannian distance d is a metric space $(\mathbb{M},d)$. A Riemannian manifold $\mathbb{M}$ is said to be complete if for all $x\in\mathbb{M}$, all geodesics emanating from x are defined for all $t\in\mathbb{R}$. The Hopf-Rinow theorem [23], posits that if $\mathbb{M}$ is complete, then any pair of points in $\mathbb{M}$ can be joined by a minimizing geodesic. Moreover, if $(\mathbb{M},d)$ is a complete metric space, then every bounded and closed subset of $\mathbb{M}$ is compact. If $\mathbb{M}$ is a complete Riemannian manifold, then the exponential map $\exp_x : T_x\mathbb{M}\to\mathbb{M}$ at $x\in\mathbb{M}$ is defined by $\exp_x v := \gamma_v(1,x)\quad\forall\, v\in T_x\mathbb{M}$, where $\gamma_v(\cdot,x)$ is the geodesic starting from x with velocity v (that is, $\gamma_v(0,x)=x$ and $\gamma_v'(0,x)=v$). Then, for any t, we have $\exp_x tv=\gamma_v(t,x)$ and $\exp_x\mathbf{0}=\gamma_v(0,x)=x$. Note that the mapping $\exp_x$ is differentiable on $T_x\mathbb{M}$ for every $x\in\mathbb{M}$. The exponential map $\exp_x$ has an inverse $\exp_x^{-1} : \mathbb{M}\to T_x\mathbb{M}$. For any $x,y\in\mathbb{M}$, we have $d(x,y)=\|\exp_y^{-1}x\|=\|\exp_x^{-1}y\|$ (see [23] for more details). The parallel transport $P_{\gamma,\gamma(b),\gamma(a)} : T_{\gamma(a)}\mathbb{M}\to T_{\gamma(b)}\mathbb{M}$ on the tangent bundle $T\mathbb{M}$ along $\gamma:[a,b]\to\mathbb{R}$ with respect to ∇ is defined by

$$P_{\gamma,\gamma(b),\gamma(a)}v=F(\gamma(b)),\quad\forall\; a,b\in\mathbb{R}\;\text{ and } v\in T_{\gamma(a)}\mathbb{M},$$

where F is the unique vector field such that $\nabla_{\gamma'(t)}v=\mathbf{0}$ for all $t\in[a,b]$ and $F(\gamma(a))=v$. If γ is a minimizing geodesic joining x to y, then we write $P_{y,x}$ instead of $P_{\gamma,y,x}$. Note that for every $a,b,r,s\in\mathbb{R}$, we have

$$P_{\gamma(s),\gamma(r)}\circ P_{\gamma(r),\gamma(a)}=P_{\gamma(s),\gamma(a)}\;\text{ and }\;P^{-1}_{\gamma(b),\gamma(a)}=P_{\gamma(a),\gamma(b)}.$$

Also, $P_{\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}\mathbb{M}$ to $T_{\gamma(b)}\mathbb{M}$, that is, the parallel transport preserves the inner product

$$\langle P_{\gamma(b),\gamma(a)}(u),P_{\gamma(b),\gamma(a)}(v)\rangle_{\gamma(b)}=\langle u,v\rangle_{\gamma(a)},\quad\forall\; u,v\in T_{\gamma(a)}\mathbb{M}.$$

A subset $C\subset\mathbb{M}$ is said to be convex if for any two points $x,y\in C$, the geodesic γ joining x to y is contained in C. That is, if $\gamma:[a,b]\to\mathbb{M}$ is a geodesic such that $x=\gamma(a)$ and $y=\gamma(b)$, then $\gamma((1-t)a+tb)\in C$ for all $t\in[0,1]$. A complete simply connected Riemannian manifold of non-positive sectional curvature is called an Hadamard manifold.

Definition 2.1 ([10]). Let C be a nonempty, closed and subset of $\mathbb{M}$ and $\{x_n\}$ be a sequence in $\mathbb{M}$. Then $\{x_n\}$ is said to be Fejér convergent with respect to C if for all $p \in C$ and $n \in \mathbb{N}$, $d(x_{n+1}, p) \leq d(x_n, p)$.

Definition 2.2 ([23]). Let $f : C \to \mathbb{R}$ be a geodesic convex. Let $p \in C$, then a vector $r \in T_p\mathbb{M}$ is said to be a subgradient of f at p if and only if $f(q) \geq f(p) + \langle r, \exp_p^{-1} q\rangle$, $\forall\, q \in C$.

Lemma 2.3 ([10]). *Let C be a nonempty, closed and closed subset of $\mathbb{M}$ and $\{x_n\} \subset \mathbb{M}$ be a sequence such that $\{x_n\}$ be a Fejér convergent with respect to C. Then, the following hold:*
(i) For every $p \in C$, $d(x_n, p)$ converges;

(ii) $\{x_n\}$ is bounded;

(iii) Assume that every cluster point of $\{x_n\}$ belongs to C. Then, $\{x_n\}$ converges to a point in C.

Definition 2.4. A mapping $S : C \to C$ is said to be
(i) contractive, if there exits a constant $k \in (0, 1)$;

$$d(Sx, Sy) \leq kd(x, y), \forall\, x, y \in C. \tag{3}$$

If $k = 1$ in (3), then S is said to be nonexpansive;

(ii) quasi-nonexpansive, if $Fix(S) \neq \emptyset$ and $d(Sx, p) \leq d(x, p)$, $\forall\, p \in Fix(S)$ and $x \in C$;

(iii) firmly nonexpansive [15] if, for all $x, y \in C$, the function $H : [0, 1] \to [0, \infty]$ defined by $H(t) := d(\exp_x t \exp_x^{-1} Sx, \exp_y^{-1} Sy)$, $\forall\, t \in [0, 1]$ is nonincreasing.

Proposition 2.5 ([15]). *Let $S : C \to C$ be a mapping. Then, the following statements are equivalent:*
(i) S is firmly nonexpansive;

(ii) for any $x, y \in C$ and $t \in [0, 1]$, $d(Sx, Sy) \leq d(\exp_x t \exp_x^{-1} Sx, \exp_y t \exp_y^{-1} Sy)$;

(iii) for any $x, y \in C$, $\langle \exp_{S(x)}^{-1} S(y), \exp_{S(x)}^{-1} x\rangle + \langle \exp_{S(y)}^{-1} S(x), \exp_{S(y)}^{-1} y\rangle \leq 0$.

Lemma 2.6 ([6]). *Let $S : C \to C$ be a firmly nonexpansive mapping and $Fix(S) \neq \emptyset$. Then, for any $x \in C$ and $p \in Fix(S)$, the following conclusion holds:*

$$d^2(Sx, p) \leq d^2(x, p) - d^2(Sx, x).$$

Proposition 2.7 ([23]). *Let $x \in \mathbb{M}$. The exponential mapping $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ is a diffeomorphism. For any two points $x, y \in \mathbb{M}$, there exists a unique normalized geodesic joining x to y, which is given by $\gamma(t) = \exp_x t \exp_x^{-1} y$, $\quad \forall\, t \in [0, 1]$.*

For any $x \in \mathbb{M}$ and $C \subset \mathbb{M}$, there exists a unique point $y \in C$, such that $d(x, y) \leq d(x, z)$ for all $z \in C$. This unique point y is called the nearest point projection of x onto the closed and convex set C and is denoted by $P_C(x)$.

LEMMA 2.8 ([26]). *Let C be a nonempty, closed and geodesic convex subset of a Hadamard manifold $\mathbb{M}$.*
(i) For any $x \in \mathbb{M}$, there exists a unique nearest point projection $y = P_C(x)$. Furthermore, the following inequality holds: $\langle \exp_y^{-1} x, \exp_y^{-1} z\rangle \leq 0, \quad \forall\ z \in C$.

(ii) $P_C : \mathbb{M} \to C$ is a firmly nonexpansive mapping. Therefore from Lemma 2.6, we have $d^2(y,p) \leq d^2(x,p) - d^2(y,x),\ \forall\ x \in \mathbb{M}$ and $p \in C$.

LEMMA 2.9 ([14]). *Let $x_0 \in \mathbb{M}$ and $\{x_n\} \subset \mathbb{M}$ be such that $x_n \to x_0$. Then, for any $y \in \mathbb{M}$, we have $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$.*

The following propositions (see [10]) are very useful in our convergence analysis.

PROPOSITION 2.10. *Let M be an Hadamard manifold and $d : \mathbb{M}\times\mathbb{M} :\to \mathbb{R}$ be the distance function. Then the function d is convex with respect to the product Riemannian metric. In other words, given any pair of geodesics $\gamma_1 : [0,1] \to \mathbb{M}$ and $\gamma_2 : [0,1] \to \mathbb{M}$, then for all $t \in [0,1]$ we have $d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1))$. In particular, for each $y \in M$, the function $d(\cdot, y) : \mathbb{M} \to \mathbb{R}$ is a convex function.*

LEMMA 2.11 ([3]). *Let $G(x_1, x_2, x_3)$ be a geodesic triangle in $\mathbb{M}$. Then, there exists a triangle $G(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ corresponding to $G(x_1, x_2, x_3)$ such that $d(x_i, x_{i+1}) = \|\bar{x}_i - \bar{x}_{i+1}\|$ with the indices taken modulo 3. This triangle is unique up to isometries of $\mathbb{R}^2$.*

The triangle $G(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in Lemma 2.11 is said to be the comparison triangle for $G(x_1, x_2, x_3) \subset \mathbb{M}$. The points $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are called comparison points for the points x_1, x_2 and x_3 in $\mathbb{M}$.

A function $h : \mathbb{M} \to \mathbb{R}$ is said to be geodesic if, for any geodesic $\gamma \in \mathbb{M}$, the composition $h \circ \gamma : [u, v] \to \mathbb{R}$ is convex, that is,

$$h \circ \gamma(\lambda u + (1-\lambda)v) \leq \lambda h \circ \gamma(u) + (1-\lambda)h \circ \gamma(v), \quad u, v \in \mathbb{R},\ \lambda \in [0,1].$$

LEMMA 2.12 ([14]). *Let $G(p,q,r)$ be a geodesic triangle in a Hadamard manifold $\mathbb{M}$ and let $G(p^{'}, q^{'}, r^{'})$ be its comparison triangle.*
(i) Let α, β, γ (resp. $\alpha^{'}, \beta^{'}, \gamma^{'}$) be the angles of $G(p,q,r)$ (resp. $G(p^{'}, q^{'}, r^{'})$) at the vertices p,q,r (resp. $p^{'}, q^{'}, r^{'}$). Then, the following inequalities hold: $\alpha^{'} \geq \alpha$, $\beta^{'} \geq \beta$, $\gamma^{'} \geq \gamma$.

(ii) Let z be a point in the geodesic joining p to q and $z^{'}$ its comparison point in the interval $[p^{'}, q^{'}]$. Suppose that $d(z,p) = \|z^{'} - p^{'}\|$ and $d(z^{'}, q^{'}) = \|z^{'} - q^{'}\|$. Then, the following inequality holds: $d(z,r) \leq \|z^{'} - r^{'}\|$.

PROPOSITION 2.13. *Let $\mathbb{M}$ be a Hadamard manifold and $x \in \mathbb{M}$. The map $\Phi_x = d^2(x,y)$ satisfies the following:*
(i) Φ_x is convex. Indeed, for any geodesic $\gamma : [0,1] \to \mathbb{M}$, the following inequality holds for all $t \in [0,1]$:

$$d^2(x, \gamma(t)) \leq (1-t)d^2(x, \gamma(0)) + td^2(x, \gamma(1)) - t(1-t)d^2(\gamma(0), \gamma(1)).$$

(ii) Φ_x is smooth. Moreover, $\partial\Phi_x(y) = -2\exp_y^{-1} x$.

Definition 2.14. Let $\mathbb{M}$ be a Hadamard manifold. A mapping $S : \mathbb{M} \to \mathbb{M}$ is said to be demiclosed at 0, if for any sequence $\{x_n\}$ in $\mathbb{M}$ such that $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} d(x_n, Sx_n) = 0$, we have that $Sp = p$.

Proposition 2.15 ([16]). *Let $S : C \to \mathbb{M}$ be a nonexpansive mapping defined on a closed convex set $C \subseteq M$. Then, the fixed point set $Fix(S)$ is closed and convex.*

Let C be a nonempty, closed and convex subset of a Hadamard manifold $\mathbb{M}$ and $\mathcal{H}(C)$ denote the set of all single valued vector fields $F : C \to T\mathbb{M}$ such that $Fx \in T_x\mathbb{M}$ for every $x \in C$. Then, a vector field $F \in \mathcal{H}(C)$ is called monotone if $\langle Fx, \exp_x^{-1} y\rangle + \langle Fy, \exp_y^{-1} x\rangle \leq 0$.

We need the following results to solve GEP (2).

Lemma 2.16 ([22]). *Let C be a nonempty, closed and convex subset of a Hadamard manifold $\mathbb{M}$. Let $F : C \to T\mathbb{M}$ be a single-valued monotone vector field and let $G : C \times C \to \mathbb{R}$ be a bifunction such that $G(x, x) = 0$ satisfying the following:*
(L1) G is monotone. That is, $G(x, y) + G(y, x) \leq 0$, for all $x, y \in C$;

(L2) For all $x \in C$, $G(x, \cdot)$ is convex;

(L3) There exists a compact subset $K \subset C$ containing $u_0 \in K$ such that $G(x, u_0) + \langle \psi x, \exp_x^{-1} u_0\rangle < 0$ whenever $x \in C \setminus K$.
Then, the GEP (2) is solvable.

The result stated below describes some properties of the resolvent operator of GEP (2).

Lemma 2.17. *[22] Let $G : C\times C \to \mathbb{R}$ be a bifunction satisfying assumptions (L1)-(L3) let $F : C \to T\mathbb{M}$ be a mapping. For $r>0$, define a set-valued mapping $R_r^{G,F} : C \to 2^C$ by*

$$R_r^{G,F}(x)=\left\{z\in C : G(z, y)+\langle Fz, \exp_z^{-1} y\rangle-\frac{1}{r}\langle\exp_z^{-1} x, \exp_z^{-1} y\rangle \geq 0\right\}, \ \forall\ y\in C, x\in\mathbb{M}.$$

Then, there hold
(i) $R_r^{G,F}$ is single-valued;

(ii) $R_r^{G,F}$ is firmly nonexpansive;

(iii) $Fix(R_r^{G,F}) = GEP(G, F)$;

(iv) $GEP(G, F)$ is closed and convex;

(v) Let $0 < r \leq s$. Then, for all $x \in C$, $d(x, R_r^{G,F} x) \leq 2d(x, R_r^{G,F} x)$;

(vi) For all $x \in C$ and $p \in Fix(R_r^{G,F})$, $d^2(p, R_r^{G,F} x) + d^2(x, R_r^{G,F} x) \leq d^2(x, p)$.

Lemma 2.18 ([11]). *Let $\{s_n\}$ be a sequence of nonnegative numbers, let $\{\alpha_n\}$ be a sequence in (0,1) and let $\{\theta_n\}$ be a sequence of real numbers satisfying the following conditions:*
(i) $s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\theta_n$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\limsup_{n\to\infty} \theta_n \leq 0$.
Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.19 ([17]). *Let $\{s_n\}$ be a real sequence which does not decrease at infinity in the sense that there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \leq s_{n_{k+1}}$, $\forall\, k \geq 0$. Define an integer sequence $\{\sigma(n)\}$, where $n > n_0$ by $\sigma(n) := \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}$. Then $\sigma(n) \to \infty$ as $n \to \infty$ and for all $n > n_0$, we have $\max\{s_{\sigma(n)}, s_n\} \leq s_{\sigma(n)+1}$.*

3. Main result

In this section, we introduce a viscosity method for solving a finite family of generalized equilibrium problems and a fixed point problem of nonexpansive mapping in a Hadamard manifold. We state and prove our convergence result:

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a Hadamard manifold $\mathbb{M}$. For $k = 1, 2, \cdots, m$, let $F_k : C \to \mathbb{TM}$ be a monotone vector field and let $G_k : C \times C \to \mathbb{R}$ be such that $G_k(u,u) = 0$ for all $u \in C$ be a bifunction satisfying (L1)-(L3). Let $\psi : C \to C$ be a contraction mapping with constant $\phi \in (0,1)$ and let $\Phi : C \to C$ be a nonexpansive mapping such that $\Upsilon := Fix(\Phi) \cap \bigcap_{k=1}^{m} Fix(R^{G_k,F_k})$ is nonempty. For an arbitrary $q_1 \in \mathbb{Z}$, let $\{q_n\}$ be generated iteratively by*

$$\begin{cases} w_n^k = R_{\lambda_n}^{G_k,F_k} q_n, k = 1,2,\ldots,N; \\ t_n \in \{w_n^k, k = 1,2,\ldots,m\} \text{ such that } d(t_n,q_n) = \max\limits_{1\leq k\leq N} d(w_n^k,q_n); \\ v_n = \exp_{t_n}(1-\eta_n)\exp_{t_n}^{-1}\Phi t_n; \\ q_{n+1} = \gamma_n^b(1-\theta_n), \ \forall\, n \geq 1; \end{cases} \tag{4}$$

where $\gamma_n^b : [0,1] \to \mathbb{Z}$ is a sequence of geodesics joining $\psi(q_n)$ to v_n, and the sequences $\{\eta_n\}, \{\theta_n\} \in (0,1)$ and $\{r_n\} \in (0,\infty)$ satisfy the following:

(i) $\lim\limits_{n\to\infty} \theta_n = 0, \quad \sum\limits_{n=1}^{\infty} \theta_n < \infty$;

(ii) $0 < a \leq \eta_n \leq b < 1$ for some $a, b > 0$ for all $n \geq 1$;

(iii) $0 < \lambda \leq \lambda_n$.

Then the sequence $\{q_n\}$ converges to an element $p \in \Upsilon$.

Proof. Let $p \in \Upsilon$. By Lemma 2.17 we have

$$d(w_n^k, p) = d(R_{\lambda_n}^{G_k,F_k} q_n, p) \leq d(q_n, p). \tag{5}$$

It is obvious from (4) and (5) that

$$d(t_n, p) = \max_{1\leq k\leq m} d(w_n^k, p) \leq d(q_n, p). \tag{6}$$

By the property of the exp function, we can re-write t_n defined in (4) as $t_n = \gamma_n(1-\eta_n)$, where $\gamma_n : [0,1] \to \mathbb{Z}$ is a geodesic sequence joining t_n to Φt_n. Using Proposition 2.13, (6) and the nonexpansive property of Φ, we obtain

$$\begin{aligned} d^2(v_n,p) &= d^2(\gamma_n(1-\eta_n),p) \\ &\leq (1-\eta_n)d^2(\eta_n^a(0),p) + \eta_n d^2(\gamma_n^a(1),p) - \eta_n(1-\eta_n)d^2(\gamma_n^a(0),\gamma_n^a(1)) \end{aligned}$$

$$
\begin{aligned}
&= (1-\eta_n)d^2(t_n,p) + \eta_n d^2(\Phi t_n,p) - \eta_n(1-\eta_n)d^2(t_n,\Phi t_n)\\
&= (1-\eta_n)d^2(t_n,p) + \eta_n d^2(t_n,p) - \eta_n(1-\eta_n)d^2(t_n,\Phi t_n)\\
&= d^2(t_n,p) - \eta_n(1-\eta_n)d^2(t_n,\Phi t_n) \qquad (7)\\
&\le d(t_n,p). \qquad (8)
\end{aligned}
$$

In view of (6) and (8), we get $d(v_n,p)\le d(t_n,p)\le d(q_n,p)$. From this, using $q_{n+1}=\gamma_n^b(1-\theta_n)$, we obtain that

$$
\begin{aligned}
&d(q_{n+1},p) = d(\gamma_n^b(1-\theta_n),p)\\
&= \theta_n d(\gamma_n^b(0),p) + (1-\theta_n)d(\gamma_n^b(1),p)\\
&\le \theta_n d(\psi(q_n),p) + (1-\theta_n)d(v_n,p)\\
&\le \theta_n\big[d(\psi(q_n),\psi(p)) + d(\psi(p),p)\big] + (1-\theta_n)d(v_n,p)\\
&\le \theta_n\big[\phi d(q_n,p) + d(\psi(p),p)\big] + (1-\theta_n)d(q_n,p)\\
&= (1-\theta_n(1-\phi))d(q_n,p) + \theta_n\big[(1-\theta_n)\frac{d(\psi(p),p)}{1-\phi}\big]\\
&\le \ldots \le \max\big\{d(q_n,p),\frac{d(\psi(p),p)}{1-\phi}\big\}.
\end{aligned}
$$

By induction, we obtain that $d(q_{n+1},p)\le \max\{d(q_1,p),\frac{d(\psi(p),p)}{1-\phi}\}$. Thus, the sequence $\{q_n\}$ is bounded. Consequently, the sequences $\{t_n\},\{v_n\},\{w_n\}$ and $\{\Phi t_n\}$ are also bounded. Using Lemma 2.17 (iv), (5) and (7), we get

$$
\begin{aligned}
d^2(v_n,p) &\le d^2(t_n,p) - \eta_n(1-\eta_n)d^2(t_n,\Phi t_n) \le d^2(w_n^k,p) - \eta_n(1-\eta_n)d^2(t_n,\Phi t_n)\\
&\le d^2(q_n,p) - d^2(w_n^k,q_n) - \eta_n(1-\eta_n)d^2(t_n,\Phi t_n). \qquad (9)
\end{aligned}
$$

For $n\ge 1$, let $a=\psi(q_n), r=\psi(p)$ and $b=v_n$. We consider the geodesic triangles with their respective comparisons $G(a,r,b)$ and $G(a',r',b')$, $G(b,r,a)$ and $G(b',r',a'), G(b,r,p)$ and $G(b',r',p')$. By applying Lemma 2.11, we have $d(a,r)=\|a'-r'\|$, $d(a,b)=\|a'-b'\|, d(a,p)=\|a'-p'\|$, $d(b,r)=\|b'-r'\|$ and $d(r,p)=\|r'-p'\|$. Thus, the comparison point of $q_{n+1}\in\mathbb{R}^2$ is $q'_{n+1}=\theta_n a'+(1-\theta_n)b'$. Let τ and τ' denote the angle and comparison angle at p and p' in the triangles $G(r,q_{n+1},p)$ and $G(w',q'_{n+1},p')$ respectively. Hence $\tau\le\tau'$ and $\cos\tau'\le\cos\tau$. By applying Lemma 2.12 and the property of ψ, we obtain

$$
\begin{aligned}
d^2(q_{n+1},p) &\le \|q'_{n+1}-p'\|^2\\
&= \|\theta_n(a'-p')+(1-\theta_n)(b'-p')\|^2\\
&\le \|\theta_n(a'-r')+(1-\theta_n)(b'-p')\|^2 + 2\theta_n\langle q'_{n+1}-p', r'-p'\rangle\\
&\le (1-\theta_n)\|b'-p'\|^2 + \theta_n\|a'-r'\|^2 + 2\theta_n\|q'_{n+1}-p'\|\ \|r'-p'\|\cos\tau'\\
&\le (1-\theta_n)d^2(b,p) + \theta_n d^2(a,r) + 2\theta_n d(q_{n+1},p)d(r,p)\cos\tau\\
&= (1-\theta_n)d^2(v_n,p) + \theta_n d^2(\psi(q_n),\psi(p)) + 2\theta_n d(q_{n+1},p)d(r,p)\cos\tau. \qquad (10)
\end{aligned}
$$

It is easy to see that $d(q_{n+1},p)d(\psi(p),p)\cos\tau = \langle \exp_p^{-1}\psi(p), \exp_p^{-1}q_{n+1}\rangle$, then on substituting (9) into (10), we get

$$
d^2(q_{n+1},p) \le (1-\theta_n)d^2(v_n,p) + \theta_n d^2(\psi(q_n),\psi(p)) + 2\theta_n\langle \exp_p^{-1}\psi(p), \exp_p^{-1}q_{n+1}\rangle
$$

$$\leq (1-\theta_n)d^2(q_n,p)-(1-\theta_n)d^2(w_n^k,q_n)-\eta_n(1-\eta_n)d^2(t_n,\Phi t_n)$$
$$+2\theta_n\langle \exp_p^{-1}\psi(p),\exp_p^{-1}q_{n+1}\rangle+\theta_n\phi d^2(q_n,p)$$
$$=(1-\theta_n(1-\phi))d^2(q_n,p)+\theta_n(1-\phi)\left[\frac{2\langle \exp_p^{-1}\psi(p),\exp_p^{-1}q_{n+1}\rangle}{(1-\phi)}\right]$$
$$-(1-\theta_n)d^2(w_n^k,q_n)-\eta_n(1-\eta_n)d^2(t_n,\Phi t_n) \tag{11}$$
$$\leq (1-\theta_n(1-\phi))d^2(q_n,p)+\theta_n(1-\phi)\left[\frac{2\langle \exp_p^{-1}\psi(p),\exp_p^{-1}q_{n+1}\rangle}{(1-\phi)}\right]. \tag{12}$$

Put $s_n := d^2(q_n,p)$ and $z_n := 2\langle \exp_p^{-1}\psi(p),\exp_p^{-1}q_{n+1}\rangle$. It follows from (11) that

$$s_{n+1}\leq (1-\theta_n(1-\phi))s_n+\theta_n(1-\phi)\left[\frac{z_n}{1-\phi}\right]. \tag{13}$$

We now establish that $s_n\to 0$ by considering two possible cases.

Case 1: Suppose $\{s_n\}$ is eventually decreasing, that is, there exists $N_0\geq 0$ such that $\{s_n\}$ is decreasing $N_0\geq 0$ such that $\{s_n\}$ is decreasing for $n\geq N_0$. In this case $\{s_n\}$ is decreasing for $n\geq N_0$. In this case $\{s_n\}$ must be convergent. Thus, we obtain from

$$(1-\theta_n)d^2(w_n^k,q_n)+\eta_n(1-\eta_n)d^2(t_n,\Phi t_n)\leq (1-\theta_n(1-\phi))s_n-s_{n+1}+\theta_n(1-\phi)\left[\frac{z_n}{1-\phi}\right].$$

Hence, using conditions (i) and (ii) of (4), we obtain that

$$\lim_{n\to\infty} d(w_n^k,q_n)=0=\lim_{n\to\infty} d(t_n,\Phi t_n). \tag{14}$$

It is obvious from (14) that $\lim_{n\to\infty} d(t_n,q_n)=0$. We can deduce from this that

$$\begin{cases}\lim\limits_{n\to\infty} d(v_n,t_n)=0,\\ \lim\limits_{n\to\infty} d(v_n,q_n)=0,\\ \lim\limits_{n\to\infty} d(q_{n+1},v_n)=0,\\ \lim\limits_{n\to\infty} d(q_{n+1},q_n)=0.\end{cases} \tag{15}$$

Since $\{q_n\}$ is bounded, there exists a subsequence $\{q_{n_l}\}$ of $\{q_n\}$ such that $q_{n_l}\rightharpoonup p$. We may assume, without any loss of generality, that $q_{n_l}\rightharpoonup x^*$. We claim that $x^*\in\Upsilon$. Indeed, since $\{q_n\}\subset C$, we have $x^*\in C$. Furthermore, since $\{v_n\}$ and $\{t_n\}$ are bounded, there exist subsequences $\{v_{n_l}\}$ of $\{v_n\}$ and $\{t_{n_l}\}$ of $\{t_n\}$ which converge weakly to $x^*\in\Upsilon$ respectively. Using the fact that $R_{\lambda_{n_l}}^{G_k,F_k}$ is nonexpansive and demiclosed at 0, we have from (14) and Lemma 2.17 (iii) that $x^*\in Fix(R_{\lambda_{n_l}}^{G_k,F_k})=\bigcap_{k=1}^{m} GEP(G_k,F_k)$. Similarly from (14), we have that $x^*\in Fix(\Phi)$. Thus, we conclude that $x^*\in\Upsilon$. Next, we claim that $\limsup_{n\to\infty} z_n\leq 0$. Since $\{q_{n_l}\}$ of $\{q_n\}$ which converges weakly to $x^*\in\Upsilon$ such that

$$\lim_{l\to\infty}\langle \exp_p^{-1}\psi(p),\exp_p^{-1}q_{n_l}\rangle=\lim_{n\to\infty}\langle \exp_p^{-1}\psi(p),\exp_p^{-1}q_n\rangle$$
$$=\langle \exp_p^{-1}\psi(p),\exp_p^{-1}x^*\rangle\leq 0. \tag{16}$$

On substituting (16) into (13) and applying Lemma 2.18, we conclude that $\{q_n\}$

converges strongly to p.

Case 2: Suppose that $\{s_n\}$ is not a monotone sequence. Then, using Lemma 2.19, we define an integer sequence $\{\sigma(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by $\sigma(n) := \max\{k \leq n : s_k < s_{k+1}\}$. Note that σ is an increasing sequence such that $\sigma(n) \to \infty$ as $n \to \infty$ and $s_{\sigma(n)} < s_{\sigma(n+1)}$ for all $n \geq n_0$. From (13), it follows that $0 < s_{\sigma(n)+1} - s_{\sigma(n)} \leq 2C_{\sigma(n)}\langle \exp_p^{-1}\psi(p), \exp_p^{-1} q_{\sigma(n)+1}\rangle$. Since $C_{\sigma(n)} \to 0$ and $\{q_n\}$ is bounded, we get

$$\lim_{n\to\infty}(s_{\sigma(n)+1} - s_{\sigma(n)}) = 0. \tag{17}$$

Following a similar approach as in **Case 1**, we obtain

$$\begin{cases} \lim\limits_{\tau(n)\to\infty} d(t_{\tau(n)}, \Phi t_{\tau(n)}) = 0, \\ \lim\limits_{\tau(n)\to\infty} d(w^k_{\tau(n)}, q_{\tau(n)}) = 0, \ k = 1, 2, \ldots, m. \end{cases} \tag{18}$$

Also, $s_{\tau(n)+1} \leq (1 - (1-\phi)C_{\sigma(n)})s_{\tau(n)} + C_{\sigma(n)}(1-\phi)\left[\frac{z_{\tau(n)}}{1-\phi}\right]$, where $\limsup\limits_{n\to\infty} z_{\tau(n)} \leq 0$. Since $s_{\tau(n)+1} > s_{\tau(n)}$ and $C_{\sigma(n)} > 0$, we have $(1-\phi)s_{\tau(n)} \leq z_{\tau(n)}$. Also, since $\limsup\limits_{n\to\infty} z_{\tau(n)} \leq 0$, we see that $\lim\limits_{n\to\infty} s_{\tau(n)} = 0$. Combining this with (17), this implies that $\lim\limits_{n\to\infty} s_{\tau(n)+1} = 0$. Since $0 \leq s_{\tau(n)} \leq \max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1} \to 0$, we obtain that $s_n \to 0$, thus the sequence $\{q_n\}$ converges strongly to $p \in \Upsilon$. □

We now state some of the consequences of our result.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a Hadamard manifold $\mathbb{M}$. For $k = 1, 2, \ldots, m$, let $F_k : C \to \mathbb{TM}$ be a monotone vector field and let $G_k : C \times C \to \mathbb{R}$ be such that $G_k(u, u) = 0$ for all $u \in C$ be a bifunction satisfying (L1)-(L3). Let $\psi : C \to C$ be a contraction mapping with constant $\phi \in (0, 1)$ and let $\Phi : C \to C$ be a nonexpansive mapping such that $\Upsilon := \bigcap\limits_{k=1}^{m} Fix(R^{G_k,F_k})$ is nonempty. For an arbitrary $q_1 \in \mathbb{Z}$, let $\{q_n\}$ be generated iteratively by*

$$\begin{cases} w^k_n = R^{G_k,F_k}_{\lambda_n} q_n, k = 1, 2, \ldots, N; \\ t_n \in \{w^k_n, k = 1, 2, \ldots, m\} \text{ such that } d(t_n, q_n) = \max\limits_{1\leq k\leq N} d(w^k_n, q_n); \\ q_{n+1} = \gamma^b_n(1-\theta_n), \ \forall\ n \geq 1; \end{cases} \tag{19}$$

where $\gamma^b_n : [0, 1] \to \mathbb{M}$ is a sequence of geodesic joining $\psi(q_n)$ to t_n and the sequences $\{\theta_n\} \in (0, 1)$ and $\{r_n\} \in (0, \infty)$ satisfy the following:

(i) $\lim\limits_{n\to\infty} \theta_n = 0$; $\sum\limits_{n=1}^{\infty} \theta_n < \infty$, *(ii)* $0 < \lambda \leq \lambda_n$.

Then, the sequence $\{q_n\}$ converges to an element $p \in \Upsilon$.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a Hadamard manifold $\mathbb{M}$. For $k = 1, 2, \ldots, m$, let $G_k : C \times C \to \mathbb{R}$ such that $G_k(u, u) = 0$ for all $u \in C$ be a bifunction satisfying $(L1) - (L3)$. Let $\psi : C \to C$ be a contraction mapping with constant $\phi \in (0, 1)$ and let $\Phi : C \to C$ be a nonexpansive mapping such* *that* $\Upsilon := Fix(\Phi) \cap \bigcap_{k=1}^{m} Fix(R^{G_k})$ *is nonempty. For an arbitrary* $q_1 \in \mathbb{Z}$, *let* $\{q_n\}$ *be generated iteratively by*

$$\begin{cases} w_n^k = R_{\lambda_n}^{G_k} q_n, k = 1, 2, \ldots, N; \\ t_n \in \{w_n^k, k = 1, 2, \ldots, m\} \text{ such that } d(t_n, q_n) = \max_{1 \le k \le N} d(w_n^k, q_n); \\ v_n = \exp_{t_n}(1 - \eta_n) \exp_{t_n}^{-1} \Phi t_n; \\ q_{n+1} = \gamma_n^b(1 - \theta_n), \ \forall \ n \ge 1; \end{cases}$$

where $\gamma_n^b : [0, 1] \to \mathbb{Z}$ *is a sequence of geodesic joining* $\psi(q_n)$ *to* v_n *and the sequences* $\{\eta_n\}, \{\theta_n\} \in (0, 1)$ *and* $\{r_n\} \in (0, \infty)$ *satisfy the following:*

(i) $\lim_{n\to\infty} \theta_n = 0 \quad \sum_{n=1}^{\infty} \theta_n < \infty$;

(ii) $0 < a \le \eta_n \le b < 1$ *for some* $a, b > 0$ *for all* $n \ge 1$;

(iii) $0 < \lambda \le \lambda_n$.
Then, the sequence $\{q_n\}$ *converges to an element* $p \in \Upsilon$.

Proof. If $F = 0$, the result follows from the proof of (19). □

4. Application

In this section, we apply our main result to determining the common solution of fixed point and convex minimization problems. We consider the following convex minimization of the sum of convex functions $\min_{x\in\mathbb{M}} h_1(x) + h_2(x)$, where $\mathbb{M}$ is a Hadamard manifold, $h_1 : \mathbb{M} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function, $h_2 : \mathbb{M} \to \mathbb{R}$ is a convex and differentiable function. It is worth-mentioning that the problem of finding $x \in \mathbb{M}$ such that $\langle Fx, \exp_x^{-1} y\rangle \ge 0$ for all $y \in \mathbb{M}$ is the optimality condition of the convex minimization problem when $G = \nabla h_2$. In addition, the Moreau-Yosida regulariazation $J_\mu^{h_1} : \mathbb{M} \to \mathbb{R}$ of a function h_1 is defined by

$$J_\mu^{h_1}(x) = arg \min_{y\in\mathbb{M}} \left(h_1(y) + \frac{1}{2\mu} d^2(x, y) \right);$$

it is the resolvent of the bifunction $G : M \times M \to \mathbb{R}$ defined by $G(x, y) = h_1(y) - h_1(x)$. It is known (see [9]) that there exists a $t_\mu = J_\mu^{h_1}(x)$ for any $x \in \mathbb{M}$ and $\mu \ge 0$ with the property $\frac{1}{\mu} \exp_{t_\mu}^{-1} x \in \partial h_1(x)$. The mapping $J_\mu^{h_1}$ is consistent, and the fixed point of $J_\mu^{h_1}(x)$ is a solution of the minimization problem $\min_{x\in\mathbb{M}} h_1(x)$.

Theorem 4.1. *Let* C *be a nonempty, closed and convex subset of a Hadamard manifold* $\mathbb{M}$. *Let* $h_1 : \mathbb{M} \to \mathbb{R} \cup \{+\infty\}$ *be proper, convex and lower semicontinuous and let* $h_2 : \mathbb{M} \to \mathbb{R}$ *be a convex and differentiable function with* $h = h_1 + h_2$. *Let* $\phi : C \to C$ *be a contraction mapping with constant* $\phi \in (0, 1)$ *and let* $\Phi : C \to C$ *be a nonexpansive mapping such that* $\Upsilon : Fix(\Phi) \bigcap arg \min_{x\in\mathbb{M}} h$ *is nonempty. For an arbitrary* $q_1 \in \mathbb{Z}$, *let* $\{q_n\}$ *be generated iteratively by*

$$\begin{cases} w_n = J^h_{\mu_n}(q_n), \\ v_n = \exp_{t_n}(1-\eta_n)\exp^{-1}_{t_n}\Phi w_n, \\ q_{n+1} = \gamma^b_n(1-\theta_n), \ \forall \ n \geq 1, \end{cases} \tag{20}$$

where $\gamma^b_n : [0,1] \to \mathbb{Z}$ *is a sequence of geodesics joining* $\psi(q_n)$ *to* v_n *and the sequences* $\{\eta_n\}, \{\theta_n\} \in (0,1)$ *and* $\{r_n\} \in (0,\infty)$ *satisfy the following:*

(i) $\lim\limits_{n\to\infty} \theta_n = 0$*;* $\sum\limits_{n=1}^{\infty} \theta_n < \infty$,

(ii) $0 < a \leq \eta_n \leq b < 1$ *for some* $a, b > 0$ *for all* $n \geq 1$*;*

(iii) $0 < \mu \leq \mu_n$.

Then, the sequence $\{q_n\}$ *converges to an element* $p \in \Upsilon$.

Proof. Since $R^{G,F}_{\lambda_n}$ and $J^h_{\mu_n}$ have the same properties, then the proof of Theorem 4.1 follows from the proof of Theorem 3.1. □

5. Numerical example

In this section, we present a numerical example in the setting of a Hadamard manifold to show the performance of Algorithm 4.

Let $\mathbb{M} = \mathbb{R}^{++} = \{x \in \mathbb{R} : x > 0\}$ and let $(\mathbb{M}, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold with $\langle\cdot,\cdot\rangle$ the Riemannian metric defined by $\langle u, v\rangle = \frac{1}{x^2}uv$, for all $u, v \in T_x\mathbb{M}$, where $T_x\mathbb{M}$ is the tangent space at $x \in \mathbb{M}$. For $x \in \mathbb{M}$, the tangent space $T_x\mathbb{M}$ at x equals $\mathbb{R}$, i.e. $T_x\mathbb{M} = \mathbb{R}$. The Riemannian distance (see [4]) $d : \mathbb{M} \times \mathbb{M} \to \mathbb{R}^+$ is defined by $d(x, y) = |\ln \frac{x}{y}|$, $\forall\ x, y \in \mathbb{M}$. Then $(\mathbb{M}, \langle.,.\rangle)$ is a Hadamard manifold and the unique geodesic $\gamma : \mathbb{R} \to \mathbb{M}$ starting from $\gamma(0) = x$ with $q = \gamma'(0) \in T_x\mathbb{M}$ is defined by $\gamma(t) = x \exp^{\frac{qt}{x}}$. Thus $\exp_x qt = x \exp^{\frac{qt}{x}}$. The inverse exponential map is defined by $\exp^{-1}_x y = \gamma'(0) = x \ln \frac{y}{x}$.

Example 5.1. Let $C = [1, +\infty)$ be a geodesically convex subset of $\mathbb{R}^{++}$ and let $G : C \times C \to \mathbb{R}$ be a bifunction defined for all $x, y \in C$ by $G_k = -\frac{k}{1+k} \ln \frac{y}{z}$ and let $F : C \to \mathbb{R}$ be a single valued vector field defined by $F_k z = kz \ln z$ for all $z \in C$. Then, it is easy to see that assumptions (L1)-(L3) are satisfied. Using Lemma 2.17 for $z \in C$, we have

$$\begin{aligned} 0 &\leq F_k(z, y) + \langle A_k z, \exp^{-1}_z y\rangle - \frac{1}{r}\langle \exp^{-1}_z x\rangle \\ &= \frac{k}{1+k} \ln \frac{y}{z} + \langle kz \ln z, z \ln \frac{y}{z}\rangle - \frac{1}{r}\langle z \ln \frac{y}{z}, z \ln \frac{x}{z}\rangle \\ &= \frac{k}{1+k} \ln \frac{y}{z} + k\left(\frac{1}{z^2}\right) z \ln z \times z \ln \frac{y}{z} - \frac{1}{r}\left(\frac{1}{z^2}\right) z \ln y \times z \ln \frac{x}{z} \end{aligned}$$

$$= \frac{k}{1+k}\ln\frac{y}{z} + k\ln z\ln\frac{y}{z} - \frac{1}{r}\ln\frac{y}{z}\ln\frac{x}{z},$$

which implies that

$$\frac{1}{r}\ln\frac{y}{z}\ln\frac{x}{z} = \frac{k}{1+k}\ln\frac{y}{z} + k\ln z\ln\frac{y}{z} \Longrightarrow \frac{(k+1)\ln x - kr}{k+1} = (kr+1)\ln z.$$

Thus,
$$z = \exp\frac{(k+1)\ln x - kr}{(k+1)(kr+1)}.$$

Let $\phi : \mathbb{M} \to \mathbb{M}$ be defined by $\phi(x) = \frac{x}{2}$. Choose $\mu_n = \frac{1}{2}, \theta_n = \frac{1}{n+1}$ and $\beta_n = \frac{1}{2n+3}$. Using $E_n = d^2(x_n, x_{n+1}) \leq \varepsilon$ with $\varepsilon = 10^{-4}$ as stopping criterion, we perform this experiment for varying values of q_1.

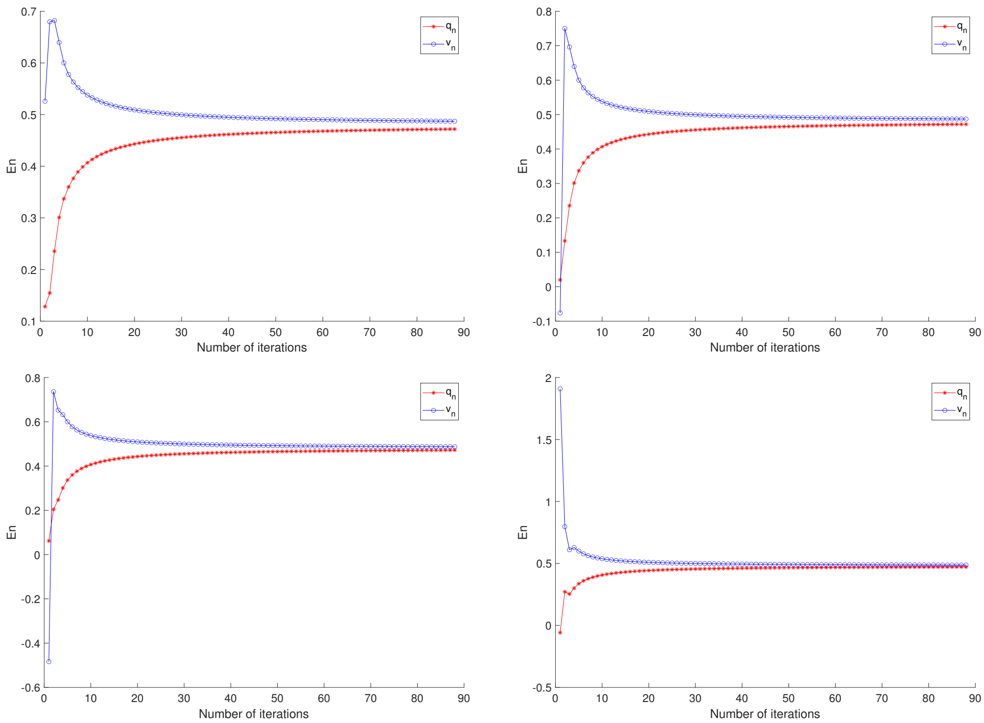

Figure 1: Example 5.1 . Top left: $m = 30$, top right: $m = 40$, bottom left: $m = 50$, bottom right: $m = 70$.

6. Conclusion

We propose an iterative method for determining a common solution of a finite family of generalized equilibrium problems and a fixed point problem for a nonexpansive mapping on Hadamard manifolds. Using a viscosity iterative algorithm, we proved that the sequence generated by our algorithm converges to a solution of the finite family of generalized equilibrium problems and a fixed point problem for a nonexpansive mapping. Lastly, we presented an application to a convex minimization problem and a numerical example to demonstrate the performance of our algorithm.

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