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FIXED POINTS OF COUPLED HYBRID CONTRACTIONS AND APPLICATIONS

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Abstract. In this paper, we prove a coupled fixed-point result for a hybrid mapping derived from generalized coupled Banach and Kannan-type contractions. We define the asymptotic regularity property for coupled maps and use it as a condition in our theorem. The results are derived in metric spaces with a preorder relation. The necessity of the hybrid contraction inequality is constrained by the preordering through the requirement that the inequality must be satisfied between points related by the preorder relation in a particular way. The main theorem is proven under several alternative additional conditions. There are several consequences of the main result, one of which is the relaxation of the contraction constant's range in the Kannan-type result. Two examples illustrate several features of the results presented herein. The paper concludes with an application to a system of integral equations.

1. Introduction

The program in this paper is to establish hybrid coupled fixed point results in a complete metric space having an additional preorder relation defined on it. We use a hybrid metric inequality which is a combination of the extended versions of coupled Banach type [10, 18] and coupled Kannan type [6] inequalities. The hybrid inequality is supposed to hold for points connected by the preorder relation in a particular way.

An interesting observation on the proofs of several fixed point theorems has been that the contraction condition is not utilized for all choices of points from the metric space. It is possible to restrict the scope of the contraction condition to certain choices of points only. Assumption of an additional order relation on the space is one approach to realize the above objective in which the contraction condition is postulated only for points appropriately related by the order relation. Several works on fixed point theory implementing the aforesaid approach with various types of relations are noted in [1, 10, 14, 18].

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The above is the motivation for introducing the preorder relation on the metric space in the present context. A preorder is weaker than a partial order. It serves our efforts towards achieving the goal of restricting the scope of the coupled inequality between pairs of points as minimally as possible while ensuring the validity of the fixed point result. Such uses of preorder relations have already appeared in the literature in the works like [12, 14].

One of the main requirements on the coupled mappings in our theorem is the property of asymptotic regularity which is defined herein for coupled mappings. This concept has been in use in the fixed point theory of ordinary functions. References [11–14] are instances of these works.

Our main result is shown to remain valid under three alternative sets of conditions. We define in this context two new definitions applicable to coupled maps in order to create these conditions.

We have two illustrative examples and an application to a system of integral equations.

Certain features of the work are summarized below.

- The definition of asymptotic regularity for coupled mappings is introduced.
- k-continuity for coupled mappings is introduced.
- Orbital continuity for coupled mappings is introduced.
- A hybrid coupled contraction condition is introduced for which fixed point results are obtained.
- The range of contraction constant in coupled Kannan type fixed point theorem is relaxed under three different conditions.
- A generalization of coupled Banach contraction by use of preorder relation is established.
- Several examples illustrate the new definitions and establish the fact that our results are proper extensions of some previous results.
- An application to a system of nonlinear integral equations is discussed.

2. Mathematical background

Let X be a nonempty set and $T : X \times X \to X$ be a map. An element $(x, y) \in X \times X$ is called a coupled fixed point of T if $T(x, y) = x$ and $T(y, x) = y$.

Definition 2.1 ([2, Coupled Banach Contraction mapping]). Let (X, d) be a metric space and $T : X \times X \to X$ be a map. T is called a coupled Banach contraction if there exists $k \in (0, 1)$ such that for all $(x_1, y_1), (x_2, y_2) \in X \times X$,

$$d\left(T(x_1, y_1), T(x_2, y_2)\right) \leq k \cdot \frac{d\left(x_1, x_2\right) + d(y_1, y_2)}{2}.$$

Bhaskar et al [2] established a fixed point result for the above coupled mapping. The result of Bhaskar has many extensions in works like [18, 21].

Definition 2.2 ([6, Coupled Kannan Contraction mapping]). Let (X, d) be a metric space and $T : X \times X \to X$ be a map. T is called a coupled Kannan contraction if there exists $k \in (0, \frac{1}{2})$ such that for all $(x_1, y_1), (x_2, y_2) \in X \times X$,

$$d\left(T(x_1, y_1), T(x_2, y_2)\right) \le k \cdot \{d(x_1, T(x_1, y_1)) + d(x_2, T(x_2, y_2))\}. \tag{1}$$

Kannan type contractions have featured prominently in fixed point theory due to their distinguished property that they can admit of discontinuities which a Banach contraction does not possess. Some prominent works, amongst others, in fixed point theory dealing with these mappings are noted in [6, 7].

Definition 2.3 ([12, Preodered set]). Let $X(\neq \emptyset)$ be a set and a binary relation $\preceq$ on X is
(a) reflexive if $x \preceq x$ for all $x \in X$,

(b) transitive if $x \preceq z$ for all $x, y, z \in X$ such that $x \preceq y$ and $y \preceq z$.
Then "$\preceq$" is called a preorder and $(X, \preceq)$ is a preordered set.

We will say that $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$.

Example 2.4. Consider a relation on $\mathbb{N}$ by $x \preceq y \iff x//4 \le y//4$ where, $x//4$ means the greatest integer that is less than or equal to x divided by 4.

The relation $\preceq$ is reflexive and transitive but not anti-symmetric. Hence $(\mathbb{N}, \preceq)$ forms a preorderd space and not a partial order.

Definition 2.5 ([2, Mixed monotone property]). Let $(X, \preceq)$ be a preordered set and $F{:}X{\times}X{\to}X$. We say that F has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, x_1 \preceq x_2 &\implies F(x_1, y) \preceq F(x_2, y), \\ y_1, y_2 \in X, y_1 \preceq y_2 &\implies F(x, y_2) \preceq F(x, y_1). \end{aligned}$$

Further, we endow the product space $X \times X$ with the following preorder: for $(x_1, y_1), (x_2, y_2) \in X \times X$, $(x_1, y_1) \preceq (x_2, y_2) \iff x_1 \preceq x_2, \quad y_2 \preceq y_1$.

Definition 2.6 ([2, Monotone regular]). $(X, d, \preceq)$ is said to be monotonic regular if the following conditions hold:
(a) if for any nondecreasing sequence $\{x_n\}$ i.e., $x_n \preceq x_{n+1}$, which is convergent to x as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;

(b) if for any nonincreasing sequence $\{y_n\}$ i.e., $y_{n+1} \preceq y_n$, which is convergent to y as $n \to \infty$, then $y \preceq y_n$ for all $n \in \mathbb{N}$.

Asymptotic regularity is a concept which has a very wide usage in fixed point theory [12–14]. We define it for coupled maps.

Let X be a non-empty set and $F : X \times X \to X$ be a coupled mapping. For any $(x, y) \in X \times X$, define $F^0(x, y) = x$ and $F^n(x, y) = F(F^{n-1}(x, y), F^{n-1}(y, x))$ for $n = 1, 2, 3 \ldots$.

DEFINITION 2.7 (Asymptotic regularity of coupled maps). Let (X, d) be a metric space. A coupled mapping $F : X \times X \to X$ is called asymptotic regular if $d(F^{n+1}(x, y), F^n(x, y)) \to 0$ as $n \to \infty$ for all $(x, y) \in X \times X$.

Ćirić [8], in 1971, introduced the definition of orbital continuity to weaken the notion of continuity for single-valued self-mappings. Another weaker form of continuity, k-continuity for single-valued maps, was introduced by Pant [19]. Now, we define the following.

DEFINITION 2.8 (k-continuity of coupled maps). Let $F : X \times X \to X$ be a coupled mapping where (X, d) is a metric space. F is called coupled k-continuous for $k = 1, 2, 3, \ldots$ if for any two sequence $\{x_n\}, \{y_n\}$ in X, $F^k(x_n, y_n) \to F(x, y)$ whenever $F^{k-1}(x_n, y_n) \to x$ and $F^{k-1}(y_n, x_n) \to y$ as $n \to \infty$.

Clearly, 1-continuity is equivalent to continuity. Moreover, k-continuity implies $(k + 1)$-continuity for all $k \geq 1$, but not conversely.

EXAMPLE 2.9. Let $X=[0, 2]$ be equipped with the usual metric. Define $F:X\times X\to X$ by

$$F(x, y) = \begin{cases} 1 & \text{if} \quad 0 \leq x, y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here $F^2(x_n, y_n) \to 1$ and $F^2(y_n, x_n) \to 1$ implies $F^3(x_n, y_n) \to 1 = F(1, 1)$ as $n \to \infty$. Hence F is 3-continuous. However, F is not 2-continuous.

DEFINITION 2.10 (Orbital continuity of coupled maps). Let (X, d) be a metric space and $F : X \times X \to X$ be a coupled mapping. Let x_0 and y_0 be any two points in X. Let us construct two sequences $\{x_n\}$ and $\{y_n\}$ by $x_n = F(x_{n-1}, y_{n-1})$ and $y_n = F(y_{n-1}, x_{n-1})$ for $n = 1, 2, 3, \ldots$. Then we define

$$O(F; x_0, y_0) = \{(x_n, y_n) : n = 0, 1, 2, 3, \ldots\} = \{(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\}$$

as the coupled orbit of F at $(x_0, y_0) \in X \times X$.

We define F to be coupled orbitally continuous if for any sequence $\{(x_n, y_n)\} \subset O(F; x_0, y_0)$ for some $(x_0, y_0) \in X \times X$, $x_n \to x$ and $y_n \to y$ as $n \to \infty$ implies $F(x_n, y_n) \to F(x, y)$ as $n \to \infty$.

Every continuous mapping is obviously coupled orbitally continuous, but not conversely.

EXAMPLE 2.11. Let $X=[0, 2]$ be equipped with the usual metric. Define $F:X\times X\to X$ by

$$F(x, y) = \begin{cases} 1 & \text{if} \quad 0 \leq x, y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here, F is coupled orbitally continuous but not continuous.

Both the above concepts of k-continuity [19] and orbital continuity [8] are used in fixed point theory for single-valued mappings in references like [3–5, 13, 14]. In the above two definitions they are extended to the case of coupled mappings.

3. Main results

In this section, our main objective is to obtain a coupled fixed point theorem for mappings satisfying a hybrid contraction inequality along with some other conditions in a preordered metric space.

Let Φ denote the class of all mappings $\varphi : [0, \infty) \to [0, \infty)$ satisfying

(i) $\varphi(t) < t$ for all $t > 0$,

(ii) φ is upper semi-continuous, that is, $\lim_{r \to t+} \varphi(r) < \varphi(t)$ for each $t > 0$.

THEOREM 3.1. *Let $(X, d, \preceq)$ be a preordered complete metric space and $F : X \times X \to X$ be an asymptotically regular mapping with mixed monotone property satisfying*

$$d(F(x,y), F(u,v)) \leq \varphi\left(\frac{d(x,u) + d(y,v)}{2}\right) + K \cdot [d(x, F(x,y)) + d(u, F(u,v))] \quad (2)$$

for any $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$ where the function $\varphi \in \Phi$ and $0 \leq K < \infty$. Suppose that one of the following conditions holds:

(i) F is coupled k-continuous for some $k \geq 1$;

(ii) F is coupled orbitally continuous;

(iii) $(X, d, \preceq)$ is monotone regular (Definition 2.6) and $0 \leq K < 1$.

If there exists $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has at least one coupled fixed point.

Proof. Let there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Since $x_0 \preceq F(x_0, y_0) = x_1$(say) and $y_0 \succeq F(y_0, x_0) = y_1$(say), letting $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$, we denote

$$F^2(x_0, y_0) = F(F(x_0, y_0), F(y_0, x_0)) = F(x_1, y_1) = x_2$$

and

$$F^2(y_0, x_0) = F(F(y_0, x_0), F(x_0, y_0)) = F(y_1, x_1) = y_2.$$

Thus we construct for $n = 1, 2, 3, \ldots$,

$$F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)) = F(x_n, y_n) = x_{n+1}$$

and

$$F^{n+1}(y_0, x_0) = F(F^n(y_0, x_0), F^n(x_0, y_0)) = F(y_n, x_n) = y_{n+1}.$$

Thus, by mixed monotone property of F, we construct two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_0 \preceq x_1 \preceq x_2 \preceq \cdots\cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$ and $y_0 \succeq y_1 \succeq y_2 \succeq \cdots\cdots \succeq y_n \succeq y_{n+1} \succeq \cdots$.

Now we prove that both the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy. Suppose, to the contrary, that at least one of them is not Cauchy. Then there exists $\epsilon > 0$ and subsequences $\{x_n\}$ and $\{y_n\}$ for integers $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ with $r_k = d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) \geq \epsilon$ for $k = 1, 2, 3, \cdots$. Also, choosing m_k small as possible, we can assume $d(x_{n_k}, x_{m_k - 1}) + d(y_{n_k}, y_{m_k - 1}) < \epsilon$. Now, for each $k \in \mathbb{N}$, using we have

$$\begin{aligned}\epsilon \leq r_k &= d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) \\ &\leq d(x_{n_k}, x_{m_k - 1}) + d(x_{m_k - 1}, x_{m_k}) + d(y_{n_k}, y_{m_k - 1}) + d(y_{m_k - 1}, y_{m_k})\end{aligned}$$

$$< \epsilon + d(x_{m_k-1}, x_{m_k}) + d(y_{m_k-1}, y_{m_k}). \tag{3}$$

Taking $k \to \infty$ in (3) and using asymptotic regularity, we get,

$$\lim_{k\to\infty} r_k = \epsilon. \tag{4}$$

Also,

$$\begin{aligned} r_k &= d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) \\ &\leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k}) \\ &+ d(y_{n_k}, y_{n_k+1}) + d(y_{n_k+1}, y_{m_k+1}) + d(y_{m_k+1}, y_{m_k}) \\ &\leq d(x_{n_k}, x_{n_k+1}) + d(x_{m_k+1}, x_{m_k}) + d(x_{n_k+1}, x_{m_k+1}) \\ &+ d(y_{n_k+1}, y_{m_k+1}) + d(y_{n_k}, y_{n_k+1}) + d(y_{m_k+1}, y_{m_k}). \end{aligned} \tag{5}$$

Using (2), we write

$$\begin{aligned} d(x_{n_k+1}, x_{m_k+1}) &= d(F(x_{n_k}, y_{n_k}), F(x_{m_k}, y_{m_k})) \\ &\leq \varphi\left(\frac{d(x_{n_k}, x_{m_k})+d(y_{n_k}, y_{m_k})}{2}\right) + K\cdot [d(x_{n_k}, x_{n_k+1})+d(x_{m_k}, x_{m_k+1})] \\ &\leq \varphi\left(\frac{r_k}{2}\right) + K\cdot [d(x_{n_k}, x_{n_k+1})+d(x_{m_k}, x_{m_k+1})]. \end{aligned} \tag{6}$$

Similarly,

$$\begin{aligned} d(y_{n_k+1}, y_{m_k+1}) &= d(F(y_{n_k}, x_{n_k}), F(y_{m_k}, x_{m_k})) \\ &\leq \varphi\left(\frac{d(y_{n_k}, y_{m_k})+d(x_{n_k}, x_{m_k})}{2}\right) + K\cdot [d(y_{n_k}, y_{n_k+1})+d(y_{m_k}, y_{m_k+1})] \\ &\leq \varphi\left(\frac{r_k}{2}\right) + K\cdot [d(y_{n_k}, y_{n_k+1})+d(y_{m_k}, y_{m_k+1})]. \end{aligned} \tag{7}$$

Using (6) and (7) in (5) we obtain

$$r_k \leq 2\varphi\left(\frac{r_k}{2}\right) + (1+K)\cdot [d(x_{n_k}, x_{n_k+1})+d(x_{m_k}, x_{m_k+1})+d(y_{n_k}, y_{n_k+1})+d(y_{m_k}, y_{m_k+1})].$$

Letting $k \to \infty$, using (4), asymptotic regularity and upper semi-continuity of φ, we obtain $0 < \epsilon = \lim_{k\to\infty} r_k \leq 2 \cdot \lim_{k\to\infty} \varphi(\frac{r_k}{2}) + 0 \leq 2 \cdot \varphi(\frac{\epsilon}{2}) < \epsilon$, which is a contradiction. Therefore, we proved that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy.

Since, X is complete $\lim_{n\to\infty} x_n = z_1$ and $\lim_{n\to\infty} y_n = z_2$. We show that (z_1, z_2) is the coupled fixed point, i.e. $F(z_1, z_2) = z_1$ and $F(z_2, z_1) = z_2$.

(i) Suppose that F is coupled k-continuous for some $k \geq 1$. Since $F^k(x_n, y_n) \to z_1$ and $F^k(y_n, x_n) \to z_2$ as $n \to \infty$, coupled k-continuity of F implies that $F^{k+1}(x_n, y_n) \to F(z_1, z_2)$ and $F^{k+1}(y_n, x_n) \to F(z_2, z_1)$ as $n \to \infty$. By uniqueness of the limit, we get $F(z_1, z_2) = z_1$ and $F(z_2, z_1) = z_2$.

(ii) Suppose that F is coupled orbitally continuous. Since $x_n \to z_1$ and $y_n \to z_2$ as $n \to \infty$, by coupled orbital continuity of F, $F(x_n, y_n) \to F(z_1, z_2)$ as $n \to \infty$, i.e. $x_{n+1} \to F(z_1, z_2)$ as $n \to \infty$.

By uniqueness of the limit, $F(z_1, z_2) = z_1$. Also, $F(y_n, x_n) \to F(z_2, z_1)$ or $y_{n+1} \to F(z_2, z_1)$ as $n \to \infty$. By uniqueness of the limit, $F(z_2, z_1) = z_2$.

(iii) Suppose $(X, d, \preceq)$ is monotonic regular and $0 \leq K < 1$. Since, x_n is monotonic non-decreasing and $x_n = F(x_{n-1}, y_{n-1}) \to z_1$, by monotonic regularity, we have $x_n \preceq z_1$ for all $n \geq 0$.

Similarly, y_n is monotonic non-increasing and $y_n = F(y_{n-1}, x_{n-1}) \to z_2$, by monotonic regularity, we have $y_n \succeq z_2$ for all $n \geq 0$. Now,

$$d(z_1, F(z_1, z_2)) \leq d(z_1, F(x_n, y_n)) + d(F(x_n, y_n), F(z_1, z_2))$$
$$\leq d(z_1, x_{n+1}) + \varphi\left(\frac{d(x_n, z_1) + d(y_n, z_2)}{2}\right) + K \cdot [d(x_n, F(x_n, y_n)) + d(z_1, F(z_1, z_2))].$$

So,

$$(1-K) \cdot d(z_1, F(z_1, z_2)) \leq d(z_1, x_{n+1}) + \varphi\left(\frac{d(x_n, z_1) + d(y_n, z_2)}{2}\right)$$
$$+ K \cdot d(F^n(x_0, y_0), F^{n+1}(x_0, y_0)) \to 0 \text{ as } n \to \infty.$$

This implies that $F(z_1, z_2) = z_1$.

Similarly,

$$d(z_2, F(z_2, z_1)) \leq d(z_2, F(y_n, x_n)) + d(F(y_n, x_n), F(z_2, z_1))$$
$$\leq d(z_2, y_{n+1}) + \varphi\left(\frac{d(y_n, z_2) + d(x_n, z_1)}{2}\right) + K \cdot [d(y_n, F(y_n, x_n)) + d(z_2, F(z_2, z_1))].$$

So,

$$(1-K) \cdot d(z_2, F(z_2, z_1)) \leq d(z_2, y_{n+1}) + \varphi\left(\frac{d(y_n, z_2) + d(x_n, z_1)}{2}\right)$$
$$+ K \cdot d(F^n(y_0, x_0), F^{n+1}(y_0, x_0)) \to 0 \text{ as } n \to \infty.$$

This implies that $F(z_2, z_1) = z_2$. Hence, (z_1, z_2) is a coupled fixed point. □

Remark 3.2. In the above theorem, the coupled inequality (2) is a hybrid combination of Banach type as well as Kannan type coupled contraction inequality. Both types of inequality have been widely considered in literature. Here the above theorem is deduced by combining the above two types of results.

Corollary 3.3. *Let $(X, d, \preceq)$ be a preordered complete metric space and $F: X \times X \to X$ be an asymptotically regular mapping with mixed monotone property satisfying*

$$d(F(x, y), F(u, v)) \leq \varphi\left(\frac{d(x, u) + d(y, v)}{2}\right) \tag{8}$$

for any $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$ where the function $\varphi \in \Phi$ and $0 \leq K < \infty$. Suppose that one of the following conditions holds:
(i) F is coupled k-continuous for some $k \geq 1$;

(ii) F is coupled orbitally continuous;

(iii) $(X, d, \preceq)$ is monotone regular.
If there exists $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has at least one coupled fixed point.

Proof. Putting $K = 0$ in the inequality (2), we get the condition (8). The result of the corollary follows by an application of Theorem 3.1. □

Corollary 3.4. *Let* $(X, d, \preceq)$ *be a preordered complete metric space and* $F{:}X\times X\to X$ *be an asymptotically regular and mixed monotone mapping satisfying*

$$d(F(x,y),F(u,v)) \le K\cdot[d(x,F(x,y))+d(u,F(u,v))] \tag{9}$$

for any $x,y,u,v\in X$ *with* $x\preceq u$ *and* $y\succeq v$ *where* $0\le K<\infty$. *Suppose that one of the following conditions holds:*
(i) F *is coupled* k*-continuous for some* $k\ge 1$*;*

(ii) F *is coupled orbitally continuous;*

(iii) $(X,d,\preceq)$ *is non-decreasing regular and* $0\le K<1$.
If there exists $x_0, y_0\in X$ *such that* $x_0\preceq F(x_0,y_0)$ *and* $y_0\succeq F(y_0,x_0)$ *then* F *has at least one coupled fixed point.*

Proof. Since for any $\varphi\in\Phi$, $\varphi(t)\ge 0$ for all $t\ge 0$, (9) holds true if (2) is satisfied. The result of the corollary follows by an application of Theorem 3.1. □

Remark 3.5. Corollary 3.4 is concerned with a generalized coupled Kannan type inequality (9). It is well known that Kannan type contractions form a very important class of mappings in the context of fixed point theory. There are several reasons for treating these contractions as important. One, and perhaps the most prominent, reason is that they can admit of discontinuities. Kannan [17] proved that in a complete metric space any self mapping T on X satisfying the inequality $d(Tx,Ty)\le K\cdot\{d(x,Tx)+d(y,Ty)\}$ for all $x,y\in X$ with $0\le K<\frac{1}{2}$ has a unique fixed point. There has been considerable research for relaxing the range of K at the cost of suitable additional conditions. Some instances of these works are [11, 13]. Corollary 3.4 is a similar result with coupled mappings.

Example 3.6. Let $X=[0,1]$ be the complete metric space equipped with the usual metric d and consider $\varphi(t)=\frac{t}{1+t}$. We consider the universal relation $x\preceq y$ for all $x,y\in X$ which is trivially a preorder.

Define the map $F: X\times X\to X$ by $F(x,y)=\frac{x+y}{1+x+y}$. F satisfies (2) with $\varphi(t)=\frac{t}{1+t}$ for $(x,y),(u,v)\in X\times X$. Also, F is k-continuous for all $k\ge 1$, thus all the conditions of Theorem 3.1 are satisfied. Hence, F has at least one coupled fixed point. Note there are actually two fixed points, namely, $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$.

Remark 3.7. We observe that in the Example 3.6 there is no $K\in[0,\infty)$ such that (9) is satisfied. As for a number $\epsilon>0$ (very small), if $(x,y)=(\epsilon,0)$ and $(u,v)=(0,0)$, then $F(x,y)-F(u,v)=\frac{\epsilon}{1+\epsilon}\le K\cdot(\epsilon-\frac{\epsilon}{1+\epsilon})$, so $K\ge\frac{1}{\epsilon}\to\infty$ as $\epsilon\to 0$.

It shows that our main result in Theorem 3.1 properly contains the Corollary 3.4.

The above example also shows that the fixed point in Theorem 3.1 is not unique.

Example 3.8. Let $X=\{0,1,4,5,8\}$ be set with usual metric. Then (X,d) is a complete metric space. Consider the preorder on X

$$(x,y)\preceq(u,v)\iff(x=u \text{ and } y=v) \text{ or } (1\le x//4\le u//4 \text{ and } y//4\ge v//4\ge 1)$$

where $x//4$ means the greatest integer that is less than or equal to x divided by 4.

Define the mapping $F : X \times X \to X$ as follows: $F(0, i) = 0$ for $i = 0, 1, 4, 5$ and $F(0, 8) = 1$; $F(1, j) = 4; F(4, j) = 5; F(5, j) = 8; F(8, j) = 8$ for $j = 0, 1, 4, 5, 8$. Here, F is asymptotically regular.

F does not satisfy (1) on X, as for $(x, y) = (0, 0)$ and $(u, v) = (8, 8)$ there does not exist any $K \in [0, \infty)$ such that $d(F(x, y), F(u, v)) \leq K \cdot [d(x, F(x, y)) + d(u, F(u, v))]$.

On the other hand, it can be shown that for all $(x, y) \preceq (u, v)$, $d(F(x, y), F(u, v)) \leq 3 \cdot [d(x, F(x, y)) + d(u, F(u, v))]$. Also, F is a k-continuous function for $k \geq 1$. Hence, by Corollary 3.4, F has coupled fixed point. In this case, F has the coupled fixed points $(0, 0)$ and $(8, 8)$.

Remark 3.9. Here, the example demonstrates the importance of preorder in our theorems. Clearly, the contractivity condition does not hold on the whole space. It is assumed to hold only for points that are related in a specific way by the preorder relation. This is a sufficient assumption for the purpose of our theorem.

4. Application to system of nonlinear integral equations

In this section, by an application to Theorem 3.1, we study the problem of solving a system of nonlinear integral equations. Several authors have attempted to solve nonlinear integral equations with the help of coupled fixed point theory. For some references see [15, 16].

Let $I = [a, b]$, $a, b \in \mathbb{R}$ with $a < b$. $\mathcal{C}(I, \mathbb{R})$ be the metric space of all real-valued continuous functions defined over $I = [a, b]$.

Problem (P) : Consider the system of nonlinear integral equations:

$$x(t) = a(t) + \int_a^b \kappa(t, s) f(t, s, x(s), y(s)) ds, \quad y(t) = a(t) + \int_a^b \kappa(t, s) f(t, s, y(s), x(s)) ds,$$

where $x, y \in \mathcal{C}(I, \mathbb{R})$ is the unknowns, $t, s \in I = [a, b]$ and $\kappa : I \times I \to \mathbb{R}$ is the kernel function.

We analyze the Problem (P) under the following assumptions:

(P1) $a \in \mathcal{C}(I, \mathbb{R})$ and $f \in \mathcal{C}(I \times I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ are continuous.

(P2) $0 \leq \kappa(t, s) < \lambda$, for all $t, s \in I$ where $\lambda > 0$.

(P3) For $x_1 \leq x_2$ and $y_1 \leq y_2$, $f(t, s, x_1, y) - f(t, s, x_2, y) \leq 0$ and $f(t, s, x, y_1) - f(t, s, x, y_2) \geq 0$.

(P4) for all $(x, y), (u, v) \in X \times X$ and $t, s \in I$

$$|f(t, s, x(s), y(s)) - f(t, s, u(s), v(s))| \leq \frac{1}{\lambda(b - a)} \cdot \frac{|x(s) - u(s)| + |y(s) - v(s)|}{2 + |x(s) - u(s)| + |y(s) - v(s)|}.$$

Let $X = \mathcal{C}(I, \mathbb{R})$ be equipped with the metric $d(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|$. Let the complete metric space (X, d) is endowed with the relation

$$x, y \in \mathcal{C}(I, \mathbb{R}), \ x \preceq y \Longleftrightarrow x(t) \leq y(t) \text{ for all } t \in I.$$

Also define on $X \times X$ the preorder for $(x, y), (u, v) \in X \times X$,

$$(x, y) \preceq (u, v) \iff x(t) \leq u(t) \text{ and } y(t) \geq v(t) \text{ for all } t \in I.$$

Now, define the mapping $F : X \times X \to X$

$$F(x, y)(t) = a(t) + \int_a^b \kappa(t, s) f(t, s, x(s), y(s)) ds \text{ for all } t, s \in I.$$

Theorem 4.1. *Under the assumptions (P1)-(P4), Problem (P) has a solution in* $\mathcal{C}(I, \mathbb{R})$.

Proof. First we show that F has mixed monotone property. For $x_1 \preceq x_2$ and $t \in I$, we have

$$\begin{aligned}
&F(x_1, y)(t) - F(x_2, y)(t) \\
=&a(t) + \int_a^b \kappa(t, s) f(t, s, x_1(s), y(s)) ds - a(t) - \int_a^b \kappa(t, s) f(t, s, x_2(s), y(s)) ds \\
=&\int_a^b \kappa(t, s) \{f(t, s, x_1(s), y(s)) - f(t, s, x_2(s), y(s))\} ds \leq 0,
\end{aligned}$$

by assumptions (P2) and (P3). Hence $F(x_1, y)(t) \leq F(x_2, y)(t)$ for all $t \in I$.

Similarly, for $y_1 \preceq y_2$ and $t \in I$, we have

$$\begin{aligned}
&F(x, y_1)(t) - F(x, y_2)(t) \\
=&a(t) + \int_a^b \kappa(t, s) f(t, s, x(s), y_1(s)) ds - a(t) - \int_a^b \kappa(t, s) f(t, s, x(s), y_2(s)) ds \\
=&\int_a^b \kappa(t, s) \{f(t, s, x(s), y_1(s)) - f(t, s, x(s), y_2(s))\} ds \geq 0,
\end{aligned}$$

by assumptions (P2) and (P3). Hence $F(x, y_1)(t) \geq F(x, y_2)(t)$ for all $t \in I$. Thus $F(x, y)$ is monotone nondecreasing in x and monotone nonincreasing in y.

Now for any $(x, y) \preceq (u, v)$ and $t \in I$,

$$\begin{aligned}
&|F(x, y)(t) - F(u, v)(t)| \\
=&\left| a(t) + \int_a^b \kappa(t, s) f(t, s, x(s), y(s)) ds - a(t) - \int_a^b \kappa(t, s) f(t, s, u(s), v(s)) ds \right| \\
\leq&\left| \int_a^b \kappa(t, s) \{f(t, s, x(s), y(s)) - f(t, s, u(s), v(s)) ds\} \right| \\
\leq&\int_a^b |\kappa(t, s)| \cdot |f(t, s, x(s), y(s)) - f(t, s, u(s), v(s))| \, ds \\
\leq&\lambda \cdot \int_a^b |f(t, s, x(s), y(s)) - f(t, s, u(s), v(s))| \, ds \qquad \text{[by (P2)]} \\
\leq&\lambda \cdot \int_a^b \frac{1}{\lambda(b-a)} \cdot \frac{|x(s) - u(s)| + |y(s) - v(s)|}{2 + |x(s) - u(s)| + |y(s) - v(s)|} ds \qquad \text{[by (P4)]}
\end{aligned}$$

$$\leq \frac{1}{(b-a)} \cdot \int_a^b \frac{d(x,u)+d(y,v)}{2+d(x,u)+d(y,v)} ds \leq \frac{1}{(b-a)} \cdot \frac{d(x,u)+d(y,v)}{2+d(x,u)+d(y,v)} \int_a^b ds$$
$$= \frac{d(x,u)+d(y,v)}{2+d(x,u)+d(y,v)} = \varphi\left(\frac{d(x,u)+d(y,v)}{2}\right) \quad \text{where} \quad \varphi(t) = \frac{t}{1+t}.$$

Thus the inequality (2) holds for $K = 0$ and $\varphi(t) = \frac{t}{1+t}$.

Therefore F satisfies all the conditions of Theorem 3.1. Hence F has a coupled fixed point in $\mathcal{C}(I, \mathbb{R})$. Consequently, there exists a solution of Problem (P). □

5. Conclusion

In this paper, we introduce the notion of coupled asymptotic regularity. In fixed-point theory, asymptotic regularity has been applied to obtain new fixed-point results in several recent works. A description of this recent development is provided by Górnicki [12]. Here, we focus on coupled fixed-point theory. There is an equivalent formulation of coupled fixed-point results on product spaces under certain circumstances. However, this approach has some intrinsic limitations. We refer to [9] for a discussion of this issue. This alternative approach is not pursued here. For results obtained by adopting such approaches, we refer to works [20, 22]. Our results are derived in preordered metric spaces, which we consider for obtaining results under a minimalistic consideration of the hybrid contractive inequality. One of our findings is that the range of the contractive constant in the coupled Kannan-type inequality, for the purpose of obtaining coupled fixed-point results, can be relaxed under the addition of suitable conditions. Such problems are considered important and have been addressed in several works. It is believed that many more coupled fixed-point results can be generalized and new results can be obtained by pursuing the research approach we have adopted here. Several weaker continuity notions for single-valued mappings have appeared in the literature, a survey of which can be found in [4]. The Definitions 2.8 and 2.10, utilized to derive the main theorem of the paper, are extensions of two such notions noted in [4] to the case of coupled mappings. Other weaker continuity versions may be extended to coupled mappings, but they may require modification of inequality (2) to ensure the existence of a coupled fixed point. This would be a new problem to address in future work.

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